MAPPINGS ON FUNCTION SPACES

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THESIS

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Introduction

In the thesis, we investigate continuous mappings on function spaces with various topologies. To study them has been one of fundamental methods of functional analysis and infinite dimensional topology from the years of the Polish school; S. Banach [5], H. Steinhaus and J. P. Schauder in 1930s'. Among many kinds of mappings on function spaces with their various ranges, continuous real-valued functions (functionals) and (linear) homeomorphisms have been studied well, because they play important roles in the research of function spaces. In the area, studies of linear continuous functionals and linear homeomorphisms between function spaces with the sup-norm and the compact-open topologies are fruitful. Some concrete results on them can be seen in many literatures (for example, see [3], [4], [5], [15], [22] and [28]). Now, we concentrate our investigations on (not necessary linear) functionals on function spaces and linear homeomorphism between function spaces with the pointwise convergent topology. This thesis consists of three parts. In the first part Chapter 1, we introduce basic notations and terminology and express our motivation for the thesis.

In Chapter 2, we are concerned with supports for continuous functionals on function spaces. For linear continuous functionals, the concept of supports was already defined. They showed us the way of finding relations between linear continuous functionals and subsets of underlying spaces of function spaces. We can see some results on them in [4]. Now, we generalize such concepts for (not necessary linear) continuous functionals. Considering general continuous functionals, we have few informations on supports. At least, for every continuous functional, we would like to take a support as small as possible. Here, we mention the existence of minimum supports. In Section 2.2, we investigate the existence of them for continuous functionals on spaces of real-valued
continuous functions on a Tychonoff space with the pointwise convergent and compact-open topologies. The result in question is stated as Theorem 2.2.1. All results in the section completely depend on the author’s papers [17] and [18]. In Section 2.3, applying our renewal methods to supports for continuous functionals on spaces of real-valued “bounded” continuous functions with various topologies, the existence of minimum supports is proved in several cases. This result, which can be seen in [19], is stated as Theorem 2.3.1. In Section 2.4, some properties of supports are also mentioned. Theorems 2.4.6, 2.4.10, 2.4.14 and 2.4.18 contribute to clarify them.

Our main part Chapter 3 is dedicated to investigate linear homeomorphisms between function spaces with the pointwise convergent topology. In the research area, we have already known some nice results. Two Tychonoff spaces are said to be l-equivalent if function spaces on them with the pointwise convergent topology are linearly homeomorphic. In 1980, Pavlovskii [23] proved that two compacta have the same topological dimension if they are l-equivalent. Here, we refer to the Lebesgue covering dimension dim as “the topological dimension”. The topological invariant dim is the most important item for us among many ones, because it decides the basic structures of spaces. Owing to him, unfortunately, the converse of his theorem is not hold. This fact says that the classification of spaces depending on the linear topological structure of function spaces on them is precisely finer than the another one depending on dim. In the same paper, he also noted that every n-dimensional finite polyhedron is l-equivalent to the n-disk. Subsequently, several general topologists investigated spaces which are l-equivalent to the n-disk (for example, see [2], [13], [14] and [26]). In particular, Arhangel’skii [2] introduced a powerful notation “S-stability” and subsequently obtained an interesting progress. In [13], Kawamura and the author gave the generalization of Pavlovskii’s result to a topological manifolds. On the other hand, Valov [26]
applied Pelczyński’s method to an investigation of the $C_p$-theory and obtained characterizations of spaces which are $l$-equivalent to the Hilbert cube, and to the universal Menger compactum. It goes out saying here that disks are fundamental for us. But their simple structures had rather prevented us from applying Valov’s methods to them. We have two purposes in the chapter. The first is to prove that every $n$-dimensional compact topological manifold is $l$-equivalent to the $n$-disk. This result, which can be seen in [13], is stated in Section 3.2 as Corollary 3.2.2. In Section 3.3, we present the another one as Theorem 3.3.3; a characterization of spaces which are $l$-equivalent to the $n$-disk, which is given in [20].
Chapter 1
Definitions and preliminaries

In the chapter, we introduce basic notations and terminology and express our motivation for the thesis.

1.1 Basic terminology

In the thesis, we assume that all spaces under consideration are Tychonoff. Let $C(X)$ be the set of all real-valued continuous functions on $X$ and $C^b(X)$ the set of all real-valued bounded continuous functions on $X$. Below, the symbol $C(X)$ is intended to mean both $C(X)$ and $C^b(X)$ when there is no need to distinguish them. We call a real-valued function on $C(X)$ a functional. The symbols $C_p(X)$ and $C_k(X)$ denote function spaces over $X$ with the pointwise convergent topology and the compact-open topology, respectively. The symbols $\mathbb{R}$, $I$, $D^n$ and $\omega$ denote the real line, the unit closed interval, the $n$-disk and the first infinite ordinal, respectively. For a function $f$ on $X$ and a subset $M$ of $X$, the restriction of $f$ to $M$ is denoted by $f|_M$. The symbol $\pi_M$ denotes the map from $\mathbb{R}^X$ to $\mathbb{R}^M$ defined by $\pi_M(f) = f|_M$. We use the same symbol as we denote its restriction to a subset of $\mathbb{R}^X$. A compact metrizable space is called a compactum. Other undefined terms can be found in [3], [9] and [10].
1.2 Our motivation for Chapter 2

First, we consider linear continuous functionals on $C_p(X)$. For any point $x$ in $X$, we can suppose that $x$ is a functional on $C(X)$, which carries $f$ into $f(x)$ for any $f$ in $C(X)$. Obviously $x$ is a linear continuous functional on $C_p(X)$. The following proposition is well-known.

**Proposition 1.2.1.** Let $\lambda$ be a non-constant linear continuous functional on $C_p(X)$. There exist a finite subset $\{x_1, \ldots, x_n\}$ of $X$ and non-zero numbers $\{\alpha_1, \ldots, \alpha_n\}$ such that $\lambda = \sum_{i=1}^{n} \alpha_i x_i$.

By Proposition 1.2.1, we have two observations as follows.

1. For any pair $(f, g)$ of functions in $C_p(X)$, if $f|_{\{x_1, \ldots, x_n\}} = g|_{\{x_1, \ldots, x_n\}}$ then $\lambda(f) = \lambda(g)$.

2. There exists a real-valued continuous function $\tilde{\lambda}$ on $\mathbb{R}^{\{x_1, \ldots, x_n\}}$ such that $\lambda = \tilde{\lambda} \circ \pi_{\{x_1, \ldots, x_n\}}$.

In (2), the continuity of $\tilde{\lambda}$ is deduced by the following proposition.

**Proposition 1.2.2 ([3, Proposition 0.4.1.(2)]).** The restriction map $\pi_F$ is an open map onto $\pi_F(C_p(X))$ if $F$ is a closed subset of $X$.

Now, we concern with non-linear functionals in general. In view of (1), (2) and Proposition 1.2.2, we define a notion.

**Definition 1.2.3.** Let $\xi$ be a functional on $C(X)$. A closed subset $S$ of $X$ is said to be a support for $\xi$ if $\xi(f) = \xi(g)$ for any pair $(f, g)$ of functions in $C(X)$ such that $f|S = g|S$. 

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By Proposition 1.2.2, if $\xi$ is a continuous functional on $C_p(X)$ and $S$ is a support for $\xi$, then there exists a real-valued continuous function $\tilde{\xi}$ on $\pi_S(C_p(X))$ such that $\xi = \tilde{\xi} \circ \pi_S$.

Moreover, we can find a property of the set $\{x_1, \ldots, x_n\}$ in Proposition 1.2.1;

(3) If $S$ is a support for $\lambda$, then $S$ contains $\{x_1, \ldots, x_n\}$.

This (3) says that the set $\{x_1, \ldots, x_n\}$ is minimum among all supports for $\lambda$. In general, we define a concept as follows.

**Definition 1.2.4.** Let $\xi$ be a functional on $C(X)$. We say that a support $S$ for $\xi$ is minimum if every support for $\xi$ contains $S$.

By properties (1) and (3), we can notice that every linear continuous functional on $C_p(X)$ has the finite minimum support. Otherwise, Kundu, McCoy and Okuyama [15] proved that the compact minimum support for every linear continuous functional on $C_k(X)$ exists. In Section 2.2 of Chapter 2, we are concerned with the existence of minimum supports for continuous (not necessary linear) functionals on $C_p(X)$ and $C_k(X)$.

In Section 2.3, applying our renewal methods to supports for continuous functionals on spaces of real-valued "bounded" continuous functions with various topologies, the existence of minimum supports is proved in several cases. In Section 2.4, we investigate the structure of minimum supports. Owing to our concern about Proposition 1.2.1, the relationship between a functional and its minimum support is also investigated (see Propositions 2.4.2 and 2.4.19).
1.3 Our motivation for Chapter 3

The space $C_p(X)$ is a locally convex linear topological space. Therefore, we may regard $C_p(\cdot)$ as a functor from the class of compacta into the class of locally convex linear topological spaces. From this point of view, for the class of compacta, we have the following general problem;

Problem 1.3.1. Classify the class of compacta by means of the functor $C_p(\cdot)$.

Now, we introduce a notion.

Definition 1.3.2. Two spaces $X$ and $Y$ are said to be $l$-equivalent if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.

In 1980, Pavlovskii [23] proved the following beautiful result;

Theorem 1.3.3 (Pavlovskii [23, Corollary 2]). If two compacta $X$ and $Y$ are $l$-equivalent, then $\dim X = \dim Y$.

Here, we denote by $\dim$ the Lebesgue covering dimension. The topological invariant $\dim$ is the most important item for us among many ones, because it decides the basic structures of spaces. At the same time, Pavlovskii pointed out that the converse of Theorem 1.3.3 is not hold. This fact says that the classification of spaces depending on the linear topological structure of function spaces on them is precisely fine than the another one depending on $\dim$. His key in [23] is the following theorem.

Theorem 1.3.4 (Pavlovskii [23, Proposition 2]). Let $X$ and $Y$ be compacta being $l$-equivalent. If $Z$ is a subset of $X$ with the Baire property, then there exists a non-empty open subset of $Z$ which is homeomorphic to a subset of $Y$. 

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We note that the results of Pavlovskii are more general than the statements in Theorems 1.3.3 and 1.3.4 (see [23, Proposition 2 and Corollary 2]). Using Theorem 1.3.4, he proved Theorem 1.3.3 and noted the existence of a 2-dimensional compactum which is not $l$-equivalent to the 2-disk. It goes out saying that disks are fundamental for us. Therefore, we must consider the following problem.

**Problem 1.3.5.** What kind of spaces can be $l$-equivalent to the $n$-disk?

For the problem, Pavlovskii [23] gave an answer in person.

**Theorem 1.3.6 (Pavlovskii [23, Theorem 2]).** Any $n$-dimensional finite polyhedron is $l$-equivalent to the $n$-disk.

Subsequently, several general topologists investigated spaces which are $l$-equivalent to the $n$-disk (for example, see [2], [13], [14] and [26]). In particular, Arhangel'skii [2] introduced a powerful notation "$S$-stability" and subsequently obtained an interesting progress. In [13], Kawamura and the author gave the generalization of Pavlovskii's method to topological manifolds. On the other hand, Valov [26] applied Pełczński's method to an investigation of the $C_p$-theory and established characterizations of spaces which are $l$-equivalent to the Hilbert cube and to the universal Menger compactum $\mu^n$ as follows.

**Theorem 1.3.7 (Valov [26, Theorem 2.8.(ii)]).** A space $X$ is $l$-equivalent to $I^n$ if and only if $X$ is a compactum containing a copy of $I^n$.

**Theorem 1.3.8 (Valov [26, Theorem 2.9]).** A space $X$ is $l$-equivalent to $\mu^n$ if and only if $X$ is an $n$-dimensional compactum containing a copy of $\mu^n$. 

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His characterizations are sufficiently simple. He could use universality of those spaces (see [10, Definition 1.3.9, page 18]. For disks, their simple structures had rather prevented us from applying Valov’s methods to them. We have two purposes in Chapter 3. The first is to prove that every $n$-dimensional compact topological manifold is $l$-equivalent to the $n$-disk. This result, which can be seen in [13], is stated in Section 3.2 as Corollary 3.2.2. In Section 3.3, we present the another one as Theorem 3.3.3; a characterization of spaces which are $l$-equivalent to the $n$-disk, which is given in [20].

Remark 1.3.9. Unfortunately, we can not set Corollary 3.2.2 as a corollary of Theorem 3.3.3, because it is difficult to check the condition (LD2) in Theorem 3.3.3 directly for several compact topological manifolds.
Chapter 2

Supports for continuous functionals

In the chapter, we are concerned with supports for continuous functionals on function spaces.

2.1 Basic terminology for the chapter

In this section, we introduce some terminology which will be needed in the later sections in the chapter.

The symbol Supp $\xi$ denotes the set of all supports for a functional $\xi$. For a family $\mathcal{A}$ of sets, we write $\bigcap \mathcal{A} = \bigcap \{A : A \in \mathcal{A}\}$ and $\bigcup \mathcal{A} = \bigcup \{A : A \in \mathcal{A}\}$. The symbol $\omega_1$ denotes the first uncountable ordinal. If we deal with ordinals as spaces, we always consider usual order topologies on them.

Here, we introduce some terminology related to various topologies on $C(X)$ and $C^b(X)$.

First, we put

$$< f, K, \varepsilon > = \{g \in C(X) : |f(x) - g(x)| < \varepsilon \text{ for any } x \in K\}$$

for $f \in C(X)$, a compact subset $K$ of $X$ and a positive number $\varepsilon$. The family of all sets of the above form is a base of $C_k(X)$.

Next, we introduce some terminology for $C^b(X)$. Let $B(X)$ be the set of all real-valued bounded (not necessary continuous) functions on $X$. For a function $\phi$ in $B(X)$, and a subset $M$ of $X$, we set;
\[ \|\phi\|_M = \sup\{ |\phi(x)| : x \in M \}, \]

and if \( M = X \), we write \( \| \cdot \| \) for \( \| \cdot \|_X \).

A function \( \phi \) in \( B(X) \) is said to vanish at infinity if, for any positive number \( \varepsilon \), there exists a compact subset \( K \) such that \( \| \phi \|_{X\setminus K} \leq \varepsilon \). A function \( \phi \) that vanishes at infinity, is said to be tame if, for any positive number \( \varepsilon \), there exists a compact subset \( K \) such that \( \| \phi \|_{X\setminus K} \leq \varepsilon \) and \( \phi|_K \) is continuous on \( K \). We fix one of such compact sets, \( SK(\phi, \varepsilon) \) for every tame function \( \phi \). Let

\( Fin(X) = \) the set of all characteristic functions of finite subsets of \( X \),

\( K(X) = \) the set of all characteristic functions of compact subsets of \( X \),

\( B_{00}(X) = \) the set of all tame real-valued functions on \( X \),

\( B_0(X) = \) the set of all real-valued functions on \( X \) that vanish at infinity.

Let \( \mathcal{F} \) be a subset of \( B(X) \) that contains \( Fin(X) \). For functions \( f \) in \( C^b(X) \), \( \phi \) in \( \mathcal{F} \), a subset \( M \) of \( X \) and a positive number \( \varepsilon \) we set

\[ < f, \phi, \varepsilon >_M = \{ g \in C^b(X) : \| \phi \cdot (g - f) \|_M < \varepsilon \}, \]

and if \( M = X \), we write \( < f, \phi, \varepsilon > \) for \( < f, \phi, \varepsilon >_X \). For a space \( X \), \( C^b_\varepsilon(X) \) denotes the space of all real-valued bounded continuous functions on \( X \) with the locally convex topology which has a neighborhood base at \( f \) in \( C^b_\varepsilon(X) \) consisting of all sets of the form

\[ < f, \phi, \varepsilon > \]

where \( \phi \) is a function in \( \mathcal{F} \) and \( \varepsilon \) is a positive number. We call this topology on \( C^b(X) \) \( \mathcal{F} \)-generated topology. Under this formulation, we note the following.

<table>
<thead>
<tr>
<th>( \mathcal{F} )</th>
<th>( \mathcal{F} )-generated topology</th>
</tr>
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<tbody>
<tr>
<td>( Fin(X) )</td>
<td>pointwise convergent topology</td>
</tr>
<tr>
<td>( K(X) )</td>
<td>compact-open topology</td>
</tr>
<tr>
<td>( B_0(X) )</td>
<td>strict topology</td>
</tr>
<tr>
<td>( B(X) )</td>
<td>sup-norm topology</td>
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</table>
Obviously, if $\mathcal{F} \subset \mathcal{F}'$ then $\mathcal{F}'$-generated topology is finer than $\mathcal{F}$-generated one. The strict topology on $C^k(X)$ was originally defined by R. C. Back [7] for a locally compact space $X$ as the $(C(X) \cap B_0(X))$-generated topology, and has been generalized for an arbitrary space $X$ as the $B_0(X)$-generated topology.

In the proofs of the results which are stated in Sections 2.3 and 2.4 except Proposition 2.3.2, the sets $B_{00}(X)$ are more convenient than the sets $B_0(X)$, because we can use the continuity of functions on compact sets.

2.2 For spaces of continuous functions

In the section, we investigate the existence of minimum supports for continuous functionals on $C_p(X)$ and $C_k(X)$. Our purpose is to give a result for any continuous functional on $C_k(X)$.

**Theorem 2.2.1 ([18, Theorem 1]).** The minimum support for any continuous functional on $C_k(X)$ exists.

Since any continuous functional on $C_p(X)$ is continuous on $C_k(X)$ obviously, we have;

**Corollary 2.2.2 ([17, Theorem 0]).** The minimum support for any continuous functional on $C_p(X)$ exists.

To prove the theorem, we need some lemmas. Below, $\xi$ denotes a non-constant functional on $C_k(X)$.

**Lemma 2.2.3.** Let $K$ be a compact subset of $X$ and $F$ a closed subset of $X$. For any pair $(f, g)$ of functions in $C_k(X)$ and a positive number $\varepsilon$, if $g \in< f, K \cap F, \varepsilon >$, then there exists a function $h$ in $C_k(X)$ such that $h \in< f, K, \varepsilon >$ and $h|_F = g|_F$. 

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Proof. Put $L = K \setminus \{x \in X : |f(x) - g(x)| < \varepsilon\}$. Then $L \cap F = \emptyset$. Since $L$ is compact, there exists a continuous function $r : X \to \mathbb{I}$ such that $r|_F = 0$ and $r|_L = 1$. Put $h = f \times r + g \times (1 - r)$. Obviously $h|_F = g|_F$. We have
\[|h(x) - f(x)| = |f(x) - g(x)| |r(x) - 1| < \varepsilon\]
for any $x \in K$.

Lemma 2.2.4. Let $F$ be a closed subset of $X$ and $\pi_F$ the restriction map from $C_k(X)$ into $C_k(F)$. Then $\pi_F$ is an open map onto $\pi_F(C_k(X))$.

Proof. By Lemma 2.2.3, we have;
\[\pi_F(< f, K, \varepsilon >) = \{g \in C_k(F) : |f(x) - g(x)| < \varepsilon \text{ for any } x \in K \cap F\} \cap \pi_F(C_k(X))\]
for any $f \in C_k(X)$, any compact subset $K$ of $X$ and any positive number $\varepsilon$.

Lemma 2.2.5. For any pair $(S, T)$ of elements of $\text{Supp } \xi$, $S \cap T$ belongs to $\text{Supp } \xi$.

Proof. Put $A = S \cap T$. Assume that there exist functions $f$ and $g$ in $C_k(X)$ such that $\xi(f) \neq \xi(g)$ and $f|_A = g|_A$. It is easily checked that the equality
\[\pi_S^{-1}(\pi_S(\xi^{-1}(\xi(g)))) = \xi^{-1}(\xi(g))\]
holds for the restriction map $\pi_S$. From this, we have $f|_S \notin \pi_S(\xi^{-1}(\xi(g)))$. By Lemma 2.2.4, $\pi_S(\xi^{-1}(\xi(g)))$ is closed in $\pi_S(C_k(X))$. There exist a compact subset $K$ of $S$ and a positive number $\varepsilon$ such that $< f, K, \varepsilon > \cap \xi^{-1}(\xi(g)) = \emptyset$. Since $K \cap T \subset A$, by the assumption, we have $g \in < f, K \cap T, \varepsilon >$. By Lemma 2.2.3, there exists $h$ in $< f, K, \varepsilon >$ such that $h|_T = g|_T$. This is a contradiction.

Lemma 2.2.6. $\bigcap \text{Supp } \xi$ is a support for $\xi$.  

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Proof. Put $S = \bigcap \text{Supp} \xi$. Assume that there exist functions $f$ and $g$ in $C_k(X)$ such that $\xi(f) \neq \xi(g)$ and $f|_S = g|_S$. Since $f$ does not belong to the closed subset $\xi^{-1}(\xi(g))$ of $C_k(X)$, there exist a compact subset $K$ of $X$ and a positive number $\varepsilon$ such that $< f, K, \varepsilon > \cap \xi^{-1}(\xi(g)) = \emptyset$. Put $L = K \setminus \{ x \in X : | f(x) - g(x) | < \varepsilon \}$. Then $L \cap S = \emptyset$. Using Lemma 2.2.5, by the definition of $S$, we can find an element $T$ of $\text{Supp} \xi$ such that $L \cap T = \emptyset$. Since $g \in < f, K, \varepsilon >$, there exists $h$ in $< f, K, \varepsilon >$ such that $h|_T = g|_T$ by Lemma 2.2.3. This is a contradiction.

Proof of Theorem 2.2.1. By Lemma 2.2.6, $\bigcap \text{Supp} \xi$ is the minimum support for $\xi$, which proves Theorem 2.2.1.

2.3 For spaces of bounded continuous functions

In the section, we investigate the existence of minimum supports for continuous functionals on $C^b_k(X)$ (for the definition, see Section 2.1). Our main theorem in this section is;

Theorem 2.3.1 ([19, Theorem 1]). For every space $X$, every subset $\mathcal{F}$ of $B_{00}(X)$ that contains $\text{Fin}(X)$ and every continuous functional $\xi$ on $C^b_k(X)$, the minimum support for $\xi$ exists.

In connection with Theorem 2.3.1, we have;

Proposition 2.3.2 ([19, Proposition 2]). Let $\mathcal{F}$ be a subset of $B(\omega_1)$ that contains $B_0(\omega_1)$, and $\xi$ the functional on $C^b_k(\omega_1)$ defined by $\xi(f) = \overline{f}(\omega_1)$ for any function $f$ in $C^b_k(\omega_1)$; where $\overline{f}$ denotes the unique continuous extension of $f$ to $\omega_1 + 1$. If $B_0(\omega_1)$ is a proper subset of $\mathcal{F}$, then $\xi$ is continuous.
Proof of Proposition 2.3.2. It is sufficient to show that $\xi$ is continuous at the zero function. Take a function $\phi_0$ in $\mathcal{F} \setminus B_0(\omega_1)$. There exists a positive number $\delta_0$ such that for any compact subset $K$ of $\omega_1 \setminus K$. For a positive number $\varepsilon$, we set $\delta = \varepsilon \delta_0$. Take a function $f$ in $< 0, \phi_0, \delta >$. There exists a countable ordinal $\alpha$ such that $f|_{[\alpha, \omega_1]} = \overline{f}(\omega_1)$. For the compact subset $[0, \alpha]$ of $\omega_1$, take a point $\beta$ in $\omega_1 \setminus [0, \alpha]$ such that $|\phi_0(\beta)| \geq \delta_0$. We have:

$$|\xi(f)| = |\overline{f}(\omega_1)| = |f(\beta)| = |f(\beta)|\frac{|\phi_0(\beta)|}{|\phi_0(\beta)|} \leq \frac{||f : \phi_0||}{\delta_0} < \frac{\delta}{\delta_0} = \varepsilon.$$ 

Proposition 2.3.2 says that, for some space $X$, the strict topology is the maximal topology on $C_b(X)$ among all topologies which are generated by subsets of $B(X)$ in the sense that every continuous functional has a minimum support for it. Indeed, for the functional $\xi$ in the above proposition, we have $\cap \text{Supp} \, \xi = \emptyset$. In addition, we can see that Theorem 2.2.1 in Section 2.2 is not valid for $C(\omega_1)$ with the sup-norm topology by Proposition 2.3.2. We note that the $B_0(\omega_1)$-generated topology on $C_b(\omega_1)$ ($= C(\omega_1)$) coincides with $B_{00}(\omega_1)$-generated one.

To prove Theorem 2.3.1, we need Lemmas 2.3.3, 2.3.4 and 2.3.5. In them, $\mathcal{F}$ is always assumed to be a subset of $B_{00}(X)$ that contains $Fin(X)$. Lemmas 2.3.3 and 2.3.4 will also be used in Section 2.4.

Lemma 2.3.3. Let $\phi$ be a function in $\mathcal{F}$ and $F_1, F_2$ closed subsets of $X$. For any pair $(f, g)$ of functions in $C_b^\phi(X)$ and a positive number $\varepsilon$, if $g < f, \phi, \varepsilon > F_1 \cap F_2$, then there exists a function $h$ in $C_b^\phi(X)$ such that $h < f, \phi, \varepsilon > F_1$ and $h|_{F_2} = g|_{F_2}$.

Proof. Put

$$K = SK(\phi, \frac{\varepsilon}{2(||f|| + ||g||)}),$$

and
\[ L = \{ x \in K \cap F_1 : |\phi(x)(g(x) - f(x))| \geq \varepsilon \}. \]

Since \[ \|\phi \cdot (g - f)\|_{F_1 \cap F_2} < \varepsilon, \] we have \( L \cap F_2 = \emptyset \). Since \( \phi \) is tame, \( L \) is a compact subset of \( X \), and so there exists a continuous function \( r : X \to \mathbb{I} \) such that \( r|_{F_2} = 0 \) and \( r|_L = 1 \). Put \( h = f \cdot r + g \cdot (1 - r) \). Then \( h \) is a function in \( C^b_{\mathcal{F}}(X) \) such that \( h|_{F_2} = g|_{F_2} \). For any \( x \) in \( F_1 \), we have
\[
|\phi(x)(h(x) - f(x))| = (1 - r(x))|\phi(x)(g(x) - f(x))| \\
\leq \begin{cases} 
\frac{\varepsilon}{2(\|f\| + \|g\|)}(\|f(x)\| + |g(x)|), & \text{if } x \in F_1 \setminus K, \\
0, & \text{if } x \in L, \\
(1 - r(x))\varepsilon, & \text{if } x \in (K \cap F_1) \setminus L.
\end{cases}
\]

Hence \( h \in < f, \phi, \varepsilon >_{F_1} \), as required. \( \square \)

Lemma 2.3.4. For any closed subset \( F \) of \( X \), the restriction map \( \pi_F \) from \( C^b_{\mathcal{F}}(X) \) to \( C^b_{\pi_F(\mathcal{F})}(F) \) is an open map onto \( \pi_F(C^b_{\mathcal{F}}(X)) \) and the set \( \pi_F(C^b_{\mathcal{F}}(X)) \) is dense in \( C^b_{\pi_F(\mathcal{F})}(F) \).

Proof. Using Lemma 2.3.3 with \( F_1 = X \) and \( F_2 = F \), we have
\[
\pi_F(< f, \phi, \varepsilon >) = \{ g \in C^b(F) : \|\phi \cdot (g - f)\|_F < \varepsilon \} \cap \pi_F(C^b_{\mathcal{F}}(X))
\]
for any function \( f \) in \( C^b_{\mathcal{F}}(X) \), any function \( \phi \) in \( \mathcal{F} \) and any positive number \( \varepsilon \). Hence \( \pi_F \) is relatively open. To prove the second statement, take a function \( f \) in \( C^b_{\pi_F(\mathcal{F})}(F) \) and \( \phi \in \mathcal{F} \) and a positive number \( \varepsilon \). Put \( M = \|f\|_F + \varepsilon \) and \( K = SK(\phi, \frac{\varepsilon}{2M}) \cap F \). There exists a function \( g \) in \( C^b_{\mathcal{F}}(X) \) such that \( g|_K = f|_K \). Put
\[
L = \{ x \in F : |g(x) - f(x)| \geq \frac{\varepsilon}{2} \}.
\]
Since \( L \) is a closed subset of \( X \), there exists a continuous function \( r : X \to \mathbb{I} \) such that \( r|_{L} = 0 \) and \( r|_{K} = 1 \). Put \( h = g \cdot r \). We have
(1) $|\phi(x)(h(x) - f(x))| = |\phi(x)(g(x) - f(x))| = 0$ for any $x \in K$,

(2) $|\phi(x)(h(x) - f(x))| = |\phi(x)f(x)| \leq \frac{\varepsilon}{2M}||f|| < \varepsilon$ for any $x \in L \setminus K$.

And, since

$$|g(x) - f(x)| < \frac{\varepsilon}{2} \text{ and } |g(x)| < ||f|| + \frac{\varepsilon}{2}$$

for any $x \in F \setminus (L \cup K)$, we have

(3) $|\phi(x)(h(x) - f(x))| \leq |\phi(x)||r(x) - 1||g(x)| + |g(x) - f(x)| \leq \frac{\varepsilon}{2M}M < \varepsilon$.

By (1), (2) and (3), we have $\pi_F(h) \in \{f' \in F : \||\phi \cdot (f' - f)\|_F < \varepsilon\}$.

\[\square\]

**Lemma 2.3.5.** Let $\xi$ be a continuous functional on $C^b_{\mathcal{F}}(X)$. For any pair $(S,T)$ of elements of $\text{Supp } \xi$, $S \cap T$ belongs to $\text{Supp } \xi$.

**Proof.** Assume that there exist functions $f$ and $g$ in $C^b_{\mathcal{F}}(X)$ such that $f|_{S \cap T} = g|_{S \cap T}$ but $\xi(f) \neq \xi(g)$. Since $S \in \text{Supp } \xi$,

$$\pi^{-1}_S(\pi_S(\xi^{-1}(\xi(g)))) = \xi^{-1}(\xi(g))$$

for the restriction map $\pi_S$. From this, we have $f|_S \notin \pi_S(\xi^{-1}(\xi(g)))$. By Lemma 2.3.4, $\pi_S(\xi^{-1}(\xi(g)))$ is closed in $\pi_S(C^b_{\mathcal{F}}(X))$. There exist a function $\phi$ in $\mathcal{F}$ and a positive number $\varepsilon$ such that $< f, \phi, \varepsilon > S \cap \xi^{-1}(\xi(g)) = \emptyset$. Since $g \in < f, \phi, \varepsilon >_{S \cap T}$, using Lemma 2.3.3 with $F_1 = S$ and $F_2 = T$, we can find a function $h$ in $C^b_{\mathcal{F}}(X)$ such that $h \in < f, \phi, \varepsilon >_S$ and $h|_T = g|_T$. This is a contradiction. \[\square\]

**Proof of Theorem 2.3.1.** We need to prove $\bigcap \text{Supp } \xi \in \text{Supp } \xi$ as we did in the proof of Lemma 2.2.6. Put $S = \bigcap \text{Supp } \xi$. Assume that there exist functions $f$ and $g$ in $C^b_{\mathcal{F}}(X)$ such that $f|_S = g|_S$, but $\xi(f) \neq \xi(g)$. Since $f$ does not belong to the closed subset $\xi^{-1}(\xi(g))$ of $C^b_{\mathcal{F}}(X)$, there exist a function $\phi$ in $\mathcal{F}$ and a positive number $\varepsilon$ such that $< f, \phi, \varepsilon > \cap \xi^{-1}(\xi(g)) = \emptyset$. Put
\[ K = SK(\phi, \frac{\varepsilon}{2(\|f\| + \|g\|)}), \text{ and} \]

\[ L = \{ x \in K : |\phi(x)(g(x) - f(x))| \geq \varepsilon \}. \]

Since \( \phi \) is tame, \( L \) is a compact subset of \( X \). Since \( S \) does not intersect \( L \), using Lemma 2.3.5, we can find an element \( T \) of Supp \( \xi \) such that \( L \cap T = \emptyset \). For any \( x \) in \( T \), we have

\[ |\phi(x)(g(x) - f(x))| < \begin{cases} \frac{\varepsilon}{2(\|f\| + \|g\|)}(|f(x)| + |g(x)|), & \text{if } x \in T \setminus K, \\ \varepsilon, & \text{if } x \in T \cap K. \end{cases} \]

Since \( g \in< f, \phi, \varepsilon >_T \), using Lemma 2.3.3 with \( F_1 = X \) and \( F_2 = T \), we can find a function \( h \) in \( C^*_p(X) \) such that \( h \in< f, \phi, \varepsilon > \) and \( h|_T = g|_T \). This is a contradiction.

\[ \square \]

2.4 Some properties of supports

In the section, we investigate properties of supports which are mentioned in Section 1.2. The following statements were stated in [17].

**Proposition 2.4.1 ([17, Lemma 4]).** For every space \( X \) and every continuous functional \( \xi \) on \( C_p(X) \), \( \bigcap \text{Supp } \xi \) is a separable subspace of \( X \).

**Proposition 2.4.2 ([17, Remark I]).** For any countable subset \( A \) of \( X \), there exists a continuous functional \( \xi_A \) on \( C_p(X) \) such that \( \bigcap \text{Supp } \xi_A = \bar{A} \).

Propositions 2.4.1 and 2.4.2 were the beginning of our research of structures of supports. The proof of Proposition 2.4.1 is completely depends on the fact that the cellularity of \( C_p(X) \) is countable (see [3]). Here, a space is said to satisfy the countable chain condition if there exists no uncountable pairwise disjoint family of non-empty open subsets of that space. The reader can see the scheme of the proof of Proposition 2.4.1 in the proofs of Lemma 2.4.5 and Theorem 2.4.6. Here, we give a proof of Proposition 2.4.2.
Proof of Proposition 2.4.2. We may assume that $A$ is indexed as $A = \{x_n : 0 < n < \omega\}$. For every $f \in C_\infty(X)$, we put

$$\xi(f) = \sum_{n=0}^{\infty} \frac{r(f(x_n))}{2^n} : 0 < n < \omega,$$

where $r : \mathbb{R} \to \mathbb{I}$ is a continuous function defined by

$$r(\alpha) = \begin{cases} 1, & 1 \leq \alpha \\ \alpha, & 0 \leq \alpha < 1 \\ 0, & \alpha < 0. \end{cases}$$

Our motivation on Proposition 2.4.2 will be reconsidered as Proposition 2.4.19. Our next target is supports for continuous functionals on $C_k(X)$. By Proposition 2.4.1, we have the following question naturally.

Question 2.4.3. Does $\bigcap \text{Supp } \xi$ have a dense $\sigma$-compact subset of it for every space $X$ and every continuous functional $\xi$ on $C_k(X)$?

Unfortunately, the answer is negative. Below, we give some result that is related to it.

Let $\tau$ be a cardinal. A space $X$ is said to be almost $\tau$-compact if for any $\alpha < \tau$, there exists a compact subset $K_\alpha$ of $X$ such that $X = \bigcup\{K_\alpha : \alpha < \tau\}$. Almost $\omega$-compact spaces are said to be almost $\sigma$-compact. The smallest cardinal $\tau$ such that $X$ is almost $\tau$-compact, is denoted by $cd(X)$.

Definition 2.4.4. A space $X$ has $\sigma$-property if, for every continuous functional $\xi$ on $C_k(X)$, the closed subset $\bigcap \text{Supp } \xi$ of $X$ is almost $\sigma$-compact.

First, we give a sufficient condition in order that $X$ has $\sigma$-property.
Lemma 2.4.5. Let $D$ be a dense subset of $C_k(S)$ which satisfies the countable chain condition and $\rho$ a real-valued continuous function on $D$. There exists a $\sigma$-compact subset $A$ of $S$ which satisfies the following condition $(*)$:

$(*)$ For any pair $(f, g)$ of functions in $D$, if $f|_A = g|_A$, then $\rho(f) = \rho(g)$.

**Proof.** Let $\{U_n : n \in \omega\}$ be a base for $\mathbb{R}$ and $\gamma_n$ the maximal disjoint family of non-empty basic open subsets of $D$ such that $\bigcup \gamma_n \subset \rho^{-1}(U_n)$. By the maximality of $\gamma_n$, $\bigcup \gamma_n$ is a dense subset of $\rho^{-1}(U_n)$. For a non-empty basic open subset $U = < f, K, \varepsilon > \cap D$ in $D$, we set $\text{supp}(U) = K$. Put

$$A = \bigcup \{\text{supp}(U) : U \in \bigcup \{\gamma_n : n \in \omega\}\}$$

Since $D$ satisfies the countable chain condition, $A$ is a $\sigma$-compact subset of $S$. To show that $A$ satisfies the condition $(*)$, it is sufficient to show that, for any pair $(f, g)$ of functions in $D$, $g \in \bigcup \gamma_n$ if $f|_A = g|_A$ and $f \in \bigcup \gamma_n$ for some $n \in \omega$. Take a compact subset $K$ of $S$ and a positive number $\varepsilon$. Since $f \in \bigcup \gamma_n$, there exist an element $U$ of $\gamma_n$ and a function $h$ in $D$ such that $h \in < f, K, \varepsilon > \cap U$. Since $h \in < g, K \cap \text{supp}(U), \varepsilon >$, by Lemma 2.2.3, there exists a function $\tilde{h}$ in $C_k(S)$ such that $\tilde{h} \in < g, K, \varepsilon >$ and $\tilde{h}|_{\text{supp}(U)} = h|_{\text{supp}(U)}$. Since $D$ is dense in $C_k(S)$, the set $< g, K, \varepsilon > \cap U$ is not empty. 

**Theorem 2.4.6 ([18, Theorem 7]).** If the space $C_k(X)$ satisfies the countable chain condition, then $X$ has $\sigma$-property.

**Proof.** Let $\xi$ be a continuous functional on $C_k(X)$. Put $S = \bigcap \text{Supp} \xi$ and $D = \pi_S(C_k(X))$. There exists a real-valued function $\rho$ on $D$ such that $\xi = \rho \circ \pi_S$. We can check that $D$ and $\rho$ satisfy the conditions in Lemma 2.4.5 easily. So there exists a $\sigma$-compact subset $A$ of $S$ which satisfies the condition $(*)$ in Lemma 2.4.5. Obviously, $\overline{A} \in \text{Supp} \xi$. By the minimality of $\bigcap \text{Supp} \xi$, we have $\bigcap \text{Supp} \xi = \overline{A}$. 

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Vidossich [27] and Nakhmanson [21] proved that $C_k(X)$ satisfies the countable chain condition if $X$ is submetrizable. Here, a space $X$ is said to be submetrizable if there exists a continuous bijection from $X$ onto a metrizable space. We have the following corollary.

**Corollary 2.4.7 ([18, Corollary 8]).** If $X$ is submetrizable (in particular, metrizable), then $X$ has $\sigma$-property.

Nakhmanson [21] noted that $C_k(\omega_1)$ does not satisfy the countable chain condition, but we have;

**Proposition 2.4.8 ([18, Proposition 9]).** The space $\omega_1$ has $\sigma$-property.

**Proof.** It is sufficient to show that, for any continuous functional $\xi$ on $C_k(\omega_1)$, there exists a countable support for $\xi$. To show this, we assume that there exists a positive number $\varepsilon$ such that, for any $\alpha < \omega_1$, there exist functions $f_\alpha$ and $g_\alpha$ in $C_k(\omega_1)$ satisfying the following two conditions;

\begin{align*}
(1) & \quad f_\alpha \big|_{[0,\alpha]} = g_\alpha \big|_{[0,\alpha]}, \\
(2) & \quad |\xi(f_\alpha) - \xi(g_\alpha)| > 2\varepsilon.
\end{align*}

Gul'ko [12] showed that $C_k(\omega_1)$ is Lindelöf. So the net $\{f_\alpha : \alpha < \omega_1\}$ has a cluster point $h$ in $C_k(\omega_1)$. By the continuity of $\xi$, there exist a $\beta < \omega_1$ and a positive number $\delta$ such that, for any function $f$ in $< h, [0,\beta], \delta >$, $|\xi(f) - \xi(h)| < \varepsilon$. Choose $\gamma < \omega_1$ such that $\beta < \gamma$ and $f_\gamma \in < h, [0,\beta], \delta >$. Then we have $g_\gamma \in < h, [0,\beta], \delta >$ by (1). This contradicts the condition (2). \qed

In special cases, we have a necessary condition for a space to have $\sigma$-property. The following lemma is well-known (for example, see [9, Exercise 3.4.H.(b), page 166]).
Lemma 2.4.9. Let $K$ be a compact space. If $C_k(K)$ is separable, then $K$ is metrizable.

Theorem 2.4.10 ([18, Theorem 11]). Let $X$ be a space which has a closed-and-open subset $Y$ such that $cd(Y) = \omega_1$. If $X$ has $\sigma$-property, then every compact subset of $X$ is metrizable.

Proof. Suppose that a non-metrizable compact subset $K$ of $X$ exists. $C_k(K)$ is metrizable but not separable by Lemma 2.4.9. By [9, Theorem 4.4.3], for any $\alpha < \omega_1$, there exists a non-constant continuous function $\xi_\alpha : C_k(K) \to \mathbb{I}$ such that $\{\xi_\alpha^{-1}((0,1]) : \alpha < \omega_1\}$ is discrete. Since $cd(Y) = \omega_1$, for any $\alpha < \omega_1$, there exists a non-empty compact subset $K_\alpha$ of $Y$ such that $Y = \bigcup\{K_\alpha : \alpha < \omega_1\}$. For any $f$ in $C_k(X)$ and an $\alpha < \omega_1$, we put

$$s(f, \alpha) = \sup\{|f(x)| : x \in K_\alpha\}.$$

We define a functional $\xi$ on $C_k(X)$ by;

$$\xi(f) = \sum \{s(f, \alpha) \times \xi_\alpha(\pi_K(f)) : \alpha < \omega_1\}$$

for any $f$ in $C_k(X)$. Obviously $\xi$ is continuous. We show that $\xi$ has no almost $\sigma$-compact support. Let $S$ be a support for $\xi$. Suppose that there exist a $\beta < \omega_1$ and a point $x$ in $K_\beta$ such that $x \notin S \cup K$. Take a function $f$ in $C_k(X)$ such that $f|_K \in \xi_\beta^{-1}((0,1])$. There exists a function $g$ in $C_k(X)$ such that $f|_{S \cup K} = g|_{S \cup K}$ and $g(x) = s(f, \beta) + 1$. We have

$$\xi(g) - \xi(f) = s(g, \beta) \times \xi_\beta(\pi_K(g)) - s(f, \beta) \times \xi_\beta(\pi_K(f)) \neq 0.$$ 

This contradicts the fact $f|_S = g|_S$. So we have

$$Y = \bigcup\{K_\alpha : \alpha < \omega_1\} \subset S \cup K.$$

If $S$ is almost $\sigma$-compact, then $S \cup K$ is also obviously. Then $Y$ is almost $\sigma$-compact because $Y$ is a closed-and-open subset of $S \cup K$. This contradicts the fact $cd(Y) = \omega_1$. □
By Theorem 2.4.10, we can find a space which does not have $\sigma$-property.

**Example 2.4.11.** Let $D(\omega_1)$ be the discrete space whose cardinality is $\omega_1$. The space $D(\omega_1) \oplus (\omega_1 + 1)$ does not have $\sigma$-property.

Next, we investigate supports for continuous functionals on function spaces $C^b_{\mathcal{F}}(X)$. For spaces $C^b_{\mathcal{F}}(X)$, we can get similar results.

The scheme of the proof of the following proposition may be already known. McCoy [16] informed the author of a proof for function spaces with the compact-open topology, which is different from Nakhmanson's one in [21]. For the completeness of the thesis, we will give a proof afterward.

**Proposition 2.4.12.** Let $\mathcal{F}$ be a subset of $B_{00}(X)$ that contains $\text{Fin}(X)$. If a space $X$ is submetrizable (in particular metrizable), then $C^b_{\mathcal{F}}(X)$ satisfies the countable chain condition.

We need a definition here.

**Definition 2.4.13.** Let $\mathcal{F}$ be a subset of $B_{00}(X)$ that contains $K(X)$. A space $X$ has $\mathcal{F}$-$\sigma$-property if for every continuous functional $\xi$ on $C^b_{\mathcal{F}}(X)$, the closed subset $\bigcap \text{Supp} \xi$ of $X$ is almost $\sigma$-compact.

**Theorem 2.4.14 ([19, Theorem 6]).** Let $\mathcal{F}$ be a subset of $B_{00}(X)$ that contains $K(X)$. If $C^b_{\mathcal{F}}(X)$ satisfies the countable chain condition, then $X$ has $\mathcal{F}$-$\sigma$-property.

By Proposition 2.4.12, we get;

**Corollary 2.4.15 ([19, Corollary 7]).** If a space $X$ is submetrizable (in particular metrizable), then $X$ has $\mathcal{F}$-$\sigma$-property.

Obviously, $\mathcal{F}$-$\sigma$-property of a space implies that it has $\sigma$-property (see Definition 2.4.4). By Theorem 2.4.10, we have;
Corollary 2.4.16 ([19, Theorem 8]). Let $X$ be a space that has a closed-and-open subset $Y$ such that $cd(Y) = \omega_1$. If $X$ has $\mathcal{F} - \sigma$-property, then every compact subset of $X$ is metrizable.

By Corollary 2.4.16, we can find a space $X$ such that there exists a continuous functional on $C^b_{K(X)}(X)$ which has no almost $\sigma$-compact support. From this point of view, the next theorem seems interesting. We need a notion here.

Definition 2.4.17. Let $\mathcal{F}$ be a subset of $B_{00}(X)$ that contains $Fin(X)$. A functional $\xi$ on $C^b_{\mathcal{F}}(X)$ is said to be uniformly continuous if for any positive number $\varepsilon$, there exist a function $\phi$ in $\mathcal{F}$ and a positive number $\delta$ such that for any pair $(f, g)$ of functions in $C^b_{\mathcal{F}}(X)$, $f - g < \varepsilon, \phi, \delta >$ implies $|\xi(f) - \xi(g)| < \varepsilon$ where $0$ is the zero function on $X$.

Theorem 2.4.18 ([19, Theorem 11]). Let $X$ be a space and $\mathcal{F}$ a subset of $B_{00}(X)$ that contains $K(X)$. If $\xi$ is a uniformly continuous functional on $C^b_{\mathcal{F}}(X)$, then the closed subset $\bigcap \text{Supp} \xi$ of $X$ is almost $\sigma$-compact.

Proof. Put $S = \bigcap \text{Supp} \xi$. Since $\xi$ is uniformly continuous, there exist a function $\phi_m$ in $\mathcal{F}$ and a positive number $\delta_m$ such that

$$f - g < 0, \phi_m, \delta_m > \text{implies } |\xi(f) - \xi(g)| < \frac{1}{m + 1}$$

for every $m < \omega$ and every pair $(f, g)$ of functions in $C^b_{\mathcal{F}}(X)$. Put

$$K_{mn} = S K(\phi_m, \frac{\delta_m}{n + 1}) \cap S,$$

and

$$T = \bigcup \{K_{mn} : m, n < \omega\}.$$
Take a pair \((f, g)\) of functions in \(C^b_f(X)\) such that \(f|_T = g|_T\). Choose a number \(n\) such that \(\|f\| + \|g\| < n + 1\). For any \(x\) in \(S\), we have
\[
|\phi_m(x)(g(x) - f(x))| \leq \left\{ \begin{array}{cl}
\frac{\delta_m}{n + 1}(|g(x)| + |f(x)|), & \text{if } x \in S \setminus K_{mn} \\
0, & \text{if } x \in K_{mn}
\end{array} \right\} < \delta_m.
\]

Hence \(g \in < f, \phi_m, \delta_m >_S\). Using Lemma 2.3.3 with \(F_1 = X\) and \(F_2 = S\), we can find a function \(h_m\) in \(C^b_f(X)\) such that \(h_m \in < f, \phi_m, \delta_m >\) and \(h_m|_S = g|_S\) for every \(m\). Since \(S\) is a support for \(\xi\), we have \(\xi(h_m) = \xi(g)\) for every \(m\). Meanwhile, we have \(|\xi(h_m) - \xi(f)| < \frac{1}{m + 1}\) for every \(m\), because \(h_m - f \in < 0, \phi_m, \delta_m >\). These facts imply that \(T\) is a support for \(\xi\). By the minimality of \(S\), we have \(S = T\). \(\Box\)

The next proposition completes Theorem 2.4.18.

**Proposition 2.4.19 ([19, Proposition 12]).** Let \(X\) be a space and \(F\) a subset of \(B_{\mathbb{K}_0}(X)\) that contains \(K(X)\). For any closed almost \(\sigma\)-compact subset \(S\) of \(X\), there exists a uniformly continuous functional \(\xi_S\) on \(C^b_f(X)\) such that \(\bigcap \text{Supp } \xi_S = S\).

**Proof.** Suppose \(S = \bigcup \{K_n : n < \omega\}\) where \(K_n\) is a compact subset of \(X\) for every \(n\). For every \(f\) in \(C^b_f(X)\), we put
\[
\xi_S(f) = \sum_{n=0}^{\infty} \frac{1}{2^n} \arctan(||f||_{K_n}).
\]

For every positive number \(\varepsilon\), we can choose a number \(m < \omega\) such that \(\frac{2\pi}{\varepsilon} < 2^m\). Let \(\phi\) be the characteristic function of the compact subset \(\bigcup \{K_n : 0 \leq n \leq m\}\). We can find a positive number \(\delta\) such that \(f - g \in < 0, \phi, \delta >\) implies
\[
|\arctan(||f||_{K_n}) - \arctan(||g||_{K_n})| < \frac{1}{m + 1}
\]
for every \(n \in \{0, 1, 2, \cdots, m\}\) and every pair \((f, g)\) of functions in \(C^b_f(X)\).

If \(f - g \in < 0, \phi, \delta >\), then

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This shows that $\xi_S$ is uniformly continuous. Obviously, $S \in \text{Supp} \, \xi_S$. If $S \neq \bigcap \text{Supp} \, \xi_S$, we can find a function $f$ in $C^b_{\mathcal{F}}(X)$ such that $f|_{\bigcap \text{Supp} \, \xi_S} = 0$ and $\xi_S(f) > 0$. This is a contradiction.

Below, we state proofs of Proposition 2.4.12 and Theorem 2.4.14 with two lemmas which are needed to prove them.

**Lemma 2.4.20.** Let $X$ be a space, $M$ a metrizable space and $\rho$ a continuous bijection from $X$ onto $M$. For any subset $\mathcal{F}$ of $B_0(X)$ that contains $F_{\text{in}}(X)$, the mapping $\rho^*$ from $C^b_{\mathcal{F}}(M)$ into $C^b_{\mathcal{F}}(X)$ defined by $\rho^*(f) = f \circ \rho$ for all $f$ in $C^b_{\mathcal{F}}(M)$, is a dense embedding, where $\mathcal{F}^* = \{ \phi \circ \rho^{-1} : \phi \in \mathcal{F} \}$.

**Proof.** It is easily checked that $\rho^*$ is an embedding. For every $g$ in $C^b_{\mathcal{F}}(X)$, $\phi$ in $\mathcal{F}$ and any positive number $\varepsilon$, by Dugundji's extension theorem [8], we can find a function $f$ in $C^b_{\mathcal{F}}(M)$ such that $(f \circ \rho)|_K = g|_K$ and $\|f \circ \rho\| = \|g\|_K$ where $K = SK(\phi, \frac{\varepsilon}{3\|g\|})$. For this $f$, $\rho^*(f) \in (g, \phi, \varepsilon)$. \qed

**Proof of Proposition 2.4.12.** By Lemma 2.4.20, we may assume that $X$ itself is metrizable. We only need to show that a family
\[ \{ < f_\alpha, \phi_\alpha, \varepsilon_\alpha > : f_\alpha \in C^b_{\mathcal{F}}(X), \phi_\alpha \in \mathcal{F}, \varepsilon_\alpha > 0, \alpha < \omega_1 \} \]
can not be pairwise disjoint. Put
\[ K_{\alpha n} = SK(\phi_\alpha, \frac{\varepsilon_\alpha}{n + 1}), \text{ and} \]
\[ F = \bigcup \{ K_{\alpha n} : \alpha < \omega_1, n < \omega \}. \]
Since $X$ is metrizable, $F$ is a metrizable space whose weight does not exceed $2^\omega$. Hence, there exist a separable metrizable space $Y$ and a continuous bijection $\rho$ from $F$ onto $Y$ (see [9, Exercise 4.4.C.(a), page 286]). Since the set $\{(\phi|_F) \circ \rho^{-1} : \phi \in \mathcal{F}\}$ is contained in $B_{00}(Y)$ and $C^b_{B_{00}(Y)}(Y)$ is separable (see [11, Theorem 3a]), by Lemma 2.4.20, $C^b_{\pi_F(F)}(F)$ is separable. By Lemma 2.3.4, $\pi_F(< f_\alpha, \phi_\alpha, \varepsilon_\alpha >)$ is a non-empty open subset of $C^b_{\pi_F(F)}(F)$ (note that $\pi_F(C^b_{\pi_F(F)}(X)) = C^b_{\pi_F(F)}(F)$, because $F$ is a $C^*$-embedded subset of $X$). Therefore, there exist two distinct indexes $\alpha$ and $\beta$ such that

$$\pi_F(< f_\alpha, \phi_\alpha, \varepsilon_\alpha >) \cap \pi_F(< f_\beta, \phi_\beta, \varepsilon_\beta >) \neq \emptyset.$$ 

Hence there is a function $g$ in $< f_\alpha, \phi_\alpha, \varepsilon_\alpha >$ such that $g \not\in < f_\beta, \phi_\beta, \varepsilon_\beta >_F$. Using Lemma 2.3.3 with $F_1 = X$ and $F_2 = F$, we can find a function $h$ in $< f_\beta, \phi_\beta, \varepsilon_\beta >$ such that $h|_F = g|_F$. Choose a number $n$ such that $\|h\| + \|f_\alpha\| < n + 1$. For any $x$ in $X$, we have

$$|\phi_\alpha(x)(h(x) - f_\alpha(x))| < \begin{cases} \frac{\varepsilon_\alpha}{n + 1}(|h(x)| + |f_\alpha(x)|), & \text{if } x \in X \setminus K_{an} \\ \|\phi_\alpha \cdot (g - f_\alpha)\|, & \text{if } x \in K_{an} \end{cases} < \varepsilon_\alpha.$$

Hence $h \in < f_\alpha, \phi_\alpha, \varepsilon_\alpha >$ holds. $\square$

**Lemma 2.4.21.** Let $\mathcal{F}$ be a subset of $B_{00}(S)$ that contains $K(S)$, and $D$ a dense subset of $C^b_{\pi_F(F)}(S)$ that satisfies the countable chain condition. For any real-valued continuous function $\rho$ on $D$, there exists a closed almost $\sigma$-compact subset $A$ of $S$ that satisfies the following condition $(\ast)$:

$$(\ast) \quad \text{For any pair } (f, g) \text{ of functions in } D, \text{ if } f|_A = g|_A, \text{ then } \rho(f) = \rho(g).$$

**Proof.** Let $\{U_k : k \in \omega\}$ be a base for $\mathbb{R}$. For every $k$, we can find a maximal disjoint countable family $\gamma_k = \{< f^k_l, \phi^k_l, \varepsilon^k_l > : l < \omega\} \cap D$ such that $\bigcup \gamma_k \subset \rho^{-1}(U_k)$. By the maximality of $\gamma_k$, $\bigcup \gamma_k$ is a dense subset of $\rho^{-1}(U_k)$. Put
\[ K_{klm} = SK(\phi_k, \frac{c^k}{m+1}), \quad \text{and} \]
\[ A = \bigcup \{K_{klm} : k, l, m < \omega\}. \]

The set \( A \) is the required closed almost \( \sigma \)-compact subset of \( S \). To show that \( A \) satisfies the condition \((*)\), it is sufficient to show that, for any pair \((f, g)\) of functions in \( D \), \( g \in \bigcup \gamma_k \) if \( f|_A = g|_A \) and \( f \in \bigcup \gamma_k \) for some \( k < \omega \). Take a function \( \phi \) in \( F \) and a positive number \( \varepsilon \). Since \( f \in \bigcup \gamma_k \), there exist a number \( l \) and a function \( h \) in \( D \) such that \( h \in < f, \phi, \varepsilon > \cap < f^l_k, \phi^l_k, \varepsilon^l_k > \). Since \( h \in < g, \phi, \varepsilon >_A \), using Lemma 2.3.3 with \( F_1 = S \) and \( F_2 = A \), we can find a function \( h \) in \( C^b_x(S) \) such that \( h \in < g, \phi, \varepsilon > \) and \( h|_A = h|_A \). Choose a number \( m \) such that \( \|f^k_l\| + \|h\| < m + 1 \). For any \( x \) in \( S \), we have
\[
|\phi^l_k(x)(h(x) - f^k_l(x))| < \begin{cases} \frac{c^k}{m+1}(|h(x)| + |f^k_l(x)|), & \text{if } x \in S \setminus K_{klm} \\ \|\phi^l_k \cdot (h - f^k_l)\|, & \text{if } x \in K_{klm} \end{cases} < \varepsilon^l_k.
\]
Hence \( h \in < f^k_l, \phi^l_k, \varepsilon^l_k > \). Since \( D \) is dense in \( C^b_x(S) \), the set
\[
< g, \phi, \varepsilon > \cap < f^k_l, \phi^l_k, \varepsilon^l_k > \cap D
\]
is not empty. \( \square \)

**Proof of Theorem 2.4.14.** Let \( \xi \) be a continuous functional on \( C^b_x(X) \), put \( S = \bigcap \text{Supp \( \xi \)} \) and \( D = \pi_S(C^b_x(X)) \). By Lemma 2.3.4, there exists a real-valued continuous function \( \rho \) on \( D \) such that \( \xi = \rho \circ \pi_S \). Since \( \pi_S(F) \) is a subset of \( B_{00}(S) \) that contains \( K(S) \), and the set \( D \) is dense in \( C^b_{\pi_S(F)}(S) \) by Lemma 2.3.4, we can use Lemma 2.4.21. So there exists a closed almost \( \sigma \)-compact subset \( A \) of \( S \) which satisfies the condition \((*)\) in Lemma 2.4.21. Obviously, \( A \in \text{Supp \( \xi \)} \) holds. By the minimality of \( \bigcap \text{Supp \( \xi \)} \), we have \( \bigcap \text{Supp \( \xi \)} = A \). \( \square \)
Chapter 3

Spaces being $l$-equivalent to the $n$-disk

This main chapter of the thesis is dedicated to investigate some properties of spaces which are $l$-equivalent to the $n$-disk.

3.1 Basic terminology and lemmas for the chapter

In this section, we introduce some terminology and lemmas which will be needed in the later sections in the chapter.

For two linear topological spaces $V$ and $W$, $V \sim W$ means that $V$ and $W$ are linearly homeomorphic. Otherwise, for two (topological) spaces $X$ and $Y$, $X \sim_1 Y$ means that $X$ and $Y$ are $l$-equivalent. The symbol $C_p(X|A)$ denotes the subspace of $C_p(X)$ consisting of all functions vanishing on a closed subset $A$ of $X$. The symbol $S$ denotes the convergent sequence. A space $X$ is said to be $S$-stable if $X \times S \sim_1 X$.

Arhangel’skiǐ stated in [2] the following useful three results.

Lemma 3.1.1 (Arhangel’skiǐ [2, Proposition 19]). If two compacta $X$ and $Y$ are $l$-equivalent and $Z$ is an arbitrary space, then $X \times Z \sim_1 Y \times Z$.

Lemma 3.1.2 (Arhangel’skiǐ [2, Proposition 20]). For any space $X$, $X \times S$ is $S$-stable.
Lemma 3.1.3 (Arhangel'skiǐ [2, Proposition 22]). Let $X$ be a compactum. If $X$ contains a closed subset $A$ such that $A \sim X \times S$, then $X$ is $S$-stable.

By Lemma 3.1.3, we can notice that;

Lemma 3.1.4. $D^n$ is $S$-stable.

Lemma 3.1.5. If a space $X$ is $S$-stable, then;

1. $C_p(X) \times C_p(X) \sim C_p(X)$,

2. for every space $Y$ with $Y \sim X$, $Y$ is $S$-stable.

Proof. For (1);

$$C_p(X) \times C_p(X) \sim C_p(X \times S) \times C_p(X)$$

$$\sim C_p(X \times S \oplus X) \sim C_p(X \times S) \sim C_p(X).$$

Lemma 3.1.1 implies (2) as follows;

$$Y \times S \sim X \times S \sim X \sim Y.$$  

The following lemma is well-known (for example, see [26, page 585]). The reader can see the scheme of the proof of the lemma in the proof of Lemma 3.2.5 in Section 3.2.

Lemma 3.1.6. Let $X$ be a compactum. If $A$ is a closed subset of $X$, then $C_p(X) \sim C_p(X|A) \times C_p(A)$. 

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3.2 Function spaces on compact topological manifolds

The purpose of this section to prove that every compact topological $n$-manifold is $l$-equivalent to the $n$-disk. To this end, we prove the following theorem.

**Theorem 3.2.1 ([13, Theorem 2.1]).** Let $X$ be a compactum and suppose that there exists a finite open cover $\{U_1, U_2, \cdots, U_m\}$ such that $\overline{U}_i$ is $l$-equivalent to an $S$-stable compactum $E$. Then $X$ is $l$-equivalent to $E$.

Since any topological $n$-manifold can be covered by finitely many open disks. Therefore, by Lemma 3.1.4, we have;

**Corollary 3.2.2 ([13, Corollary 2.2]).** Every compact topological $n$-manifold is $l$-equivalent to $D^n$.

Below, we try to prove Theorem 3.2.1. We need some terminology and lemmas.

**Definitions 3.2.3.** (1) Suppose that linear topological spaces $V$ and $W$ have some norms $\| \cdot \|_V$ and $\| \cdot \|_W$ respectively, which are not necessarily compatible with their topologies. A linear continuous map $\phi : V \to W$ is said to be bounded if there exists a positive constant $K$ such that $\|\phi(x)\|_W \leq K \cdot \|x\|_V$ for each $x \in V$. Two spaces $V$ and $W$ are said to be norm-equivalent if there exists linear homeomorphism $\phi : V \to W$ such that $\phi$ and $\phi^{-1}$ are bounded with respect to $\| \cdot \|_V$ and $\| \cdot \|_W$. The homeomorphism $\phi$ is called a norm-equivalence.

(2) The product $\prod_{i=1}^m V_i$ of linear topological spaces $V_i$ with their norms, is always assumed to be endowed with the norm defined by;

$$\|(x_i)\| = \max\{\|x_i\|_{V_i} : i = 1, 2, \cdots, m\}.$$
(3) For the power \( V^\omega \) of a linear topological space \( V \) with its norm, the symbol \( V^\omega_0 \) denotes the linear subspace of it defined by
\[
V^\omega_0 = \{ (x_n) \in V^\omega : \lim_{n \to \infty} \|x_n\|_V = 0 \}.
\]

The proofs of the next lemma are easy and omitted.

**Lemma 3.2.4.** Let \( W_i \) and \( \widehat{W_i} \) \((i = 1, 2, \ldots, m)\) be linear topological spaces with norms \( \| \cdot \|_{W_i} \) and \( \| \cdot \|_{\widehat{W_i}} \), respectively. Then we have the following.

1. If \( W_1 \) and \( \widehat{W_1} \) are norm-equivalent with respect to the norms \( \| \cdot \|_{W_1} \) and \( \| \cdot \|_{\widehat{W_1}} \), then \( (W_1)^\omega_0 \sim (\widehat{W_1})^\omega_0 \).
2. If \( W_i \) and \( \widehat{W_i} \) are norm-equivalent with respect to \( \| \cdot \|_{W_i} \) and \( \| \cdot \|_{\widehat{W_i}} \) for every \( i \), then \( \prod_{i=1}^{m} W_i \) and \( \prod_{i=1}^{m} \widehat{W_i} \) are too, with respect to their norms.
3. For the product \( W_1 \times \widehat{W_1} \), \( (W_1 \times \widehat{W_1})^\omega_0 \sim (W_1)^\omega_0 \times (\widehat{W_1})^\omega_0 \).
4. \( (W_i)^\omega_0 \sim (W_1)^\omega_0 \times (W_1)^\omega_0 \).

Below, function spaces over compactum and their subspaces are always assumed to be endowed with the sup-norm.

**Lemma 3.2.5.** For any compactum \( X \) and any closed subset \( Y \) of \( X \), \( C_p(X) \) and \( C_p(Y) \times C_p(X|Y) \) are norm-equivalent.

**Proof.** By Dugundji's extension theorem [8], there exists a bounded linear continuous map \( u : C_p(Y) \rightarrow C_p(X) \) such that \( u(f)|_Y = f \) for every \( f \in C_p(Y) \). Here, we define a map \( \phi : C_p(X) \rightarrow C_p(Y) \times C_p(X|Y) \) by \( \phi(g) = (g|_Y, g - u(g|_Y)) \) for every \( g \in C_p(X) \). We can see easily that the map \( \phi \) is a norm-equivalence. \( \square \)
Lemma 3.2.6. Let \( \{U_1, U_2, \cdots, U_m\} \) be a finite open cover of a compactum \( X \), and \( Z = \bigoplus_{i=1}^{m} \overline{U}_i \). There exists a linear subspace \( Q \) of \( C_p(Z) \) such that \( C_p(Z) \) and \( C_p(X) \times Q \) are norm-equivalent.

Proof. By [26, Proposition 2.13], we can find a linear homeomorphism in question. The norm condition easily follows from the proof. \( \Box \)

Lemma 3.2.7 ([1, Theorem 1]). Let \( Y \) and \( Z \) are compacta. Then every linear homeomorphism from \( C_p(Y) \) onto \( C_p(Z) \) is a norm-equivalence.

Lemma 3.2.8 ([26, Lemma 2.5]). \( C_p(Y \times S) \sim (C_p(Y))_0^\omega \) for a compactum \( Y \).

Lemma 3.2.9. (1) For compacta \( Z_1, Z_2, \cdots, Z_m \),
\[
C_p(\bigoplus_{i=1}^{m} Z_i) \text{ and } \prod_{i=1}^{m} C_p(Z_i) \text{ are norm-equivalent.}
\]

(2) Let \( Z \) be an \( S \)-stable compactum. Then \( C_p(Z \oplus Z \oplus \cdots \oplus Z) \) and \( C_p(Z) \) are norm-equivalent.

Proof. (1); We can easily see that the usual homeomorphism is a norm-equivalence.

(2); We have;
\[
C_p(Z \oplus Z) \sim C_p(Z) \times C_p(Z) \sim C_p(Z)
\]
by using Lemma 3.1.5.(1), and
\[
C_p(Z \oplus Z \oplus \cdots \oplus Z) \sim C_p(Z)
\]
by an easy induction. For the rest, it is sufficient to apply Lemma 3.2.7. \( \Box \)

Lemma 3.2.10. Suppose two compacta \( X \) and \( E \) satisfy the assumption of Theorem 3.2.1. Then, \( X \times S \sim_I E \times S \).

Lemma 3.2.10 completes the proof of Theorem 3.2.1.
Proof of Theorem 3.2.1. Since $\overline{U}_1 \sim_1 E$ and $E$ is $S$-stable, $\overline{U}_1 \sim_1 E \times S$. By Lemma 3.2.10, $\overline{U}_1 \sim_1 X \times S$. Moreover, we can see that $X$ is $S$-stable by Lemma 3.1.3. Since $X$ and $E$ are $S$-stable, we have:

$$X \sim_1 X \times S \sim_1 E \times S \sim_1 E$$

$\square$

Proof of Lemma 3.2.10. The isomorphism between $C_p(X \times S)$ and $C_p(E \times S)$ is established by a sequence of linear homeomorphisms. The following diagram indicates the sequence. The definitions of each linear homeomorphisms will follow after the diagram.
\[
\begin{array}{c}
C_p(X \times S) \\
\downarrow \phi_1 \\
(C_p(X \times S))^\omega_0 \\
\downarrow \phi_2 \\
(C_p(E \times S) \times P)^\omega_0 \\
\downarrow \phi_3 \\
(C_p(E \times S))^\omega_0 \times P^\omega_0 \\
\downarrow \phi_4 \\
(id \times \phi_3^{-1}) \\
(C_p(E \times S))^\omega_0 \times (C_p(E \times S))^\omega_0 \times P^\omega_0 \\
\downarrow \psi_2 \times \text{id} \\
(C_p(X \times S) \times Q)^\omega_0 \times (C_p(E \times S) \times P)^\omega_0 \\
\uparrow \text{trivial} \\
(C_p(E \times S) \times P)^\omega_0 \times (C_p(X \times S) \times Q)^\omega_0 \\
\downarrow \phi_2 \times \text{id} \\
(C_p(X \times S))^\omega_0 \times (C_p(X \times S))^\omega_0 \times Q^\omega_0 \\
\downarrow \psi_4 \\
(C_p(X \times S))^\omega_0 \times Q^\omega_0 \\
\uparrow \psi_3 \\
(C_p(X \times S) \times Q)^\omega_0 \\
\uparrow \psi_2 \\
(C_p(E \times S))^\omega_0 \\
\uparrow \psi_1 \\
C_p(E \times S)
\end{array}
\]

$\phi_1$ and $\psi_1$; Since $X \times S$ and $E \times S$ are $S$-stable by Lemma 3.1.2, the existence of these isomorphisms follows from Lemma 3.2.8.

$\phi_2$; Since $\overline{U_1} \sim E$, $C_p(\overline{U_1} \times S)$ and $C_p(E \times S)$ are norm-equivalent by Lemmas 3.1.1 and 3.2.7. Moreover, $C_p(X \times S)$ and $C_p(\overline{U_1} \times S) \times C_p(X \times S)[\overline{U_1} \times S]$ are norm-equivalent by Lemma 3.2.5. Thus, $C_p(X \times S)$ and $C_p(E \times S) \times P$ are norm-equivalent,
where $P = C_p(X \times S | \overline{U_i} \times S)$. The existence of this isomorphism follows from Lemma 3.2.4.(1).

$\psi_2$; We can easily see that $X \times S = \bigcup_{i=1}^{m} U_i \times S$. By Lemma 3.2.6, there exists a linear subspace $Q$ of $C_p(Z)$ such that $C_p(X \times S) \times Q$ are $C_p(Z)$ are norm-equivalent, where $Z = \bigoplus_{i=1}^{m} \overline{U_i} \times S$. Since $\overline{U_i} \times S = \overline{U_i} \times S$, $C_p(Z)$ and $\prod_{i=1}^{m} C_p(\overline{U_i} \times S)$ are norm-equivalent by Lemma 3.2.9.(1). Since $\overline{U_i} \sim E$ for every $i$, $C_p(\overline{U_i} \times S)$ and $C_p(E \times S)$ are norm-equivalent by Lemmas 3.1.1 and 3.2.7. Moreover, $\prod_{i=1}^{m} C_p(\overline{U_i} \times S)$ and $(C_p(E \times S))^m$ are norm-equivalent by Lemma 3.2.4.(2). Applying Lemma 3.2.9.(1) again, we can see that $(C_p(E \times S))^m$ and $C_p((E \times S) \oplus \cdots \oplus (E \times S))$ are norm-equivalent. Since $C_p((E \times S) \oplus \cdots \oplus (E \times S))$ and $C_p(E \times S)$ are norm-equivalent by Lemmas 3.1.2 and 3.2.9.(2), $C_p(X \times S) \times Q$ and $C_p(E \times S)$ are norm-equivalent. The existence of this isomorphism follows from Lemma 3.2.4.(1).

$\phi_3$ and $\psi_3$; The existence of these isomorphisms follows from Lemma 3.2.4.(3).

$\phi_4$ and $\psi_4$; There exist isomorphisms $\hat{\phi}_4$ and $\hat{\psi}_4$ such that

$$
\hat{\phi}_4 : (C_p(E \times S))_0^o \to (C_p(E \times S))_0^o \times (C_p(E \times S))_0^o \quad \text{and} \\
\hat{\psi}_4 : (C_p(X \times S))_0^o \to (C_p(X \times S))_0^o \times (C_p(X \times S))_0^o,
$$

by Lemma 3.2.4.(4). We can get these isomorphisms by;

$$
\phi_4 = \hat{\phi}_4 \times \id_{P_0^o}, \quad \psi_4 = \hat{\psi}_4 \times \id_{Q_0^o}.
$$

\[\square\]
3.3 A characterization theorem

The purpose of the section is to present a characterization of spaces which are $l$-equivalent to the $n$-disk, and three corollaries of the theorem. For a subset $Y$ of $X$ and a subset of $Z$ of $Y$, we set;

\[
\text{Int}_Y Z = \text{the interior of } Z, \\
\text{Fr}_Y Z = \text{the boundary of } Z
\]

in the subspace $Y$, respectively; if $Y = X$, we write $\text{Int } Z$ for $\text{Int}_X Z$. Here, we need two definitions.

**Definition 3.3.1.** Let $X$ be a finite-dimensional compactum. We put;

\[
\text{DK}(X) = \{x \in X : \text{ind}_x X = \dim X\}, \text{ and} \\
\text{M}(X) = \text{the closure of } \text{DK}(X) \text{ in } X.
\]

For the definition of $\text{ind}_x X$, see [10, Problem 1.1.B, page 7]. The subset $\text{DK}(X)$ of $X$ is a known one as the *dimensional kernel* of $X$.

**Definition 3.3.2.** An $n$-dimensional space $X$ is said to be an MU($n$)-space if every subset $Y$ of $X$ with $\dim Y = n$ satisfies $\text{Int } Y \neq \emptyset$ and contains a copy of $D^n$.

Euclidean $n$-space is an MU($n$)-space; this classical result which is due to Menger and Urysohn is well-known. By using the above two definitions, we can now formulate a theorem which gives a characterization in question.

**Theorem 3.3.3 ([20, Theorem 2.3]).** For a space $X$ and a natural number $n$, $X$ is $l$-equivalent to $D^n$ if and only if $X$ is an $n$-dimensional compactum with;

(LD1) there exists a non-empty open subset of $M(X)$ which is an MU($n$)-space,
(LD2) every non-empty open subset of $M(X)$ contains a subset which is $l$-equivalent to $X$.

To prove the theorem, we need some lemmas. Below, we state them.

The following lemma is due to Menger (see, [10, Problem 1.5.H.(c), page 38]).

**Lemma 3.3.4 (Menger).** Let $X$ be an $n$-dimensional compactum. For every point $x$ in $DK(X)$, $\text{ind}_x DK(X) = n$.

By Lemma 3.3.4, we have;

**Lemma 3.3.5.** Let $X$ be an $n$-dimensional compactum. If $U$ is a non-empty open subset of $M(X)$, then $\dim U = n$.

As Lemma 3.3.5 may be known, for the completeness of the thesis, we give a proof.

**Proof.** Suppose that $\dim U \leq n - 1$ and take a point $x$ in $U \cap DK(X)$. Since $\text{ind}_x DK(X) = n$ by Lemma 3.3.4, there exists a neighbourhood $V$ of $x$ in $DK(X)$ such that, for every neighbourhood $W$ of $x$ in $DK(X)$ with $W \subset V$, $\text{ind}_{DK(X)} W > n - 2$. Take a neighbourhood $N$ of $x$ in $M(X)$ such that $N \cap DK(X) = V$. By our assumption, there exists an open subset $G$ of $U$ with $x \in G \subset N \cap U$ and $\text{ind}_{U} G \leq n - 2$. Otherwise, obviously, we have $\text{Fr}_{DK(X)}(G \cap DK(X)) \subset \text{Fr}_{U} G$. This is a contradiction. □

**Lemma 3.3.6.** If a space $X$ is $l$-equivalent to some compactum, then $X$ is a compactum.

**Proof.** Since the network weight and compactness are preserved by $l$-equivalence (see [3, Theorem I.1.3, page 26] and [25]), $X$ is a compactum. □
Proof of Theorem 3.3.3. ('if' part) Suppose that an $n$-dimensional compactum $X$ satisfies the conditions (LD1) and (LD2) in Theorem 3.3.3. First, we show that $X \times S \sim_l \mathbb{D}^n$. By the condition (LD1), there exists a non-empty open subset $U$ of $M(X)$ that is an $\text{MU}(n)$-space. Furthermore, by the condition (LD2), which contains a subset $Y_1$ with $Y_1 \sim_l X$. Since, by Theorem 1.3.3, $\dim Y_1 = n$ and $Y_1$ contains a copy $A$ of $\mathbb{D}^n$. Then $\text{Int}_U A \neq \emptyset$ because $\dim A = n$. Applying the condition (LD2) again, we can find a subset $Y_2$ of $\text{Int}_U A$ such that $Y_2 \sim_l X$. Then $Y_2 \times S \subset A \times S \subset Y_1 \times S$, and, by Lemma 3.1.1, we have;

$$Y_1 \times S \sim_l X \times S \sim_l Y_2 \times S.$$ 

It follows that;

$$C_p(X \times S) \sim C_p(Y_1 \times S)$$

$$\sim C_p(Y_1 \times S|A \times S) \times C_p(A \times S) \ (\text{by Lemma 3.1.6})$$

$$\sim C_p(Y_1 \times S|A \times S) \times C_p(A \times S) \times C_p(A \times S) \ (\text{by Lemmas 3.1.2 and 3.1.5}(1))$$

$$\sim C_p(Y_1 \times S) \times C_p(A \times S) \ (\text{by Lemma 3.1.6})$$

$$\sim C_p(Y_2 \times S) \times C_p(A \times S)$$

$$\sim C_p(Y_2 \times S) \times C_p(Y_2 \times S) \times C_p(A \times S|Y_2 \times S) \ (\text{by Lemma 3.1.6})$$

$$\sim C_p(Y_2 \times S) \times C_p(A \times S|Y_2 \times S) \ (\text{by Lemmas 3.1.2 and 3.1.5}(1))$$

$$\sim C_p(A \times S) \ (\text{by Lemma 3.1.6})$$

$$\sim C_p(\mathbb{D}^n \times S)$$

$$\sim C_p(\mathbb{D}^n) \ (\text{by Lemma 3.1.4}).$$
Hence $X \times S \sim_l \mathbb{D}^n$. Thereby $X$ is $S$-stable because of condition (LD1) and Lemma 3.1.3. It follows that $X \sim_l X \times S \sim_l \mathbb{D}^n$.

('only if' part) By Theorem 1.3.3 and Lemma 3.3.6, $X$ is an $n$-dimensional compactum. Now we proceed to check the conditions (LD1) and (LD2). By Theorem 1.3.4, there exists a non-empty open subset $U$ of $M(X)$ which is homeomorphic to a subset of $\mathbb{D}^n$. By Lemma 3.3.5, $\dim U = n$. It is easy to see that $U$ is an $MU(n)$-space. To check the condition (LD2), take a non-empty open subset $V$ of $M(X)$. Since $M(X)$ is closed in $X$ and $V$ is open in $M(X)$, $V$ has the Baire property. By Theorem 1.3.4 again, there exists a non-empty open subset $W$ of $V$ which is homeomorphic to a subset of $\mathbb{D}^n$. Since $\dim W = n$ by Lemma 3.3.5, $W$ contains a copy of $\mathbb{D}^n$ which is $l$-equivalent to $X$ by the assumption. □

Corollary 3.3.7 ([20, Corollary 3.1]). If $X \sim_l \mathbb{D}^n$ then $X \sim_l M(X)$.

Proof. By Theorem 3.3.3, $X$ is a compactum. Moreover, by the condition (LD2) in Theorem 3.3.3 and Lemma 3.3.6, there exists a compact subset $Z$ of $M(X)$ such that $Z \sim_l X$. Now we have;

\[
\begin{align*}
C_p(M(X) \times S) \\
\sim C_p(M(X) \times S|Z \times S) \times C_p(Z \times S) \quad \text{(by Lemma 3.1.6)} \\
\sim C_p(M(X) \times S|Z \times S) \times C_p(Z \times S) \times C_p(Z \times S) \quad \text{(by Lemmas 3.1.2 and 3.1.5.(1))} \\
\sim C_p(M(X) \times S) \times C_p(Z \times S) \quad \text{(by Lemma 3.1.6)} \\
\sim C_p(M(X) \times S) \times C_p(X \times S) \quad \text{(by Lemma 3.1.1)} \\
\sim C_p(M(X) \times S) \times C_p(M(X) \times S) \times C_p(X \times S| M(X) \times S) \quad \text{(by Lemma 3.1.6)} \\
\sim C_p(M(X) \times S) \times C_p(X \times S| M(X) \times S) \quad \text{(by Lemmas 3.1.2 and 3.1.5.(1))}
\end{align*}
\]
\[ \sim C_p(X \times S) \text{ (by Lemma 3.1.6)} \sim C_p(X) \text{ (by Lemma 3.1.5.(2))} \]

\[ \sim C_p(Z). \]

By Lemma 3.1.3, we have \( X \sim_1 M(X). \)

Now, according to Remark 1.3.9, we introduce a class of spaces for that we have no difficulty in checking the condition (LD2) in Theorem 3.3.3.

**Definition 3.3.8.** A space \( X \) is said to be **densely self-embeddable** if every non-empty open subset of \( X \) contains a copy of \( X \).

**Corollary 3.3.9 ([20, Corollary 3.3]).** For a densely self-embeddable space \( X \) and a natural number \( n \), \( X \) is \( l \)-equivalent to \( \mathbb{D}^n \) if and only if \( X \) is an \( n \)-dimensional MU(\( n \))-compactum.

**Proof.** First, notice that \( M(X) = X \). Suppose that \( X \) is an \( n \)-dimensional MU(\( n \))-compactum. It is easy to see that \( X \) satisfies the conditions (LD1) and (LD2). Now we proceed to show the converse. By Theorem 3.3.3, \( X \) satisfies the condition (LD1). Since \( X \) is densely self-embeddable, we may assume that \( X \) is a subset of the MU(\( n \))-space \( U \). Notice that every \( n \)-dimensional subset of an MU(\( n \))-space is an MU(\( n \))-space. \( \square \)

**Corollary 3.3.10 ([20, Corollary 3.4]).** For every densely self-embeddable space \( X \), the space \( X \times 2^\omega \) is not \( l \)-equivalent to \( \mathbb{D}^n \), where \( 2^\omega \) is the Cantor set.

**Proof.** Suppose that the space \( X \times 2^\omega \) is \( l \)-equivalent to \( \mathbb{D}^n \). By Theorem 3.3.3, \( X \times 2^\omega \) is an \( n \)-dimensional compactum. Furthermore, \( X \times 2^\omega \) is densely self-embeddable. The interior of the \( n \)-dimensional compact subset \( X \times \{p\} \) of \( X \times 2^\omega \) is empty, where \( p \) is a point of \( 2^\omega \). This contradicts Corollary 3.3.9. \( \square \)

We conclude this section with the following remarks.
Remark 3.3.11. Bestvina [6] proved that the $n$-dimensional universal Menger compactum $\mu^n$ is densely self-embeddable. Thus, we have $M(\mu^1) = \mu^1$. By these facts, we can see that the space $\mu^1$ satisfies the condition (LD2), but not (LD1) for $n = 1$. Moreover, by Valov’s theorem (Theorem 1.3.8), we can see that $\mu^1$ is not $l$-equivalent to $\mathbb{I}(= D^1)$, and thus, the converse of Corollary 3.3.7 does not hold. Obviously, the space $\mu^1 \oplus \mathbb{I}$ satisfies the condition (LD1). By theorem 1.3.8, $\mu^1 \oplus \mathbb{I}$ is $l$-equivalent to $\mu^1$, and thus, it does not satisfy the condition (LD2). Thus, we can not drop each one of the conditions (LD1) and (LD2) in Theorem 3.3.3. Next, we refer to the existence of a space $X$ that is not $l$-equivalent to $M(X)$ (cf. Corollary 3.3.7). Let $X$ be the space $\mu^1 \oplus D^2$. Obviously, we have $M(X) = D^2$. Suppose that $X$ is $l$-equivalent to $D^2$. By Theorem 1.3.4, there exists a non-empty open subset of $\mu^1$ that is homeomorphic to a subset of $D^2$. This contradicts the fact that $\mu^1$ is densely self-embeddable. In conclusion, we point out the existence of a space $X$ with $X \sim_1 I$ that is not an MU(1)-space (cf. Corollary 3.3.9). Let $X$ be the space $I \times S$. By Lemma 3.1.4, $X$ is $l$-equivalent to $I$. The interior of the 1-dimensional compact subset $I \times \{p\}$ of $X$ is empty, where $p$ is the limit point of $S$. Notice that $M(X) = X$ for this space $X$. 

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Bibliography


