SHRINKING PROPERTIES of OPEN COVERS

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THESIS

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CONTENTS

Conventions

CHAPTER 0: INTRODUCTION 1

CHAPTER 1: RELATIONS AMONG VARIOUS COVERING PROPERTIES 8

CHAPTER 2: FUNDAMENTALS OF PROPERTIES $\mathcal{B}$ AND $\mathcal{D}$ 18

CHAPTER 3: CHARACTERIZATIONS OF PROPERTY $\mathcal{B}$ 21
   1. Property $\mathcal{B}$ and certain open covers 21
   2. Property $\mathcal{B}$ and the normality of product spaces 30

CHAPTER 4: SHRINKING PROPERTY 35
   1. Spaces having a shrinking property 35
   2. Gruenhage and Michael's Problem 40
   3. Countably many product spaces 44

REFERENCES 47
Conventions

All spaces are assumed to be Hausdorff topological spaces and all mappings are continuous.
An ordinal number is equal to the set of its predecessors and cardinal numbers are initial ordinals.
ω and ω₁ are used to denote the first infinite ordinal and the first uncountable ordinal respectively.
The letter I will always denote the closed unit interval [0, 1].
|A| is the cardinality of a set A.
For a subset A of a space X, clXA (or clA) denotes the closure of A in X.

Let ℱ be a cover of X. A refinement ℶ of ℱ is a cover which refines ℱ.

Undefined notions and terminologies will follow R. Engelking [1989].
CHAPTER 0
INTRODUCTION

The notions of shrinkages of open covers have played an important role in the development of many areas of set-theoretic topology (R. Engelking [1989]).

An open cover \( \mathcal{U} = \{ U_\alpha \mid \alpha \in A \} \) of a space \( X \) is said to be shrinkable if there exists an open cover \( \{ V_\alpha \mid \alpha \in A \} \) of \( \mathcal{U} \) such that \( \text{cl} V_\alpha \subseteq U_\alpha \) for each \( \alpha \in A \).

The following theorem is the first one which shows the usefulness of shrinkages of some open covers in normal spaces and has an influence on related properties.

Recall that a space is countably paracompact if every countable open cover admits a locally finite open refinement.

0.1. Theorem (C. H. Dowker [1951])

The following conditions are equivalent for a normal space \( X \):

\( (1) \) \( X \) is countably paracompact.

\( (2) \) \( X \times I \) is normal.

\( (3) \) Every countable open cover of \( X \) is shrinkable.
If we do not assume the normality of $X$ in the above theorem, the following one which contributed the development in covering properties is well-known and shows the values of shrinkages of increasing open covers.

A collection $\mathcal{A} = \{A_\lambda | \lambda \in \Lambda\}$ of subsets of a space $X$ is increasing (resp. decreasing) if $\Lambda$ is well-ordered and $A_\lambda \subseteq A_\mu$ (resp. $A_\lambda \supseteq A_\mu$) for any $\lambda, \mu \in \Lambda$ with $\lambda < \mu$.

0.2. Theorem (F. Ishikawa [1955] and J. Mack [1967])

The following conditions are equivalent for a space $X$:

(1) $X$ is countably paracompact.

(2) Every countable increasing open cover of $X$ is shrinkable.

(3) Every countable increasing open cover $\{U_n | n < \omega\}$ of $X$ has an increasing open cover $\{V_n | n < \omega\}$ such that $\text{cl}V_n \subseteq U_n$ for each $n < \omega$.

Under considerations of the above conditions (2) and (3) of Theorem 0.2, we have the following notions which have the rich developments in wide areas of topology:

0.3. Definition

Let $X$ be a space and $\kappa$ an infinite cardinal number.
Then $X$ has property $\mathcal{D}(\kappa)$ (resp. property $\mathcal{B}(\kappa)$) if every increasing open cover $\{U_\alpha | \alpha < \kappa\}$ of $X$ has an (resp. an increasing) open cover $\{V_\alpha | \alpha < \kappa\}$ such that $\text{cl}V_\alpha \subseteq U_\alpha$ for each $\alpha < \kappa$.

If $X$ has property $\mathcal{B}(\kappa)$ (resp. property $\mathcal{B}(\kappa)$) for any infinite cardinal number $\kappa$, $X$ is said to have property $\mathcal{B}$ (resp. property $\mathcal{B}$).

Clearly property $\mathcal{B}(\kappa)$ implies property $\mathcal{D}(\kappa)$, and in case $\kappa = \omega$, properties $\mathcal{B}(\omega)$ and $\mathcal{D}(\omega)$ coincide with countable paracompactness.

Property $\mathcal{B}$ was introduced by P. Zenor [1970] and property $\mathcal{D}$ was defined in Yasui's paper [1972] under the name of weak $\mathcal{B}$-property.

It is difficult to check for a lot of spaces whether every (increasing) open cover is shrinkable or not, even if we know their normality. For instance the proof that every open cover of $\Sigma$-products of metric spaces, or compact or $p$-spaces with countable tightness is shrinkable is more involved and technical than that of normality (A. P. Kombarov [1978], K. Chiba [1982] and Y. Yajima [1984]).

Also we have another difficulty in finding spaces which are normal countably paracompact but not having property $\mathcal{D}(\kappa)$; indeed, the only such an example is obtained by M. E. Rudin.
Therefore to study the shrinkability of (increasing) open covers is one of the subjects which we cannot fail to notice.

Most covering properties are defined by refining arbitrary open cover with an appropriate open cover such as compactness, paracompactness, Lindelöfness, subparacompactness, submetacompactness (see Definitions 4.2 and 4.3).

One of our purposes is to characterize property $B$ along this line without using increasing open covers. Indeed we shall prove that a space $X$ has a property $B$ if and only if every infinite open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ with the following property: Each $x \in X$ has a nbd $O$ such that the cardinality of $\{ V \in \mathcal{V} : O \cap V \neq \emptyset \}$ is less than $|\mathcal{U}|$ (Theorem 3.3).

This characterization is useful when we apply property $B$ and study the relations among covering properties (Corollaries 3.4 and 3.5).

To study the normality of product spaces is one of basic and difficult problems in general topology.

Next we will characterize property $B(\kappa)$ in terms of normality of product spaces. As is known, for a normal space $X$ the product $X \times \mathcal{L}^\kappa$ is normal if and only if $X$ is $\kappa$-paracompact (K. Morita [1961]).

Since every normal $\kappa$-paracompact space has property $B(\kappa)$
(see Theorem 1.1), it seems to be desirable to find a specific space $Y$ with the property that for a normal space $X$, $X \times Y$ is normal for this space $Y$ if and only if $X$ has property $B(\kappa)$.

In Chapter 3 we will construct such a space which is denoted by $I_\kappa$. That is, $I_\kappa$ is a test space for property $B(\kappa)$.

Several compact spaces that are test spaces for $\kappa$-paracompactness or $\kappa$-collectionwise normality have been considered (see C. H. Dowker [1951], k. Morita [1961] and O. T. Alas [1971]). We note that $I_\kappa$ is not compact and besides these compact spaces and our $I_\kappa$ other specific spaces that can be test spaces for other covering properties are not yet obtained.

As to shrinkability of (not necessarily increasing) open covers, there has not been any other equivalent condition.

In 1984, K. Chiba obtained a sufficient one that if a space is either normal subparacompact or perfectly normal, then every open cover is shrinkable. Our Theorem 4.7 will extend this result to normal submetacompact spaces. Notice that a class of submetacompact spaces includes both subparacompact spaces and perfectly normal spaces, and that a wider class of spaces than that of our case is not known for which the same result is true.

Another purpose is to study shrinkability of a certain kind of open cover. In [1983], Gruenhage and Michael showed that every cover of a regular space by open subsets with Lindelöf
boundaries is shrinkable. In their paper they posed the following problem: "Is every cover of a regular space by open subsets with metrizable closures shrinkable?"

We shall show the following theorem on shrinkability (Theorem 4.8): "Every cover of a space by open subsets with perfectly normal closures is shrinkable." This theorem contains a positive answer to the problem above.

Finally we shall discuss on the product spaces of countably many spaces having property \( R \).
CHAPTER 1
RELATIONS AMONG VARIOUS COVERING PROPERTIES

In this chapter we see how our properties \( \mathcal{Z} \) and \( \mathcal{G} \) relate to other covering ones.

Basically the following implications hold:

1.1. **Theorem**

Paracompactness \( \Rightarrow \) property \( \mathcal{Z} \) \( \Rightarrow \) property \( \mathcal{G} \)
\( \Rightarrow \) countable paracompactness.

Main purpose of this chapter is to see that all the reverse implications of Theorem 1.1 do not hold.

Before we list the spaces in illustration of their gaps, we observe a proposition for later use; the proof is easy and omitted.

1.2. **Proposition**

Let \( X \) be a space and \( \kappa \) an infinite cardinal number. Then \( X \) has property \( \mathcal{G}(\kappa) \) (resp. property \( \mathcal{B}(\kappa) \)) if and only if, for any decreasing collection \( \{ F_\alpha \mid \alpha < \kappa \} \) of closed subsets of \( X \)
with $\alpha^\alpha \mathcal{C}_\alpha = \emptyset$, there is a (resp. a decreasing) collection
\[ \{ G_\alpha | \alpha < \kappa \} \]
of open subsets of $X$ such that
\[
(1) \quad F_\alpha \subseteq G_\alpha \text{ for each } \alpha < \kappa
\]
and
\[
(2) \quad \alpha^\alpha \mathcal{C}_\alpha = \emptyset.
\]

As the first example, we shall show the following:

1.3. Example (Y. Yasui [1972])
Let $\omega_1$ be the first uncountable ordinal number.
If $\omega_1$ has an order topology, then $\omega_1$ has property $\mathfrak{D}$, but does not have property $\mathfrak{B}$ (strictly speaking, $\omega_1$ does not have property $\mathfrak{B}(\omega_1)$).

Proof $X$ has property $\mathfrak{D}$.

It is known that this space $\omega_1$ is normal. In order to show by Proposition 1.2 that $\omega_1$ has property $\mathfrak{D}$, let $\kappa$ be any infinite cardinal and $\mathcal{I} = \{ F_\lambda | \lambda < \kappa \}$ any decreasing collection of closed subsets of $X$ with $\lambda^\alpha \mathcal{C}_\alpha = \emptyset$. We may assume each $F_\lambda$ is not empty.

For each $\alpha \in \omega_1$, we let
\[
f(\alpha) = \text{the first of } \{ \lambda | \lambda < \kappa, [0, \alpha] \cap F_\lambda = \emptyset \}.
\]
Since $[0, \alpha]$ is compact and $\{ F_\lambda | \lambda < \kappa \}$ is the decreasing
collection with \( \bigcap_{\alpha} F_\lambda = \emptyset \), \( f(\alpha) \) is well-defined.

We shall show that, if we let \( \Lambda = \{ f(\alpha) \mid \alpha < \omega_1 \} \), then \( \Lambda \) is cofinal in \( \kappa \). For this purpose, we assume that \( \Lambda \) is not cofinal in \( \kappa \), that is, there exists some \( \lambda_0 < \kappa \) such that \( f(\alpha) \leq \lambda_0 \) for any \( \alpha < \omega_1 \). This means that \( [0, \alpha] \cap F_{\lambda_0} = \emptyset \) for any \( \alpha < \omega_1 \), that is, \( F_{\lambda_0} = \emptyset \). This is contradictory.

Now, we select any point \( \alpha_\lambda \) of \( f^{-1}(\lambda) \) for each \( \lambda \in \Lambda \). Then it is seen that \( \{ \alpha_\lambda \mid \lambda \in \Lambda \} \) is cofinal in \( \omega_1 \). If we let
\[
G_\lambda = (\alpha_\lambda, \omega_1) \quad \text{if} \ \lambda \in \Lambda
\]
\[
= [0, \omega_1) \quad \text{if} \ \lambda \in \kappa - \Lambda,
\]
then \( \{ G_\lambda \mid \lambda \in \kappa \} \) is a collection of open subsets of \( \omega_1 \) such that \( F_\lambda \subseteq G_\lambda \) for each \( \lambda < \kappa \), and furthermore \( \bigcap_{\lambda \in \kappa} G_\lambda \) is empty by the cofinality of \( \{ \alpha_\lambda \mid \lambda \in \Lambda \} \). Hence \( X \) has property \( \mathcal{D} \).

**Space \( \omega_1 \) does not have property \( \mathcal{B}(\omega_1) \).**

For each \( \alpha < \omega_1 \), we let \( F_\alpha = [\alpha, \omega_1) \). Then \( \{ F_\alpha \mid \alpha < \omega_1 \} \) is a decreasing collection of closed subsets of \( \omega_1 \) with \( \bigcap_{\alpha < \omega_1} F_\alpha = \emptyset \).

If \( \omega_1 \) has property \( \mathcal{B}(\omega_1) \), there exists a decreasing collection \( \{ G_\alpha \mid \alpha < \omega_1 \} \) of open subsets of \( \omega_1 \) by Proposition 1.1 such that
\[
(1) \quad F_\alpha \subseteq G_\alpha \quad \text{for each} \ \alpha < \omega_1
\]
and
\[
(2) \quad \alpha < \omega_1 \cap \cl G_\alpha = \emptyset.
\]

For each \( \alpha < \omega_1 \), we let
\[
f(\alpha) = \text{the first of} \ \{ \beta < \omega_1 \mid (\beta, \omega_1) \subseteq G_\alpha \}.
\]

- 10 -
Then \( f \) is a mapping from \([1, \omega_1]\) to \( \omega_1 \) by (1) such that

\( f(\alpha) < \alpha \) for each \( \alpha \) with \( 1 \leq \alpha < \omega_1 \)

and

\( f(\beta) \leq f(\alpha) \) for \( \alpha, \beta < \omega_1 \) with \( \beta < \alpha \).

By definition of \( f(\alpha) \), we have \( [f(\alpha)+1, \omega_1] \subset G_\alpha \) for each \( \alpha < \omega_1 \) and hence \( \bigcap_{\omega_1} [f(\alpha)+1, \omega_1] \subset \bigcap_{\omega_1} G_\alpha = \emptyset \). This means that

\( \{f(\alpha) \mid \alpha < \omega_1\} \) is cofinal in \( \omega_1 \).

By pressing down lemma (see K. Kunen [1980]) and (3), there exists some \( \alpha_0 < \omega_1 \) (where we may assume that \( \alpha_0 \geq 1 \)) such that

\( \{\alpha \mid \alpha < \omega_1, f(\alpha) \leq \alpha_0\} \) is cofinal in \( \omega_1 \).

On the other hand, there exists some \( \alpha_1 < \omega_1 \) by (5) such that

\( \alpha_0 < f(\alpha_1) \).

For \( \alpha_1 \), there exists some \( \alpha_2 < \omega_1 \) by (6) such that

\( \alpha_1 < \alpha_2 \) and \( f(\alpha_2) \leq \alpha_0 \).

By (4), (7) and (8), \( \alpha_0 < f(\alpha_1) \leq f(\alpha_2) \leq \alpha_0 \). This is contradictory. \( \Box \)

**Remarks**

1. W. M. Fleishman ([1970]) proved the following theorem almost simultaneous with our Example 1.3: "Every open cover of a linealy ordered space is shrinkable." So it is seen that every open cover of \( \omega_1 \) is shrinkable.
2. As in the proof of the above example, we can show that:
   For any regular cardinal number \( \kappa \), there exists a normal space which has property \( \mathcal{D} \) but does not have property \( \mathcal{B}(\kappa) \).

   In fact, the space \( \kappa \) with order topology is such a space. Furthermore it is seen that there is a normal space which has property \( \mathcal{B}(\lambda) \) for every \( \lambda < \kappa \), but does not have property \( \mathcal{B}(\kappa) \).

   To discuss the gap between property \( \mathcal{D} \) and countable paracompactness, let us first mention the following theorem due to M. E. Rudin.

1.4. Theorem (M. E. Rudin [1978])

   For each infinite cardinal number \( \kappa \), there exists a normal space \( X_\kappa \) without property \( \mathcal{B}(\kappa) \).

   A normal space without property \( \mathcal{B}(\kappa) \) is generally called a \( \kappa \)-Dowker space which was previously defined by Rudin [1971] means a normal but not countably paracompact space. Thus, an \( \omega \)-Dowker space is nothing but a Dowker space.

   Let \( X_\kappa \) be the \( \kappa \)-Dowker space given in Theorem 1.4. Let \( \mathcal{U} \) be any open cover of \( X_\kappa \) with \( |\mathcal{U}| < cf(\kappa) \). Then \( \mathcal{U} \) is shown to be refined by a cover of mutually disjoint open sets (that is, \( X_\kappa \) is ultra \( \lambda \)-paracompact for any \( \lambda < \text{cf}(\kappa) \)). Therefore,
taking $k = \omega_1$, we have the following:

1.5. Example (M. E. Rudin [1983-a] and [1985])

There is a normal space $X$ which is countably paracompact but does not have property 2.

Next we shall introduce a Navy's space which is repeatedly quoted in this paper. This space is normal and paracompact, but it is not paracompact (K. Navy [1981]), where a space $X$ is called to be paracompact if every open cover of $X$ has a locally countable open refinement.

We present this space due to the definition which is appeared in Rudin's paper [1983-a].

**Navy's space $S$**

Let $F$ be the set of all functions from $\omega$ into $\omega_1$. For $n (1 < n < \omega)$, let $\Sigma_n = \{ f|n : f \in F \}$, and $P_n$ be the set of all subsets of $\Sigma_n$. Let $\Sigma = \bigcup_{\omega} \Sigma_n$ and $P$ be the set of all finite subsets of $\bigcup_{\omega} P_n$.

Let $\Delta = \{(\sigma, \tau, d) | (1) \ d \in P$, and $\sigma, \tau \in \Sigma_n$ for some $n \}

(2) $A \in d \cap P_m$ for some $m \leq n$, then $\sigma|m \in A$ iff $\tau|m \in A$

(3) $\sigma(0) < \tau(0) < \sigma(1) < \tau(1) < \ldots < \sigma(n-1) < \tau(n-1)$$

For $\rho \in \Sigma$, and $\mathcal{B} \in P$, let

$B(\rho, \mathcal{B}) = \{ f \in F | f \supset \rho \} \cup \{ <\sigma, \tau, d> \in \Delta | (1) \ \sigma \supset \rho \text{ or } \tau \supset \rho \}$
where \( f \supset \rho \) means that \( f \) extends \( \rho \).

Navy’s space \( S \) is \( F \cup \Delta \) topologized by having \( \{B(\rho, B) | \rho \in \Sigma \) and \( B \in P \} \cup \{(\sigma, \tau, \mathcal{A}) \mid (\sigma, \tau, \mathcal{A}) \in \Delta \} \) as an open base.

1.6. Example (M. E. Rudin [1983-a] and [1985])

Navy’s space \( S \) is a normal space with property \( \mathcal{B} \) and every open cover of \( S \) is shrinkable, but is not paracompact.

If we apply our theorems of a later chapter, it is very easier than Rudin’s one to prove that every open cover of \( S \) is shrinkable and \( S \) has property \( \mathcal{B} \). So we shall show them.

**proof of some part of Example 1.6**

Since it is known that \( S \) is countably paracompact and paralindelöf (K. Navy [1981]), \( S \) has property \( \mathcal{B} \) by the below Corollary 3.4.

Next we can show that every open cover of \( S \) is shrinkable. By Fleissner [1984] the subspace \( F \) of \( S \) is ultra paracompact. So every open cover of \( F \) is shrinkable. On the other hand, \( \Delta \) is a discrete subspace of \( S \). Hence every open cover of \( S \) is shrinkable (by Proposition 4.1). ⊓⊔

In 1951, Bing introduced the concept of collectionwise shrinkable.
normality and constructed the valuable example $G$.

Example $G$ and its subspaces have proved to be rich source of examples among topological properties (ref. I. W. Lewis [1977]).

For completeness let us recall Bing's example $G$:

Let $P$ be uncountable set and $D$ two-point set $\{0, 1\}$. Furthermore let $\mathcal{P}$ be the power set of $P$ and $F$ the product space of $\mathcal{P}$-copies of $D$, that is, $F = \{f | f$ is a mapping from $\mathcal{P}$ to $D\}$.

For each $p \in P$, define $f_p \in F$ as follows:

$$f_p(q) = \begin{cases} 1 & \text{if } p \in Q \\ 0 & \text{otherwise.} \end{cases}$$

Let $F_0$ as $\{f_p | p \in P\}$. For each $R \in \mathcal{P}$ and $p \in P$, let $U(p, R) = \{f \in F | f(R) = f_p(R) \text{ for any } R \in R\}$, where $\mathcal{P}$ denotes the set of all finite subsets of $\mathcal{P}$. Let us define a nbhd base $\mathcal{U}(f)$ of $f \in F$ as follows:

$$\mathcal{U}(f) = \begin{cases} \{f\} & \text{if } f \in F - F_0 \\ \{U(p, R) | R \in \mathcal{P}\} & \text{if } f = f_p \in F_0. \end{cases}$$

Therefore each point $f \in F - F_0$ is isolated in $F$ and each point $f_p \in F_0$ is isolated in $F_0$ (but $f_p$ is not isolated in $F$).

1.7. Example (K. Chiba [1984], or ref. Y. Yasui [1989])

Let $F$ be a Bing's example $G$ and $X$ any subspace of $F$. Then every open cover of $X$ is shrinkable, but $F$ does not have property $\mathcal{B}$. 

- 15 -
proof

Since the shrinkability of any open cover of $X$ is due to Proposition 4.1, we shall sketch the outline that $F$ does not have property $\mathcal{F}$.

Without loss of generality, we may assume that $P = \omega_1$. So we let $F_0 = \{ f_\alpha | \alpha < \omega_1 \}$. If we let $H_\alpha = \{ f_\beta | \alpha < \beta < \omega_1 \}$ for each $\alpha < \omega_1$, then $\{ H_\alpha | \alpha < \omega_1 \}$ is a decreasing collection of closed sets with $\bigcap_{\alpha < \omega_1} H_\alpha = \emptyset$. It is seen that if $\{ U_\alpha | \alpha < \omega \}$ is a decreasing open collection such that $H_\alpha \subseteq U_\alpha$ for any $\alpha$, then $\bigcap_{\alpha < \omega_1} \text{cl} U_\alpha \neq \emptyset$. □

As the last example of this chapter, we shall consider that in the definition of property $\mathcal{F}(\omega) = \text{property } \mathcal{D}(\omega)$ (Theorem 0.2), we cannot weaken to the following: Any countable increasing open cover $\{ U_n | n < \omega \}$ of $X$ has a countable increasing closed cover $\{ F_n | n < \omega \}$ such that $F_n \subseteq U_n$ for each $n$.

1.8. Example

Let $X = \{ (x, y) | x, y \text{ are real numbers with } y \geq 0 \}$ with Niemytzki's tangent disc topology. Then every countable increasing open cover $\{ U_n | n < \omega \}$ of $X$ has an increasing closed cover $\{ F_n | n < \omega \}$ such that $F_n \subseteq U_n$ for each $n < \omega$. 

- 16 -
proof

We recall Niemytzki's tangent disc topology. Let \( L = \{(x, 0) | x: \text{real}\} \) and \( \tau \) be the Euclidean topology for \( X \).

We generate a topology \( \tau^* \) on \( X \) by adding to \( \tau \) all sets of the form \( \{p\} \cup D \), where \( p \in L \) and \( D \) is an open disc in \( X - L \) which is tangent to \( L \) at the point \( p \). Then \( \tau^* \) is called the Niemytzki's tangent disc topology.

Let \( \{U_n | n < \omega\} \) be any countable increasing open cover of \( X \). We select a countable open base \( \{B_n | n < \omega\} \) for open half-plane \( X - L \) such that \( (\text{cl}_{X}B_n) \cap L = \emptyset \) and each \( \text{cl}_{X}B_n \) is contained in some \( U_m \). Let \( f \) be a mapping from \( \omega \to \omega \) as follows:

\[
f(m) = \min \{n | \text{cl}_{X}B_m \subseteq U_n\} \quad \text{for each } m < \omega.
\]

If we let

\[
F_n = \bigcup \{\text{cl}_{X}B_m | f(m) < n, m < n\}
\]

for each \( n < \omega \), then it is seen that \( \{F_n | n < \omega\} \) is an increasing closed cover of \( X \) such that \( F_n \subseteq U_n \) for each \( n < \omega \). \( \square \)
In Chapter 2 we shall see the fundamentals of properties $\mathscr{D}$ and $\mathscr{O}$ and their applications.

It is seen that properties $\mathscr{D}$ and $\mathscr{O}$ are closed hereditary, that is, every closed subspace of a space with property $\mathscr{D}$ (resp. property $\mathscr{O}$) has property $\mathscr{D}$ (resp. property $\mathscr{O}$), but it does not hold for open subspace.

We shall next consider the closed images of spaces with property $\mathscr{D}$ (resp. $\mathscr{O}$). It is known that the closed image of a paracompact space is necessarily paracompact, but M. E. Rudin [1983-a] and [1985] (resp. H. Ohta [1985]) showed that property $\mathscr{D}$ (resp. property $\mathscr{O}$) is not preserved under a closed mapping. As a matter of course, in a class of normal spaces, property $\mathscr{O}$ is preserved under closed mappings.

Really Rudin used Navy's space $S$ for this example (see Example 1.6). Using the notation in the front of Example 1.6, for $\alpha < \omega_1$, let $F_\alpha = \{ f \in F | f(0) = \alpha \}$. Let $T$ be the quotient space gotten from $S$ by idetifying the terms of $F_\alpha$ for each $\alpha < \omega_1$.

Furthermore $f$ is a quotient mapping from $S$ to $T$. Then Rudin showed that $f$ is closed but $T$ is a normal space which does not have property $\mathscr{D}$.
On the other hand, Ohta's example showed that the countable paracompactness is not preserved by closed continuous mapping. Such a space was assured by P. Zenor [1969], but the range space of Zenor's mapping is not regular. But Ohta showed that countable paracompactness is not an invariant of closed mappings in the realm of Tychonoff spaces.

As mentioned above, any closed continuous image of property $\mathcal{E}$ does not necessarily have property $\mathcal{B}$, but the following is seen:

2.1. Proposition

Every perfect image of a space with property $\mathcal{E}(\kappa)$ has also property $\mathcal{B}(\kappa)$ for any infinite cardinal number $\kappa$.

A mapping is called to be perfect if it is closed and every inverse image of any one-point set is compact.

From this fact, we have easily the following corollary with respect to the union of spaces with property $\mathcal{B}$:

2.2. Corollary

If a space $X$ has a locally finite closed cover
\[ \mathcal{F} = \{ F_\alpha \mid \alpha \in A \} \text{ such that each } F_\alpha \text{ has property } \mathcal{Z}, \text{ then } X \text{ has also property } \mathcal{Z}. \]

2.3. Corollary

Let \( \{ U_n \mid n < \omega \} \) be a countable open cover of a space \( X \). If \( \text{cl} U_n \) has property \( \mathcal{Z} \) for any \( n < \omega \), then \( X \) has property \( \mathcal{Z} \).

proof

If we let \( F_n = \text{cl} U_n - \bigcup_{i<n} U_i \) for \( n \geq 2 \) and \( F_1 = \text{cl} U_1 \), then \( \{ F_n \mid n < \omega \} \) is a locally finite closed cover of \( X \) each of which has property \( \mathcal{Z} \). So \( X \) has property \( \mathcal{Z} \) by Corollary 2.2. \( \square \)
CHAPTER 3
CHARACTERIZATIONS OF PROPERTY \( \mathcal{B} \)

The purpose of Chapter 3 is to give characterizations of property \( \mathcal{B} \) and to apply them and furthermore to clear up this concept.

1. Property \( \mathcal{B} \) and certain open covers

Most covering properties are defined by refining an arbitrary open cover with an open cover having an appropriate local property such as paracompactness, metacompactness and subparacompactness (see Definitions 4.2 and 4.3).

One of the purposes of this section is to study the following problem: "For any increasing open cover of a space having \( \mathcal{B} \)-property, can we take its refinement with a 'nice' local property?"

3.1. Theorem (Y. Yasui [1986])

Let \( X \) be a space and \( \kappa \) an infinite cardinal number. Then the following conditions are equivalent:

(1) \( X \) has property \( \mathcal{B}(\kappa) \).
(2) Every increasing open cover \( \{ U_\alpha \mid \alpha < \kappa \} \) of \( X \) has an open cover \( \{ V_\alpha \mid \alpha < \kappa \} \) of \( X \) such that

\[
(2-1) \quad V_\alpha \subset U_\alpha \quad \text{for any } \alpha < \kappa,
\]

and

\[
(2-2) \quad \text{for each } x \in X, \text{ there exist some open nbd } O_x \text{ of } x \text{ and some } \alpha_x \in \kappa \text{ such that } O_x \cap ( \cup_{\alpha_x} V_\alpha ) = \emptyset.
\]

(3) Every increasing open cover \( \{ U_\alpha \mid \alpha < \kappa \} \) of \( X \) has an open cover \( \{ V_\alpha \mid \alpha < \kappa \} \) of \( X \) such that

\[
(3-1) \quad \text{cl}(V_\alpha) \subset U_\alpha \quad \text{for any } \alpha < \kappa,
\]

and

\[
(3-2) \quad \text{for each } x \in X, \text{ there exist some open nbd } O_x \text{ of } x \text{ and some } \alpha_x \in \kappa \text{ such that } O_x \cap ( \cup_{\alpha_x} V_\alpha ) = \emptyset.
\]

**proof** \((1) \rightarrow (3)\):

Let \( \{ U_\alpha \mid \alpha < \kappa \} \) be an increasing open cover of \( X \). Then we have two increasing open covers \( \{ T_\alpha \mid \alpha < \kappa \} \) and \( \{ S_\alpha \mid \alpha < \kappa \} \) of \( X \) such that \( \text{cl} S_\alpha \subset T_\alpha \subset \text{cl} T_\alpha \subset U_\alpha \) for each \( \alpha \).

Without loss of generality, we may assume that

\[
(\ast) \quad T_\alpha = \cup \{ T_\beta \mid \beta < \alpha \} \quad \text{for any limit ordinal } \alpha < \kappa.
\]

Let

\[
V_\alpha = T_\alpha - \text{cl}(S_{\alpha-1}) \quad \text{if } \alpha \text{ is not limit}
\]

\[
= \emptyset \quad \text{if } \alpha \text{ is limit}.
\]

Then the collection \( \{ V_\alpha \mid \alpha < \kappa \} \) of open sets will be a cover of \( X \). Let \( x \) be any point of \( X \) and \( \alpha_0 \) the first of \( \{ \alpha \mid \quad - 22 - \)
$x \in T_\alpha$}, then $\alpha_0$ is not limit by (*). So $x \notin \text{cl}(S_{\alpha_0-1})$.

Therefore we have $x \in V_{\alpha_0}$.

To see that $\{V_\alpha \mid \alpha < \kappa\}$ satisfies (3-2), let $x$ be any point of $X$. Since $\{S_\alpha \mid \alpha < \kappa\}$ is a cover of $X$, there exists some $\alpha_x < \kappa$ with $x \in S_{\alpha_x}$. Then for any non-limit ordinal $\alpha$ with $\alpha_x < \alpha < \kappa$, we have $S_{\alpha_x} \cap V_\alpha \subseteq S_{\alpha_x} - \text{cl}(S_{\alpha_x-1}) \subseteq S_{\alpha_x} - \text{cl}(S_{\alpha_x}) = \emptyset$.

(3) $\rightarrow$ (2): clear

(2) $\rightarrow$ (1):

Let $\{U_\alpha \mid \alpha < \kappa\}$ be an increasing open cover of $X$ and $\{V_\alpha \mid \alpha < \kappa\}$ as given in (2).

If we let

$$T_\alpha = \bigcup \{O \mid O \text{ is open and } O \cap (\bigcup_{\beta \geq \alpha} V_\beta) = \emptyset\}$$

for each $\alpha < \kappa$, then $\{T_\alpha \mid \alpha < \kappa\}$ is an increasing open cover of $X$. So we shall show that $\text{cl}T_\alpha \subseteq U_\alpha$ for any $\alpha$.

We have $T_\alpha \cap (\bigcup_{\beta \geq \alpha} V_\beta) = \emptyset$ and hence $(\text{cl}T_\alpha) \cap (\bigcup_{\beta \geq \alpha} V_\beta) = \emptyset$ for each $\alpha < \kappa$.

Therefore

$$\text{cl}T_\alpha \subseteq X - (\bigcup_{\beta \geq \alpha} V_\beta) = \beta \setminus \alpha \cap V_\beta \subseteq \beta \setminus \alpha \cup V_\beta \subseteq U_\alpha \subseteq U_\alpha.$$  \[
\]

Almost all the covering properties are defined by terms that an arbitrary open cover has a refinement with some property (see Definitions 4.2 and 4.3). If it is possible to have such characterizations of property $\mathcal{R}$, its utility will be
exploited in more areas.

The second purpose of this section is to have such characterizations:

3.2. Theorem  (Y. Yasui [1986], [1987] and [1989])

Let $X$ be a space and $\kappa$ an infinite cardinal number. Then the following conditions are equivalent:

1. $X$ has property $\mathfrak{F}(\kappa)$.

2. Every open cover $\{U_\alpha \mid \alpha < \kappa\}$ of $X$ has an open cover $\{U_{\alpha\beta} \mid \beta < \alpha; \alpha < \kappa\}$ of $X$ such that

   (2-1) $V_{\alpha\beta} \subset U_\beta$ for any $\beta, \alpha$ with $\beta < \alpha$,

   and

   (2-2) each $x \in X$ has some nbd $0$ and some ordinal $\alpha_x < \kappa$ such that $0 \cap (\cup \{V_{\alpha\beta} \mid \beta < \alpha; \alpha > \alpha_x\}) = \phi$.

3. Every open cover $\mathcal{U}$ of $X$ with cardinality $\kappa$ has an open refinement $\mathcal{V}$ which satisfies the following:

   (*) Each $x \in X$ has some nbd $0$ such that the cardinality of $\{V \in \mathcal{V} \mid 0 \cap V \neq \phi\}$ is less than $\kappa$.

proof  $(1) \to (2)$:

Let $\mathcal{U} = \{U_\alpha \mid \alpha < \kappa\}$ be an open cover of $X$. If we let $W_\alpha = \beta < \alpha \ U_\beta$ for each $\alpha < \kappa$, then $\{W_\alpha \mid \alpha < \kappa\}$ is an increasing open cover of $X$ such that $W_\alpha = \beta < \alpha \ W_\beta$ for each limit ordinal $\alpha < \kappa$. 

- 24 -
Since $X$ has property $\mathcal{B}(\kappa)$, there exists an increasing open cover $\mathcal{V} = \{V_\alpha \mid \alpha < \kappa\}$ of $X$ such that $\text{cl}V_\alpha \subset W_\alpha$ for each $\alpha$. We may assume that $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$ for each limit $\alpha$.

For each $\alpha$, $\beta < \kappa$ with $\beta < \alpha$, let

$$
V_{\alpha\beta} = U_\beta - \text{cl}(V_{\alpha-1}) \quad \text{if} \ \alpha \ \text{is not limit and} \ \beta < \alpha
$$

$$
= \varnothing \quad \text{otherwise}.
$$

Then it is clear that $V_{\alpha\beta} \subset U_\beta$ for each $\alpha$, $\beta$ with $\beta < \alpha$.

To see that $\{V_{\alpha\beta} \mid \beta < \alpha\}$ is a cover of $X$, let $x$ be any point of $X$. If $\alpha_x$ is the first of $\{\alpha \mid \alpha < \kappa, x \in W_\alpha\}$, then $\alpha_x$ is not limit and $x \notin W_{\alpha_x-1}$. Hence $x \notin \text{cl}(V_{\alpha_x-1})$.

Since $x \in W_{\alpha_x} = \bigcup_{\beta < \alpha_x} U_\beta$, there is some $\beta < \alpha_x$ with $x \in U_\beta$, and hence $x_\beta \in U - \text{cl}(V_{\alpha_x-1}) = V_{\alpha_x\beta}$. This means that $\{V_{\alpha\beta} \mid \beta < \alpha\}$ is a cover of $X$.

Since $V_{\alpha\beta} \subset U_\beta$ for any $\beta$, $\alpha$ with $\beta < \alpha$, it is sufficient to show that $\{V_{\alpha\beta} \mid \beta < \alpha\}$ satisfies the condition (2-2).

Let $x \in X$ and $\alpha_0 < \kappa$ with $x \in V_{\alpha_0}$. We have $V_{\alpha_0} \cap (X - \text{cl}V_\alpha)$ for any $\alpha$ with $\alpha_0 < \alpha < \kappa$, because $\{V_\alpha \mid \alpha\}$ is increasing. Then, for any non-limit ordinal $\alpha$ with $\alpha > \alpha_0 + 1$ and any ordinal $\beta$ with $\beta < \alpha$, it follows that

$$
V_{\alpha_0} \cap V_{\alpha\beta} \subset V_{\alpha_0} \cap (X - \text{cl}V_{\alpha-1}) = \varnothing.
$$

$(2) \rightarrow (3)$:

Let $\mathcal{U}$ be an open cover of $X$ with cardinality $\kappa$ and $\mathcal{U}$ express as $\{U_\alpha \mid \alpha < \kappa\}$. 

- 25 -
By (2), there exists an open cover \( \{ V_{\alpha\beta} \mid \beta < \alpha; \alpha < \kappa \} \) of \( X \) that

\[(2-1) \quad V_{\alpha\beta} \subseteq U_\beta \quad \text{for any} \quad \beta < \alpha\]

and

\[(2-2) \quad \text{each} \quad x \in X \quad \text{has some open nbd} \quad O_x \quad \text{and some ordinal} \quad \alpha_x \quad \text{such that} \quad O_x \cap (\cup \{ V_{\alpha\beta} \mid \beta < \alpha; \alpha \geq \alpha_x \}) = \emptyset.\]

Then the cover \( \{ V_{\alpha\beta} \mid \beta < \alpha \} \) is a refinement of \( \mathcal{U} \) by (2-1).

On the other hand, let \( x \) be any point of \( X \) and \( O_x, \alpha_x \) be as given in (2-2). Then the cardinality of \( \{ V_{\alpha\beta} \mid O_x \cap V_{\alpha\beta} \neq \emptyset \} \) is less than or equal to the cardinality of \( \{ (\alpha, \beta) \mid \beta < \alpha; \alpha \leq \alpha_x \} \). Then the cardinality of \( \{ V_{\alpha\beta} \mid O_x \cap V_{\alpha\beta} \neq \emptyset \} \) is less than \( \kappa = |\mathcal{U}|. \)

\((3) \rightarrow (1): \quad \text{It is clear by Theorem 3.1.}\)

These complete the proof. \( \square \)

By Theorem 3.2, we have the following:

3.3. **Theorem** (Y. Yasui [1986], [1987] and [1989])

The following conditions are equivalent for a space \( X \):

1. \( X \) has property \( \mathcal{B} \).
2. Every infinite open cover \( \{ U_\alpha \mid \alpha < \tau \} \) of \( X \) has an open cover \( \{ V_{\alpha\beta} \mid \beta < \alpha; \alpha < \tau \} \) of \( X \) such that

\[(2-1) \quad V_{\alpha\beta} \subseteq U_\beta \quad \text{for any} \quad \beta, \alpha \text{ with} \quad \beta < \alpha,\]
and

(2-2) Each $x \in X$ has some nbd $O$ and some ordinal $\alpha_x < \tau$
such that $O \cap (\cup \{V_{\alpha|} \beta < \alpha; \alpha \geq \alpha_x\}) = \emptyset$.

(3) Every infinite open cover $\mathcal{U}$ of $X$ has an open
refinement $\mathcal{V}$ which satisfies the following:

(*) Each $x \in X$ has a nbd $O$ such that the cardinality of
$\{V \in \mathcal{V} | O \cap V \neq \emptyset\}$ is less than $|\mathcal{U}|$.

In Chapter 0, we recalled a Navy's space. A value for
the existence of the space is to show that paracompactness is
stronger than paralindelöfness in normal spaces.

Furthermore such a space is only one as far as I know, and
M. E. Rudin showed that this space has also property $\mathcal{B}$
([1983-a] and [1985]).

If we show the following as a corollary of Theorem 3.3, our
proof that Navy's space has property $\mathcal{B}$ is very simpler than
Rudin's one.

3.4. Corollary

Every countably paracompact and paralindelöf space has
property $\mathcal{B}$.
Let $\mathcal{U}$ be an infinite open cover of $X$ and $\mathcal{W}$ express as $\{U_\alpha | \alpha < \tau\}$ for some $\tau$, where $\tau$ is the minimal ordinal whose cardinality is equal to $|\mathcal{U}|$.

**proof**

Let $\mathcal{U}$ be an infinite open cover of $X$ and $\mathcal{W}$ express as $\{U_\alpha | \alpha < \tau\}$ for some $\tau$, where $\tau$ is the minimal ordinal whose cardinality is equal to $|\mathcal{U}|$.

**case 1** Assume $\text{cof}(\tau) (= \text{the cofinality of } \tau)$ is countable.

Let $\{\alpha_n | n < \omega\}$ be an increasing sequence of ordinals which converges to $\tau$. Since $\{W_{\alpha_n} | n < \omega\}$ is a countable open cover of $X$, where $W_{\alpha_n} = V_{\alpha_n} \cup U_{\alpha_n}$, there exists a locally finite open cover $\{V_n | n < \omega\}$ of $X$ such that $V_n \subset W_{\alpha_n}$ for each $n$.

Then each $x \in X$ has an open nbd $O_x$ such that the cardinality of $\{W_{\alpha_n} \cap U_\alpha | \alpha < \alpha_n$ and $O_x \cap (W_{\alpha_n} \cap U_\alpha) \neq \emptyset\}$ is less than $|\mathcal{U}|$.

**case 2** Assume $\text{cof}(\tau)$ is not countable.

By paraLindelöf property of $X$, $\mathcal{U}$ has a locally countable open refinement $\mathcal{V}$. Hence each $x \in X$ has an open nbd $O_x$ which intersects $V$ for at most countably many $V \in \mathcal{V}$, this means that the cardinality of $\{V \in \mathcal{V} | O_x \cap V \neq \emptyset\}$ is less than $|\mathcal{U}|$. □

As a corollary of the above theorem, the following will be seen.
3.5. Corollary (T. Tani and Y. Yasui [1972])

A space $X$ has property $\mathcal{B}$ if and only if every increasing open cover of $X$ has a cushioned open refinement.

proof "only if" part:

Let $\mathcal{U} = \{U_\alpha \mid \alpha < \tau\}$ be an increasing open cover of $X$.

By Theorem 3.1, there exists an open cover $\mathcal{V} = \{V_\alpha \mid \alpha < \tau\}$ of $X$ such that

(1) $V_\alpha \subset U_\alpha$ for any $\alpha < \tau$

and

(2) each $x \in X$ has an open nbd $O_x$ and some $\alpha_x < \tau$ such that

$$O_x \cap (\bigcup_{\alpha \geq \alpha_x} U_\alpha) = \emptyset.$$ 

We let $W_\alpha = \bigcup \{O_x \mid \alpha_x = \alpha\}$ for each $\alpha$. To see that a cover $\mathcal{W} = \{W_\alpha \mid \alpha\}$ is cushioned in $\mathcal{U}$, let $A$ be any subset of $\tau$, where we may assume that $A$ is not cofinal in $\tau$, and let $\alpha_0$ be the sup of $A$ (and hence $\alpha_0 < \tau$).

We have $(\bigcup_{\alpha \in A} W_\alpha) \cap (\bigcup_{\alpha \geq \alpha_0} U_\alpha) = \emptyset$, and

$$\text{cl}(\bigcup_{\alpha \in A} W_\alpha) \subset \bigcup_{\alpha \leq \alpha_0} U_\alpha \subset \bigcup_{\alpha \in A} U_\alpha.$$ 

This means $\mathcal{W}$ is cushioned in $\mathcal{U}$.

"if" part:

Let $\mathcal{U} = \{U_\alpha \mid \alpha < \tau\}$ be an increasing open cover of $X$.

Then we have an open cover $\mathcal{V} = \{V_\alpha \mid \alpha < \tau\}$ which is cushioned in $\mathcal{U}$.
Let \( W_\alpha = \beta \cup_{\alpha} V_\beta \) for each \( \alpha \), then it is seen that the collection \( \{ W_\alpha \mid \alpha < \tau \} \) is an increasing open cover of \( X \) such that \( \text{cl} W_\alpha \subset U_\alpha \) for any \( \alpha \). \( \square \)

**Remark**

It is well-known that a regular space \( X \) is paracompact if and only if every open cover of \( X \) has a cushioned open refinement (E. Michael [1959]). Since paracompactness is not equivalent to property \( \mathfrak{P} \), we cannot exchange "every increasing open cover" for "every open cover" in Corollary 3.5.

2. Property \( \mathfrak{P} \) and the normality of product spaces

Some interesting and useful characterizations of separations and covering properties of a space \( X \) are given in terms of the normality of the product spaces \( X \times Y \) with all members \( Y \) of some class of spaces. In other words, this means the existences of test spaces of covering properties.

We know them for the classes of paracompact spaces, \( \kappa \)-paracompact spaces and collectionwise normal countably paracompact spaces (Theorem 0.1, K. Morita [1961], O. T. Alas
[1971] and H. Tamano [1960]).

So we shall find a test space of normal spaces with property $\mathcal{Z}(\kappa)$ in this section.

Before we give such a characterization of property $\mathcal{Z}$, we shall define some terminology.

For an infinite cardinal number $\kappa$, we let a space $I_\kappa$ as follows:

1. $I_\kappa$ is the set of all ordinals $\leq \kappa$.
2. $I_\kappa$ has an open base $\{ (\alpha) \mid \alpha < \kappa \} \cup \{ (\alpha, \kappa) \mid \alpha < \kappa \}$.

Therefore each $\alpha < \kappa$ is isolated in $I_\kappa$ and the point $\kappa$ has a usual order nbd base in $I_\kappa$.

2.6. Theorem (Y. Yasui [1983])

Let $X$ be a normal space and $\kappa$ an infinite cardinal number. Then $X$ has property $\mathcal{Z}(\kappa)$ if and only if $X \times I_\kappa$ is normal.

proof "if" part:

To see that $X$ has property $\mathcal{Z}(\kappa)$, let $\{ U_\alpha \mid \alpha < \kappa \}$ be an increasing open cover of $X$.

Let $H = \bigcup \{ (X - U_\alpha) \times (\alpha) \mid \alpha < \kappa \}$ and $K = X \times (\kappa)$.

Since $H$ and $K$ are disjoint closed sets of $X \times I_\kappa$, there are disjoint open sets $W$ and $V$ of $X \times I_\kappa$ such that $H \subset W$ and $K \subset V$. 

- 31 -
For each $\alpha < \kappa$, let

$$V_\alpha = \{ x \in X \mid O \times (\alpha, \kappa) \subset V \text{ for some nbd } O \text{ of } x \}. \]$$

Then it is seen that $\{ V_\alpha \mid \alpha < \kappa \}$ is an increasing open cover of $X$ which satisfies $\text{cl} V_\alpha \subset U_\alpha$ for $\alpha < \kappa$. Hence $X$ has property $\mathcal{E}$.

"only if" part:

Let $H$ and $K$ be any disjoint closed sets of $X \times I_\kappa$.

Since $H \cap (X \times (\kappa))$ and $K \cap (X \times (\kappa))$ are closed and disjoint in $X \times (\kappa)$, there is an open set $O$ of $X$ such that

(a) $H \cap (X \times (\kappa)) \subset O \times (\kappa)$

and

(b) $((\text{cl} O) \times (\kappa)) \cap (K \cap (X \times (\kappa))) = \emptyset$.

For each $\alpha < \kappa$, we let

(c) $U_\alpha = (X - \text{cl} O) \cup (\bigcup \{ P \mid P \text{ open, } P \times [\alpha, \kappa) \cap K = \emptyset \})$.

Since $\{ U_\alpha \mid \alpha < \kappa \}$ is an increasing cover of $X$, we have an increasing open cover $\{ V_\alpha \mid \alpha < \kappa \}$ of $X$ such that

(d) $\text{cl} V_\alpha \subset U_\alpha$ for each $\alpha$.

If we let $P = \bigcup \{ (V_\alpha \cap O) \times (\alpha, \kappa) \mid \alpha < \kappa \}$, then $P$ is an open set of $X \times I_\kappa$ such that

(e) $(X \times (\kappa)) \cap H \subset P$ (by (c))

and

(f) $(\text{cl}_{X \times I_\kappa} P) \cap K = \emptyset$ (by (d)).

Quite similarly we have an open set $Q$ of $X \times I_\kappa$ such that

(e) $(X \times (\kappa)) \cap K \subset Q$
and
\[(f)^- \quad (\text{cl}_{X\times I^\kappa} \, Q) \cap H = \emptyset.\]

On the other hand, there is an open set \( P_\alpha \) of \( X\times(\alpha) \) for each \( \alpha \) such that
\[(g) \quad H \cap (X\times(\alpha)) \subseteq P_\alpha \]
and
\[(h) \quad (\text{cl}_{X\times(\alpha)} \, P_\alpha) \cap (K \cap (X\times(\alpha))) = \emptyset,\]
because \( H \cap (X\times(\alpha)) \) and \( K \cap (X\times(\alpha)) \) are disjoint closed sets of \( X\times(\alpha) \) which is normal.

By the topology of \( I^\kappa \), each \( P_\alpha \) is open in \( X\times I^\kappa \) and \( \text{cl}_{X\times(\alpha)} \, P_\alpha = \text{cl}_{X\times I^\kappa} \, P_\alpha \). Hence if we let \( U = P \cup (\bigcup_{\alpha \in \kappa} \, P_\alpha) - \text{cl} \, Q \), then \( U \) is an open set of \( X\times I^\kappa \) which contains \( H \) (by (e), (f)\(^-\) and (g)), and \( \text{cl} \, U \) is disjoint from \( K \) (by (e)\(^-\), (f) and (h)). \( \square \)

By the above theorem, we have a class of test spaces of property \( \mathcal{B} \) as follows:

3.7. Theorem (Y. Yasui [1983])

Let \( X \) be a normal space. Then \( X \) has property \( \mathcal{B} \) if and only if \( X\times I^\kappa \) is normal for any infinite cardinal number \( \kappa \).

Remarks
1. On "only if" part of the above Theorem 3.7, it is
sufficient to show only the case $K = X \times (\kappa)$ by M. Starbird ([1974]). But in the above proof we did not use his theorem because we wanted to explain the characteristic property of space $I_{\kappa}$.

2. When we characterize the countable paracompactness and paracompactness etc. by using the terms of normality of some product spaces, all of the spaces which are test spaces are compact. For countable paracompactness, the test spaces are $\omega + 1$, 1 and all compact metric spaces (see Theorems 0.1) and for paracompactness, their test spaces are Stone–Čech compactification and all compact spaces (see H. Tamano [1960]).

But our space $I_{\kappa}$ which is used in Theorems 3.6 and 3.7 is not compact for any $\kappa > \omega$. Therefore the following question will be raised: "Is there any class $\mathcal{F}$ of compact spaces such that a normal space $X$ has property $B$ if and only if $X \times P$ is normal for any $P \in \mathcal{F}$?" But this question answered negatively by M. E. Rudin under some set-axiom ([1983-a] and [1985]).

In fact, let $S$ be Navy's space and $T$ some quotient space of $S$ which is described in explanation of Chapter 2. Since $T$ is the closed continuous image of $S$, $T \times C$ is normal for compact $C$ whenever $S \times C$ is normal (M. E. Rudin [1975]). As mentioned above, $S$ has property $B$ but $T$ does not have property $B$. 

- 34 -
1. Spaces having shrinking property

In 1984, K. Chiba proved that every open cover of a normal subparacompact or perfectly normal space is shrinkable.

One of the purposes of this section is to generalize the above theorem.

At first we shall study the shrinkability of open covers of specific spaces which are useful to study some examples:

4.1. Proposition (ref. Y. Yasui [1985])

Let $X$ be a normal space having a disjoint cover $\{A, B\}$. If $A$ is a discrete subset of $X$ and every open cover of $B$ is shrinkable, then every open cover of $X$ is shrinkable.

Proof

Let $\{U_\lambda | \lambda \in \Lambda\}$ be an open cover of $X$. Since $\{U_\lambda \cap B | \lambda \in \Lambda\}$ is an open cover of the subspace $B$, there exists an open cover
\{V_\lambda \mid \lambda \in \Lambda\} of B such that \(cl_B V_\lambda \subseteq U_\lambda\) for any \(\lambda\). Since \(cl_B V_\lambda = cl_X V_\lambda \subseteq U_\lambda\), we have some open set \(W_\lambda\) of X such that
\[
cl_X V_\lambda \subseteq W_\lambda \subseteq cl_X W_\lambda \subseteq U_\lambda \quad (\text{for } \lambda \in \Lambda).
\]
If we let \(W = \bigcup \lambda \in \Lambda W_\lambda\) and \(O_\lambda = W_\lambda \cup ((X - W) \cap U_\lambda)\), then \(\{O_\lambda \mid \lambda \in \Lambda\}\) is an open cover of X. Since W is an open set containing B and each point of \(X - B\) is isolated in X, we have;
\[
cl_X O_\lambda = cl_X W_\lambda \cup cl_X ((X - W) \cap U_\lambda) \subseteq U_\lambda \cup ((X - W) \cap U_\lambda)
\]
Hence \(cl_X O_\lambda \subseteq U_\lambda\). This completes the proof. \(\square\)

Secondly we shall study a class of spaces whose open covers are shrinkable.

To begin with, we shall define the terminologies:

4.2. Definition

A space X is subparacompact if every open cover of X has a \(\sigma\)-discrete closed refinement.

4.3. Definition

A space X is submetacompact if every open cover of X has a sequence \(\{U_n \mid n < \omega\}\) of open refinements such that for each \(x \in X\), there is some \(n\) such that \(\text{ord}(x, U_n)\) is finite, where \(\text{ord}(x, U)\) denotes the cardinality of \(\{U \mid x \in U, U \in U\}\).

Furthermore the above sequence \(\{U_n \mid n < \omega\}\) is called \(\Theta\)-sequence or \(\Theta\)-refinement of \(U\).
Subparacompactness was introduced by McAuley who called it as $F_\sigma$-screenable ([1958]). He showed that every collectionwise normal $F_\sigma$-screenable space is paracompact.

A submetacompact space was introduced by Worrell and Wicke who called it as $\theta$-refinable space ([1965]). After then Junnila called it to be 'submetacompact' ([1978]). Junnila showed that a space $X$ is subparacompact if and only if every open cover of $X$ has a sequence $\{U_n\} n < \omega$ of open refinements such that for each $x \in X$, there is some $n$ with $\text{ord}(x, U_n) = 1$.

Therefore it is seen that every subparacompact space is submetacompact.

Next, we recall the following:

4.4. Definition

A space $X$ is said to be perfectly normal if $X$ is normal and every open subset of $X$ is a union of countably many closed subsets of $X$, that is, every open subset is a $F_\sigma$-set.

Whenever we discuss the shrinkage of all open covers of a space, the space must be normal.

Before we shall have a generalization of Chiba's theorem, we shall show some lemmas:
Lemma (Y. Yasui [1984-b])

Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of a normal space $X$ and $X_f = \{x \in X \mid \text{ord}(x, \mathcal{U}) \text{ is finite}\}$.

Then for each $\alpha \in A$, there exists a sequence $\{U_{\alpha_n} \mid n < \omega\}$ of open subsets such that

1. $\text{cl}U_{\alpha_n} \subset U_\alpha$ for any $n$.

and

2. $X_f \subset \bigcup \{U_{\alpha_n} \mid \alpha \in A, n < \omega\}$.

Proof

Let $X_n = \{x \in X \mid \text{ord}(x, \mathcal{U}) \leq n\}$ for each $n < \omega$. Then $X_n$ is closed in $X$ and $X_f = \bigcup_n X_n$.

For each $\alpha \in A$, we let $F_{\alpha_1} = U_\alpha - \bigcup \{U_\beta \mid \beta \neq \alpha\}$. Then $F_{\alpha_1}$ is closed and is contained in $U_\alpha$.

Since $X$ is normal, we have an open set $U_{\alpha_1}$ of $X$ such that $F_{\alpha_1} \subset U_{\alpha_1} \subset \text{cl}U_{\alpha_1} \subset U_\alpha$. Then $\{U_{\alpha_1} \mid \alpha \in A\}$ is a cover of $X_i$.

We assume that for some $n$, there exist open sets $H_{\alpha_i}$ of $X$ for $\alpha \in A$ and $i=1,2,\ldots, n-1$ such that $\text{cl}U_{\alpha_i} \subset U_\alpha$ for each $\alpha$ and each $i \leq n-1$, and $X_{n-1} \subset \bigcup \{U_{\alpha_i} \mid \alpha \in A, i \leq n-1\}$.

Let for each $\alpha \in A$,

$$F_{\alpha_n} = U_\alpha \cap X_n - \bigcup \{U_{\beta_i} \mid \beta \in A, i \leq n-1\}.$$ 

Since $F_{\alpha_n}$ is a closed subset of $X$ which is contained in $U_\alpha$, there exists an open set $U_{\alpha_n}$ such that $F_{\alpha_n} \subset U_{\alpha_n} \subset \text{cl}U_{\alpha_n} \subset U_\alpha$.

Then $\{U_{\alpha_i} \mid \alpha \in A, i \leq n\}$ is an open cover of $X_n$. By induction on $n$, we complete the proof of Lemma. □
4.6. Lemma  (ref. A. Bešlagić [1986])

Assume that every open cover \( \{ U_\alpha \mid \alpha \in A \} \) of a space \( X \) has an open cover \( \{ V_{\alpha n} \mid \alpha \in A, \ n < \omega \} \) of \( X \) such that \( c_1 V_{\alpha n} \subset U_\alpha \) for any \( \alpha \in A \), and any \( n < \omega \). Then every open cover of \( X \) is shrinkable.

Every normal subparacompact or perfectly normal space is submetacompact. We shall prove the following theorem by using the above Lemmas:

4.7. Theorem  (Y. Yasui [1984-b])

Every open cover of a normal submetacompact space is shrinkable.

proof

Let \( X \) be a normal submetacompact space and \( \mathcal{U} = \{ U_\alpha \mid \alpha \in A \} \) an open cover of \( X \).

Then there exists a sequence \( \{ \mathcal{V}_n \mid n < \omega \} \) of open refinements of \( \mathcal{U} \) satisfying that, for each \( x \in X \), there is some \( n \) such that \( \text{ord}(x, \mathcal{V}_n) \) is finite. Since each \( \mathcal{V}_n \) is a refinement of \( \mathcal{U} \), we may assume that \( \mathcal{V}_n = \{ V_{\alpha n} \mid \alpha \in A \} \) and \( V_{\alpha n} \subset U_\alpha \) for any \( \alpha \) and any \( n \).
We let, for each $n$

$$X_n = \{ x \in X \mid \text{ord}(x, \mathcal{V}_n) \text{ is finite} \}.$$  

Then by Lemma 4.6, there exist open subsets $H_{\alpha n i} (i=1,2,\ldots)$ of $X$ such that $\text{cl}H_{\alpha n i} \subset V_{\alpha n}$ for any $\alpha$ and any $i$, and

$$X_n \subset \bigcup \{ H_{\alpha n i} \mid \alpha \in A, i < \omega \}.$$  

Then $\{ H_{\alpha n i} \mid \alpha \in A; n, i < \omega \}$ is a cover of $X$ which satisfies the condition of Lemma 4.6, and hence $\mathcal{U}$ is shrinkable.  

By the above theorem, every normal submetacompact space has property $\mathcal{D}$, and so we have the following question:

"Does every normal submetacompact space have property $\mathcal{D}$?"

But this does not hold (see Example 1.7).

2. Gruenhage and Michael's problem

On the shrinkability of certain open cover each of which has some property, G. Gruenhage and E. Michael ([1983]) raised some question:

"Is every cover of a regular space by open subsets with metrizable closures shrinkable?"

We shall give an affirmative answer to the above question as a
corollary of the following:

4.8. Theorem (Y. Yasui [1984-a])

Every cover of a space by open subsets with perfectly normal closures is shrinkable.

Proof

Let \( \mathcal{U} = \{ U_\alpha \mid \alpha \in A \} \) be an open cover of a space \( X \) such that \( \text{cl} U_\alpha \) is perfectly normal for any \( \alpha \in A \).

Since \( \text{cl} U_\alpha \) is perfectly normal, there exists a collection \( \{ U_{\alpha n} \mid n < \omega \} \) of open subsets of \( X \) (for each \( \alpha \)) such that

1. \( \text{cl}_X U_{\alpha n} \subseteq U_{\alpha n+1} \) for \( n < \omega \),
2. \( U_\alpha = \bigcup \{ U_{\alpha n} \mid n < \omega \} \)

and

3. \( U_{\alpha 0} = \emptyset \).

For each \( \alpha \in A \) and each \( n = 1, 2, \ldots \), we let

\[ F_{\alpha n} = U_{\alpha n} - \{ U_{\beta n-1} \mid \beta \in A \}. \]

Furthermore, let for each \( \alpha \in A \),

\[ F_\alpha = \text{cl}_X ( \bigcup \{ F_{\alpha n} \mid n = 1, 2, \ldots \} ). \]

Then we shall show that the collection \( \mathcal{F} = \{ F_\alpha \mid \alpha \in A \} \) of closed subsets of \( X \) satisfies the following claims:

Claim 1 \( F_\alpha \subseteq U_\alpha \) for any \( \alpha \in A \).

Let \( x \in X - U_\alpha \). Since \( \{ U_\beta \mid \beta \in A \} \) is a cover of \( X \), there
is some $\beta_0 \in A$ with $x \in U_{\beta_0}$ (hence $\alpha \neq \beta_0$).

By (2) there is some $n_0 \in \{1, 2, \ldots\}$ with $x \in U_{\beta_0 n_0}$, and by (1) and the definition of $F_{\alpha n}$, it is seen that

(4) $U_{\beta_0 n_0} \cap F_{\alpha m} = \emptyset$ for any $m > n_0$.

For $n_0$, we have $x \in X - cl_{X} U_{\alpha n_0}$ (by (2)) and

(5) $(X - cl_{X} U_{\alpha n_0}) \cap F_{\alpha m} = \emptyset$ for any $m \leq n_0$

by (1).

By (4) and (5), $U_{\beta_0 n_0} \cap (X - cl_{X} U_{\alpha n_0})$ is an open nbd of $x$ which does not intersect with $\bigcup_{m} F_{\alpha m}$.

This shows $x \notin cl_{X} (\bigcup_{m} F_{\alpha m})$, that is, $x \notin F_{\alpha}$.

**Claim 2** $\mathcal{F}$ is a closed cover of $X$.

Since the closedness of $F_{\alpha}$ is clear, it suffices to show that $\mathcal{F}$ is a cover of $X$. We let $x \in X$ and $n_x$ be the first number of $\{n \mid x \in \bigcup_{\alpha} U_{\alpha n}\}$, and $\alpha_x$ any point of $A$ with $x \in U_{\alpha n_x}$.

Then we have $x \in F_{\alpha_x n_x} \subseteq F_{\alpha_x}$. Hence $\mathcal{F}$ is a cover of $X$.

Lastly, since $cl_{X} U_{\alpha}$ is normal, we can find an open set $V_{\alpha}$ in $cl_{X} U_{\alpha}$ such that

(6) $F_{\alpha} \subseteq V_{\alpha} \subseteq cl_{X} V_{\alpha} \subseteq U_{\alpha}$,

where $cl_{X} V_{\alpha}$ denotes the closure of $V_{\alpha}$ in $cl_{X} U_{\alpha}$. It is seen that $V_{\alpha}$ is open in $X$ and $cl_{X} V_{\alpha} = cl_{X} V_{\alpha}$. Therefore by (6) and claims 1 and 2, $\{V_{\alpha} \mid \alpha \in A\}$ is an open cover of $X$ such that $cl_{X} V_{\alpha} \subseteq U_{\alpha}$ for each $\alpha \in A$. Then $\mathcal{U}$ is shrinkable. $\square$
Remark

There are several results for shrinkage of open covers in \( \Sigma \)-products. Let us recall a \( \Sigma \)-product which was introduced by H. H. Corson [1959]. Let \( \{X_\lambda | \lambda \in \Lambda\} \) be a collection of spaces and \( X = \prod X_\lambda \) the product space of \( \{X_\lambda | \lambda \in \Lambda\} \) and \( f = (f_\lambda)_\lambda \in X \). If we let \( \Sigma = \{x \in X | (\lambda | x_\lambda \neq f_\lambda) \text{ is at most countable}\} \), then the subspace \( \Sigma \) of \( X \) is called \( \Sigma \)-product of spaces \( \{X_\lambda | \lambda \} \) with the base point \( f \in X \). On the shrinkage of open covers of \( \Sigma \)-products, there are many results (M. E. Rudin [1983-b], A. L. Donne [1985] and Y. Yajima [1986]).

As characterizations of many covering properties, it is enough to show that every increasing open cover has a corresponding refinement. For example (J. Mack [1967]):

"A space \( X \) is paracompact if and only if every increasing open cover of \( X \) has a locally finite open refinement."

For submetacompactness (resp. metacompactness), such a theorem was proved by H. J. K. Junnila [1978] (resp. W. B. Sconyers [1970]).

So we have the following question:

"If any increasing open cover of \( X \) is shrinkable, then is any open cover of \( X \) shrinkable ? "

- 43 -
This question means:

"If X has property $\mathcal{D}$, then is any open cover of X shrinkable?"

This question was answered negatively under some set axiom:

4.10. **Example (\(j^{++}(E)\))** (A. Bešlagić and M. E. Rudin [1985])

Let $\kappa$ be an infinite regular cardinal number.

Then, there is a space $\Delta$ such that $\Delta$ is ultra $\kappa$-paracompact and collectionwise normal, and every increasing open cover $\{U_\alpha | \alpha \in A\}$ of $\Delta$ has a refinement $\{V_\alpha | \alpha \in A\}$ consisting of open and closed sets, but there is an open cover $\{W_\lambda | \lambda \in \Lambda\}$ which has no closed refinement $\{F_\lambda | \lambda \in \Lambda\}$ with $F_\lambda \subseteq W_\lambda$.

3. **Countably many product spaces**

It is seen that two many product space of the spaces with property $\mathcal{B}$ (resp. $\mathcal{D}$) does not have property $\mathcal{B}$ (resp. $\mathcal{D}$), because the square of Sorgenfrey line which is paracompact is not countably paracompact.

Hence we must add some conditions whenever we discuss property $\mathcal{B}$ of product spaces.

In this section we shall comment about the countably many product spaces of property $\mathcal{B}$. 

- 44 -
Let $X = \prod \{X_n \mid n < \omega\}$ be a product space. As is well-known:

(*) "If $X_n$ has some property $P$ for each $n$, then $X$ also has one."

does not hold for almost all the covering properties $P$. But:

(**) "$\prod_{i \leq n} X_i$ has property $P$ and is perfectly normal for any $n$, then $X$ has property $P."$


For shrinkage of open covers, the following theorem is known:

4.11. Theorem (A. Bšlagić [1986])

Let $X$ be a normal product space of $\{X_n \mid n < \omega\}$. Then every open cover of $X$ is shrinkable if and only if every open cover of $\prod \{X_i \mid i \leq n\}$ is shrinkable for all $n < \omega$.

We shall replace 'is shrinkable' with 'has property $\mathcal{B}$'. Though its proof is the almost same way but the last part, some characterization of property $\mathcal{B}$ is useful:
4.12. Theorem

Let $X$ be a normal product space of $\{X_n | n < \omega\}$. Then $X$ has property $\mathcal{B}$ if and only if $\pi(X_i | i \leq n)$ has property $\mathcal{B}$ for all $n < \omega$.

proof

Let $\{U_\alpha | \alpha < \tau\}$ be an increasing open cover of $X$. If we let for each $\alpha < \tau$ and each $n < \omega$,

$$U_\alpha^n = U \{O | O \text{ is open in } \prod_{i \leq n} X_i \text{ and } O \times \prod_{i > n} X_i \subset U_\alpha\},$$

then $\{U_\alpha^n | \alpha\}$ is an increasing collection of open sets of $\prod_{i \leq n} X_i$ for each $n$. Furthermore we let $O_n = (\prod_{i \leq n} U_\alpha^i \times \prod_{i > n} X_i)$ for each $n$. Then $\{O_n | n < \omega\}$ is an increasing open cover of $X$. Since $X$ is countably paracompact (T. C. Przymusinski [1984]), there is an increasing open cover $\{S_n | n < \omega\}$ of $X$ such that $\text{cl} S_n \subset O_n$ for any $n$ (Theorem 0.2). Let $p_n$ be the projection from $X$ to $\prod_{i \leq n} X_i$ and $T_n = \prod_{i \leq n} X_i - p_n(X - \text{cl} S_n)$ for any $n < \omega$, then $T_n$ is a closed subset of $\prod_{i \leq n} X_i$ and $T_n \subset \bigcup_\alpha U_\alpha^n$.

Since $T_n$ has property $\mathcal{B}$, there is an increasing open cover $\{V_\alpha^n | \alpha < \tau\}$ of $T_n$ such that $\text{cl}_{T_n} (V_\alpha^n) \subset U_\alpha^n$ for each $\alpha$ (where the closure of $V_\alpha^n$ in $T_n$ is the closure of it in $\prod_{i \leq n} X_i$).

We let for each $n$ and each $\alpha$,

$$W_\alpha^n = (V_\alpha^n \cap \text{Int}(T_n)) \times \prod_{i > n} X_i.$$

Then $\{W_\alpha^n | \alpha\}$ is an increasing collection of open subsets of $X$ such that $\text{cl} W_\alpha^n \subset U_\alpha$ for any $\alpha$ and any $n$. Since it is
seen that $(W_\alpha \mid \alpha < \tau, \eta < \omega)$ is a cover of $X$, $X$ has property $\mathcal{B}$
(T. Tani and Y. Yasui [1972]). □
REFERENCES

Alas, O. T.

[1971] On a characterization of collectionwise normality,

Atsuji, M.

[1977] On normality of the product of two spaces,
in General Top. and its Relations to Modern Analysis
and Algebra, part B, IV: Fourth Prague Top. Sympo.,
(25-27)

Bešlagić, A.

[1986] Normality in products,
Topology Appli., 22 (49-54)

Bešlagić, A. and Rudin, M. E.

[1985] Set theoretic constructions of non-shrinking open
covers,
Topology Appli., 20 (167-177)

Bing, R. H.

[1951] Metrization of topological spaces,
Can. J. Math., 3 (175-186)

Borsuk, K.

[1937] Sur les prolongments des transformations continues,
Fund. Math., 28 (99-110)
Chiba, K.

[1982] On the weak $\mathcal{B}$-property and $\Sigma$-products, 
Math. Japonica., 27 (737-746)

[1984] On the weak $\mathcal{B}$-property, 
Math. Japonica, 29 (551-567)

Corson, H. H.

[1959] Normality in subsets of product spaces, 
Amer. J. Math., 81 (785-796)

Donne, A. L.

[1985] Shrinking property in $\Sigma$-products of paracompact p-spaces, 
Topology Appli., 19 (95-101)

Dowker, C. H.

[1951] On countably paracompact spaces, 
Canad. J. Math., 3 (219-224)

Engelking, R.

[1989] General topology, 
Heldermann Verlag Berlin

Fleishman, W. M.

[1970] On coverings of linearly ordered spaces, 
Washington State Univ. Topology Conf., (52-55)

Fleissner, W. G.

[1984] The normal Moore space conjecture and large cardinals, 
in Handbook of Set-Theoretic Topology 
(North-Holland Publishing Company) (733-760)
Gruenhage, G. and Michael, E.

[1983] A result on shrinkable open covers,
Topology Proc., 8 (37-43)

Ishikawa, F.

[1955] On countably paracompact spaces,
Proc. Japan Acad., 31 (686-687)

Junnila, H. J. K.

[1978] On submetacompactness,
Topology Proc., 3 (375-405)

Kombarov, A. P.

[1978] On tightness and normality of $\Sigma$-products,

Kunen, K.

[1980] Set theory, An introduction to independence proofs,
North-Holland Publishing Company

Lewis, I. W.

[1977] On covering properties of subspaces of R. H. Bing's example G,
General Top. and its Appli., 7 (109-122)

Mack, J.

[1967] Directed covers and paracompact spaces,

McAuley, L. F.

[1958] A note on complete collectionwise normality and paracompactness,
Michael, E.

[1959] Yet another note on paracompact spaces,

[1971] Paracompactness and the Lindelöf property in finite and countable cartesian products,
Comp. Math., 23 (199-214)

Morita, K.

[1961] Paracompactness and product spaces,
Fund. Math., 53 (223-236)

[1977] Some problems on normality of products of spaces,
in General Top. and its Relations to Modern Analysis and Algebra, part B, IV: Fourth Prague Top. Sympo.,
(296-297)

Navy, K.

[1981] ParaLindelöfness and paracompactness,
Ph. D. Thesis, Univ. of Wisconsin,

Ohta, H.

[1983] On a part of problem 3 of Y. Yasui,
Questions & Answers in General Top., 1-2 (142-143)

Okuyama, A.

[1968] Some generalizations of metric spaces, their metrization theorems and product spaces,

Przymusinski, T. C.

[1984] Products of normal spaces,
in the Handbook of set theoretic top.,
(North-Holland Publishing Company)

Rudin, M. E.

[1971] A normal space X for which XXI is not normal,
Fund. Math., 73 (179-186)

[1975] The normality of products with one compact factor,
General Top. and its Appli., 5 (45-59)

[1978] k-Dowker spaces,
Czech. Math. J., 28 (103) (324-326)

[1983-a] Yasui's questions
Questions & Answers in General Top., 1-2 (122-127)

[1983-b] The shrinking property,

[1985] k-Dowker spaces,
in London Math. Soc. Lecture note Series 93
(Cambridge Univ. Press, Cambridge) (175-195)

Sconyers, W. B.

[1970] Metacompact spaces and well-ordered open coverings,
Notice Amer. Math. Soc., 18 (230)

Starbird, M.

[1974] The normality of products with a compact or a metric
factor,
Ph. D. Thesis, Univ. of Wisconsin

Tamano, H.

[1960] On paracompactness,
Tani, T. and Yasui, Y.

[1972] On the topological spaces with the $\mathfrak{B}$-property,
Proc. Japan Acad., 48-2 (81-85)

Worrell, J. M. W. and Wicke, H. H.

[1965] Characterizations of developable spaces,
Canad. J. Math., 17 (820-830)

Yajima, Y.

[1984] $\Sigma$-products of $\Sigma$-spaces,
Fund. Math., 123 (29-37)

[1986] The shrinking property of $\Sigma$-products,
Questions and Answers in General Top., 4-1 (85-96)

Yasui, Y.

[1972] On the gaps between the refinements of the increasing open coverings,
Proc. Japan Acad., 48 (86-90)

[1983] On the characterization of the $\mathfrak{B}$-property by the normality of product spaces,
Top. and its Appli., 15 (323-326)

[1984-a] A question of G. Gruenhage and E. Michael concerning the shrinkable open covers,
Questions and Answers in General Top., 2-2 (124-126)

[1984-b] A note on shrinkable open coverings,
Questions and Answers in General Top., 2 (143-146)

[1985] Some remarks on the shrinkable open covers,
Math. Japonica, 30-1 (127-131)
[1986] Some characterizations of a ξ-property,
Tsukuba J. Math., 10-2 (243-247)

[1987] Note on the characterizations of a ξ-property,
Questions & Answers in General Top., 5-1 (195-201)

[1989] Generalized paracompactness,
in Topics in General Topology,
Elsevier Scie. Publishers, North-Holland

Zenor, P.

[1969] On countable paracompactness and normality,
Prace. Math., 13 (23-32)

[1970] A class of countably paracompact spaces,
Proc. Amer. Math. Soc., 42 (258-262)