CLASSICAL THEORY OF DIRAC'S MONPOLES

by

KOHJI HIRATA

Dissertation submitted to the Faculty of the Graduate School of the University of Tsukuba in partial fulfilment of the requirements for the degree of Doctor of Science
January 1979
Abstract:

Although the magnetic monopole first proposed by Dirac in 1931 has been studied extensively up to recently, yet there has been presented no classical Lagrangian theory which is as transparent and tractable as the usual classical electrodynamics for electrically charged particles. The purpose of the present paper is to present such a formulation taking account of recent results in the field.

We redefine Dirac's action for a charge-monopole system as a certain limit of Wu-Yang's action in a way which is similar to, but different from, the work of Brandt and Primack. The Lagrangian theory thus obtained reproduces, in a simple and straightforward manner, all of the Maxwell-Lorentz equations that are of the form naturally expected for the system concerned. The theory need not make use of the so-called Dirac's veto and is applicable to the dyon model. Furthermore it appears very promising to construct the corresponding classical Hamiltonian formalism. All this is the first step necessary in constructing a satisfactory — classical and quantum — theory of Dirac's monopole.

It is also possible, on the other hand, to construct an action-at-a-distance theory of such systems. This subject will be dealt with in the latter part of the present paper.
J. Proof of independence of $L_{\mu \nu}$ on the string

References

Figures
I. Introduction

It has been a well-known fact that the Maxwell equations in the absence of matter*

\[ F^{\mu, \nu}(x) = 0 \quad , \quad (I-1) \]
\[ \tilde{F}^{\mu, \nu}(x) = 0 \quad , \quad (I-2) \]

are symmetric under the duality transformation

\[ F_{\mu, \nu} \rightarrow \tilde{F}_{\mu, \nu} \quad , \quad \tilde{F}_{\mu, \nu} \rightarrow -F_{\mu, \nu} \quad (I-3) \]

where \( F_{\mu, \nu} \) is the electromagnetic field strength. We may interpret this situation as a symmetry between electricity and magnetism. This symmetry, however, seems to be broken in the real world; while a single electric charge may be loaded on a particle, a single magnetic pole does not have been observed to be loaded on a particle. However, in order to be able to judge whether something exists or not we must have a complete, or at least reliable, theory.

If nature has the duality symmetry even in the presence of matter, there must be a particle with magnetic charge \( g \): magnetic counterpart of the conventional electric charge \( e \). Following the traditional terminology a particle with electric (magnetic) charge may be referred to as a charge (a magnetic monopole, or

*) For notations, see Appendix A.
simply, a pole) for the sake of brevity. Now a charge-pole system is expected to obey the following equations:

the Maxwell-Lorentz equations

\[ F_{\mu\nu}(x) = j_{\mu}(x) = e \int ds \; \delta^\mu(s) \; \delta^\nu(x-s), \quad (I-4) \]

\[ F_{\mu\nu},(x) = k_{\mu}(x) = \gamma \int d\tau \; \tilde{z}^\mu(\tau) \; \delta^\nu(x-z), \quad (I-5) \]

\[ m \; \ddot{z}^\mu(s) = - e \; F_{\mu\nu}(z) \; \dot{z}^\nu, \quad (I-6) \]

\[ M \; \dddot{z}^\mu(\tau) = - \gamma \; \tilde{F}_{\mu\nu}(z) \; \dot{z}^\nu, \quad (I-7) \]

where \( z^\mu(z) \), \( m(M) \) and \( j^\mu(k^\mu) \) stand for the coordinate, mass and the current of a charge(pole), respectively. We will study the classical electrodynamics of such a system.

Before going into detailed discussions of our problem it will be useful to describe a brief history of the theory up to the present time. It was Dirac\(^1\) who first studied magnetic monopoles in the framework of modern physics. He showed that the principles of quantum mechanics can be so arranged as to permit the existence of magnetic monopoles, and further that the essential feature of magnetic monopoles may be revealed only in quantum theory. In order to have a better understanding of such systems, it seems to us to be very desirable to study its
classical Lagrangian formalism. Here a difficult problem arises, however.

In the quantum-theoretical treatment given by Dirac, a pole was introduced as an endpoint of a string-like nodal line of the wave function \( \psi(x) \) of a charge: the vector potential \( \vec{A}(x) \) was supposed to be singular along this line. This line is not physical; for it can be arbitrarily moved when a (singular) gauge transformation \(^*\) is performed.

In the classical Lagrangian formalism also we must use the vector potential to describe the electromagnetic field. Dirac\(^2\) had to introduce an unphysical variable to describe the line along which the vector potential is singular; this line to be referred to, hereafter, as a string extends outward from a pole. The string then sweeps a two-dimensional sheet in space-time, the sheet being described by expressing a general point \( y_\mu \) on it as a function of two parameters \( \tau \) and \( \sigma \),

\[
\begin{align*}
y_\mu &= y_\mu(\tau, \sigma) \quad (I-8)
\end{align*}
\]

Here the line \( y_\mu(\tau,0) \) is supposed to be the world line of the pole:

\[
\begin{align*}
\bar{Z}_\mu(\tau) &= y_\mu(\tau, 0) \quad (I-9)
\end{align*}
\]

Dirac was able to construct an action for the charge-pole system,

\(^*\) See chapter II, §1.
using the string variable as an unphysical subsidiary variable. However, he showed also that his action gives the Maxwell-Lorentz equations (4)-(7) only when the string does not pass through a charged particle, that is, only when the condition

\[ \mu (y) = 0 \]  \hspace{1cm} (I-10)

holds **. Following Wentzel \(^3\) we shall call this condition Dirac's veto.

We should say, however, that Dirac's veto is an unnatural condition in the context of the classical theory; for the string is to be regarded as an unphysical object. Recently, Rosenbaum \(^4\) and Rohrlich \(^5\) have shown independently that as far as one employs the conventional method it is not possible to construct an action such that the formalism is free from Dirac's veto.

More recently Wu and Yang \(^6\) have shown that the concept of a fibre bundle is very useful and even essential in the consideration of magnetic monopoles. They were able to construct a well-defined action without using the string to reproduce the Maxwell-Lorentz equations without Dirac's veto in the framework of the action principle. \(^7\) Although the Wu-Yang Theory is complete and transparent conceptually, it is somewhat complicated and untractable from practical point of view.***

*) Equation (I-2), for example, will be referred to simply as (2) when quoted in chapter I.

**) See chapter II, §2.

***) See chapter II. §3.
The main purpose of the present paper is to construct a classical theory of a charge-monomole system, which is as transparent and tractable as the usual classical electrodynamics for a system consisting only of electrically charged particles. This will be a first necessary step to be taken when going over to canonical quantization, second quantization etc. We shall always confine ourselves to the system described by the Maxwell-Lorentz equations (4)-(7) only. A model with Dirac's veto or something alike will be left out of consideration. Generalization to the case of more than one charge and one pole or to the case of dyons (particles with both electric charge and magnetic charge) is easy and straightforward.

The present paper is arranged as follows. In chapter II a fairly detailed and critical review of the conventional theory is given for convenience of the following discussions. The next chapter III deals with construction of a Lagrangian formalism which straightforwardly reproduces the Maxwell-Lorentz equations without recourse to Dirac's veto. The theory thus obtained is then applied to the action-at-a-distance formalism in chapter IV. It will be shown there that such a theory can explicitly be constructed, contrary to the assertion made by some authors. The last chapter V is devoted to general discussions, including those concerning the quantum theory. Some of the mathematical proofs are collected in Appendices.
II. Magnetic monopole and string.

§1. The Maxwell-Lorentz equations

Throughout the present paper, we shall be concerned with systems which can be described only by the Maxwell-Lorentz equations (I-4)-(I-7). Especially here, in this section, some properties of such a system are discussed, which are derivable from the above-mentioned equations.

In order to put the equations into the form of an action principle, we need the vector potential $A_{\mu}(x)$. The usual way of introducing it consists of putting

$$ F_{\mu\nu}(x) = A_{\mu,\nu}(x) - A_{\nu,\mu}(x) , \quad (II-1) $$

But this is no longer possible when there are magnetic monopoles, since (1) leads to (I-2) and thereby contradicts (I-5).

One way of overcoming the difficulty is to introduce a string variable $y_{\mu}(\tau,\sigma)$ and to assume that (II-1) fails along the string. Thus Dirac$^2$ replaces (II-1) by an equation of the form

$$ F_{\mu\nu}(x) = A_{\mu,\nu}(x) - A_{\nu,\mu}(x) - \tilde{G}_{\mu\nu}(x) , \quad (II-2) $$

where $G_{\mu\nu}$ is a field quantity which vanishes everywhere except on $y_{\mu}(\tau,\sigma)$, and which is given by

$$ G_{\mu\nu}(x) = g \int_{-\infty}^{\infty} \int_{0}^{\infty} S(x-y) \left\{ \frac{\partial y_{\mu}}{\partial \tau} \frac{\partial y_{\nu}}{\partial \sigma} - \frac{\partial y_{\nu}}{\partial \tau} \frac{\partial y_{\mu}}{\partial \sigma} \right\} , \quad (II-3) $$
Using Stokes' theorem\(^*\), we can easily obtain

\[ G^{\mu\nu}(x) = k^\mu(x) \quad \text{(II-4)} \]

Thus in this case the equation

\[ \widetilde{H}^{\mu\nu}(x) = k^\mu(x) \quad \text{(I-5)} \]

is merely a kinematical consequence of the relation (2) between \( F_{\mu\nu} \) and \( A_\mu \). This resembles the situation of the usual electrodynamics where the equation

\[ \widetilde{F}^{\mu\nu}(x) = 0 \quad \text{(I-2)} \]

is a consequence of the relation (1).

We may interpret (2) as follows\(^3\); imagine that there lies an infinitely long and thin solenoid along the location of the string, an endpoint of which looks like a magnetic monopole: the vector potential \( A_\mu(x) \) in (2) can be regarded as the one generated by the solenoid, and then the magnetic flux flowing through the solenoid should be subtracted from \( F_{\mu\nu} \) by the term \(-\varepsilon_{\mu\nu}\). When the world lines of particles are given, the retarded field is given by solving (I-4); if we adopt the Lorentz gauge,

\[ * \) See Appendix B. \]
\[ A^{\mu e}_{\mu}(x) = e \int_{-\infty}^{\infty} ds \ \mathcal{D}^+(x - \delta) \partial_t \Delta(s), \quad (II-5) \]

\[ A^{\mu g}_{\mu}(x) = -g \epsilon_{\mu \nu \rho \sigma} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial}{\partial x^\nu} \mathcal{D}^+(x - y) \frac{\partial y^\rho}{\partial x} \frac{\partial y^\sigma}{\partial \sigma}, \quad (II-6) \]

where \( A^{\mu e}_{\mu}(A^{\mu g}_{\mu}) \) is the contribution of the charge (pole) to the retarded field and \( \mathcal{D}^+(x) \) is the retarded Green's function

\[ \mathcal{D}^+(x) = \frac{1}{4\pi} \frac{1}{|\vec{x}|} \Theta(x^0) \delta(x^0 - \tau) \quad (II-7) \]

It may be appropriate here to show simple examples of quantities given above. When a monopole is at rest at the origin and the string is chosen to lie on the \(-z\) axis, then we have

\[ A^{1}_{+\bar{g}} = \frac{g}{4\pi} \frac{-\gamma}{r(r+\delta)}, \quad A^{2}_{+\bar{g}} = \frac{g}{4\pi} \frac{\gamma}{r(r-\delta)}, \quad A^{3}_{+\bar{g}} = A^{\bar{0}}_{+\bar{g}} = 0, \quad (II-8) \]

where \( r = |\vec{x}| \),

*) Superscripts and subscripts will not be distinguished when and where no confusion arises.
\[ G^{03}(x) = -G^{30}(x) = \mathcal{J}(\theta(-3) s(x) \delta(y) \} \]
other components of \( G^{\mu \nu} = 0 \)

\[
\begin{align*}
\mathcal{F}^{0i}(x) &= \frac{g}{4\pi} \frac{x^i}{r^3} , \\
\mathcal{F}^{ai}(x) &= 0 ,
\end{align*}
\]

and

Now we discuss the gauge degrees of freedom. As is well
known in the usual electrodynamics, the electromagnetic field
strength \( F_{\mu \nu}(x) \) remains unchanged under the (non-singular) gauge
transformation

\[
A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu f(x) ,
\]

where \( f(x) \) is a nonsingular scalar function and obeys

\[
\square f(x) = 0 ,
\]
as we always work in the Lorentz gauge. This is true in the
present case. Furthermore, \( F_{\mu \nu} \) remains unchanged under a wider
class of transformations. When we change the location of the
string

\[
\mathcal{G}_\mu(\tau, \sigma) \longrightarrow \mathcal{G}'_\mu(\tau, \sigma) = \mathcal{G}_\mu(\tau, \sigma) + \delta \mathcal{G}_\mu(\tau, \sigma) ,
\]

with

\[
\delta \mathcal{G}_\mu(\tau, 0) = 0 .
\]

\( A_\mu(x) \) should also be changed according to the singular gauge
transformation

\[ A_\mu (x) \rightarrow A_\mu (x) = A_\mu (x) + \delta A_\mu (x) \quad (II-15) \]

\[ \delta A_\mu (x) = \delta A_\mu^\nu (x) \]

\[ = \frac{\varepsilon}{2} \epsilon_{\mu \lambda \rho \sigma} \int d^4 x \, \left( \frac{\partial}{\partial \sigma} \right) \left( G_{\sigma}^{\nu} (x) \right) G_{\lambda}^{\rho} \frac{\partial G_{\mu}^{\tau}}{\partial \sigma} \]

\[ - \frac{\varepsilon}{2} \epsilon_{\nu \lambda \rho \sigma} \int d^4 x \, \left( \frac{\partial}{\partial \sigma} \right) \left( G_{\sigma}^{\mu} (x) \right) G_{\lambda}^{\rho} \frac{\partial G_{\nu}^{\tau}}{\partial \sigma} \quad (II-16) \]

This transformation displaces the location of the singularity of 
\( A_\mu \) in such away as to cancel the variation of \( G_{\mu \nu} \). It is easily verified then that \( F_{\mu \nu} \) is independent of the string variable:

\[ \frac{\delta F_{\mu \nu} (x)}{\delta \gamma_\Lambda (\tau, \sigma)} = 0, \quad \text{with} \quad \delta \gamma_\Lambda (\tau, 0) = 0 \quad (II-17) \]

when use is made of Stokes' Theorem. This is consistent with the anticipation that the string variable is unphysical.

Lastly we must briefly touch on the so-called Rosenbaum paradox. From the Maxwell-Lorentz equations, we can confirm that the total angular momentum \( \mathbf{J} \) is conserved. With the monopole fixed at the origin \( \mathbf{J} \) is given as

\[ \delta J^\mu = 0 \]

*) For derivation of (16), see Appendix C.
On the other hand, the Maxwell-Lorentz equations tell us that no force acts between the charge and the pole if they approach along a straight line. In this case no interaction occurs even when the two particles pass through each other; then the angular momentum changes by $eg/4\pi$ before and after the collision. Hence a paradox.

There are several ways to avoid this paradox, but they are not the main subject of the present paper. We will come back to this paradox in the following discussions.

§2. The Dirac theory

In the present section we discuss the action formalism proposed by Dirac. The action of Dirac $I^D$ is given in the form:

$$I^D = I_1 + I_2 + I_3^D,$$

$$I_1 = -m \int ds - M \int dt,$$

$$I_2 = -\frac{1}{4} \int d^4 x \ F_{\mu\nu}(x) F^{\mu\nu}(x),$$

$$I_3^D = -e \int a^{\tau\mu} A_{\mu}(\tau).$$
where $F_{\mu\nu}$ is given by (2).

Taking $z^\mu$, $y^\mu$ and $A^\mu$ as independent variables, Dirac applied the action principle, with the result that

\[
\delta I^D = \int d\tau \left\{ m \ddot{z}_\mu + e (A_{\mu,\nu}(\delta) - A_{\nu,\mu}(\delta)) \dot{z}^\nu \right\} \delta z^\nu
\]

\[
+ \int d\tau \left\{ M \ddot{z}_\mu + \mathcal{G} \tilde{F}_{\mu\nu}(Z) \dot{z}^\nu \right\} \delta z^\mu
\]

\[
- \int d\tau \left\{ i_{\mu}(x) - F_{\mu\nu}(x) \right\} \delta A^\mu(x)
\]

\[
- \int d\tau \int_0^\infty d\sigma \sum_{\rho} \dot{Z}^\rho \dot{Z}^\sigma \frac{\partial y^1}{\partial \tau} \frac{\partial y^2}{\partial \sigma} \delta y^\rho \delta y^\mu \delta y^\nu \delta y^\sigma , \quad (II-23)
\]

where $\delta z^\mu(\tau) = \delta y^\mu(\tau, 0)$. By equating to zero the coefficients of $\delta z^\mu$, $\delta z^\nu$, $\delta A^\mu$ and $\delta y^\mu$, we obtain the equations of motion, which, however, do not agree with the Maxwell-Lorentz equations. Of course, (I-5) is a consequence of (2), i.e. a kinematical condition which need not necessarily be reproduced by action principle. In the first term of (23), the expression $A_{\mu,\nu}^\rho A_{\nu,\mu}^\rho$ appears instead of $F_{\mu\nu}^\rho$; both of these coincide with each other only when $G_{\mu\nu}(z) = 0$. Furthermore the last term of (23) leads to $F_{\mu\nu,\nu}(y) = 0$. In order for these equations to hold we must require (I-10), that is, Dirac's veto. Thus the action $I^D$ leads to the Maxwell-Lorentz equations only when the string does not pass through the charge.
In this respect it should be remarked that the last term of (23) does not necessarily lead to \( j^\mu(y) = 0 \), since it only demands

\[
\epsilon_{\mu\nu\lambda\rho} \partial(y) \frac{\partial y^\lambda}{\partial \tau} \frac{\partial y^\rho}{\partial \sigma} = 0 .
\]  

(II-24)

This condition is, however, too weak to imply \( G_{\mu\nu}(z) = 0 \). Thus we must impose Dirac's veto before carrying out the action principle. Thus the action \( I^D \) is not equivalent to the Maxwell-Lorentz equations.

The necessity of Dirac's veto in the Dirac theory can also be seen from the fact that if a set of solutions of the Maxwell-Lorentz equations are substituted in the action \( I^D \), then \( I^D \), or more precisely \( I_3^D \) will diverge indefinitely for the case when the string passes through the charge. It is easy to realize that the problem of Dirac's veto is closely related to this divergence or indefiniteness.

In this respect it may be of interest to see whether or not the action \( I^D \) is invariant under the singular gauge transformation (15)-(16). Using Stokes' Theorem and (16), we easily obtain

\[
\delta I^D = \oint_{\gamma} d\sigma \epsilon_{\mu\nu\lambda\rho} \partial(y) \frac{\partial y^\lambda}{\partial \tau} \frac{\partial y^\rho}{\partial \sigma} \delta y^\mu ,
\]  

(II-25)

for any infinitesimal variation \( \delta y^\mu(\tau, \sigma) \) with \( \delta y^\mu(\tau, 0) = 0 \). Thus \( I^D \) is not invariant under this transformation, as far as the string passes through the charge. Dirac's veto is necessitated
again.

The above discussions clearly show that the Dirac theory is not appropriate for our present purpose. If we wish to dispense with Dirac's veto in the classical theory, we must look for a theory different from that of Dirac. The latter theory, however, seems to be a complete and satisfactory one, as far as Dirac's veto is accepted even in the classical theory. It is worth noting here that with Dirac's veto we would not have to bother about Rosenbaum's paradox; this is because Dirac's veto includes the condition

$$\dot{\mathcal{H}}(\mathbf{Z}) = 0$$  \hspace{1cm} (II-26)

which we shall hereafter refer to as Rosenbaum's veto.

§3. The Wu-Yang theory

As mentioned in §1 of this chapter, no vector potential is possible such that $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ holds true everywhere on a closed surface enclosing the monopole. On every such surface there should exist at least a point at which the above relation fails: a succession of such points forms a string. Needless to say, the above relation does hold true except for this point, however.

Wu and Yang\textsuperscript{6}) pointed out that we need not employ the string if we abandon the notion of the single vector potential. Let $R_a$ and $R_b$ be two spatial domains such as those shown in Fig.1:

$$R_a = \left\{ \mathbf{r} \mid 0 \leq \theta < \frac{\pi}{2} + \delta, \ 0 < \delta \leq \frac{\pi}{2}, \ \gamma > 0 \right\}$$  \hspace{1cm} (II-27)

where the spherical coordinates around the pole is used. Further we define \( R_{ab} \) as

\[
R_{ab} = R_a \cap R_b.
\] (II-29)

Note here that the whole space \( R \) is covered by \( R_a, R_b \) and the origin (the position of the pole). A vector potential can be non-singular in \( R_a \) or in \( R_b \). Thus we define two types of vector potentials, \( A^a_\mu \) and \( A^b_\mu \), which are regular in \( R_a \) and \( R_b \), respectively. They are related to \( F_{\mu \nu} \) as

\[
F_{\mu \nu}(x) = A^a_\mu(x) - A^a_\nu, \mu(x) \quad \text{in } R_a, \quad (II-30)
\]

and

\[
F_{\mu \nu}(x) = A^b_\mu(x) - A^b_\nu, \mu(x) \quad \text{in } R_b. \quad (II-31)
\]

These two vector potentials should define the same \( F_{\mu \nu} \) in \( R_{ab} \). Thus there should be a function \( \alpha_\mu(x) \), regular in \( R_{ab} \), such that

\[
A^a_\mu(x) - A^b_\mu(x) = \alpha_\mu(x), \quad (II-32)
\]

and

\[
\alpha_\mu(x) - \alpha_\nu, \mu(x) = 0, \quad (II-33)
\]
in $R_{ab}$. Given $F_{\mu\nu}$ satisfying the Maxwell-Lorentz equations, especially (I-5), (30)-(33) imply that the integral in the direction of increasing azimuthal angle $\phi$

$$\oint \alpha_\mu(x) \, dx^\mu$$  \hspace{1cm} \text{(II-34)}

around a loop in $R_{ab}$ (as shown in Fig.II-a) gives $g$;

$$\oint \alpha_\mu(x) \, dx^\mu = g$$  \hspace{1cm} \text{(II-35)}

Clearly a loop with its winding number $n$ (cf. Fig.II-b) gives $ng$ for (34).

Thus any $F_{\mu\nu}$ satisfying (I-5) can be described in terms of $A^a_\mu$ and $A^b_\mu$ satisfying (30), (31), (32), (33) and (35). Wu and Yang have also proved the converse of the above statement. Then (I-5) becomes a kinematical equation in the same manner as in the Dirac theory.

As a simple example let us consider a case in which a monopole is placed at the origin (cf. (8)):

---

*) The proof given by Wu and Yang\textsuperscript{7}) does not seem to us to be satisfactory. We shall construct a solution of (33) and (35) later in (III-13), which explicitly gives (I-5). Hereafter we shall confine ourselves to this solution.
\[ A_a^1 = \frac{g}{4\pi} \frac{-y}{r(r+\delta)}, \quad A_a^2 = \frac{g}{4\pi} \frac{-x}{r(r+\delta)}, \quad A_a^3 = A_a^0 = 0, \quad (II-36) \]

\[ A_b^1 = \frac{g}{4\pi} \frac{y}{r(r-\delta)}, \quad A_b^2 = \frac{g}{4\pi} \frac{x}{r(r-\delta)}, \quad A_b^3 = A_b^0 = 0, \quad (II-37) \]

and

\[ \alpha^1 = \frac{g}{2\pi} \frac{-y}{x^2+y^2}, \quad \alpha^2 = \frac{g}{2\pi} \frac{x}{x^2+y^2}, \quad \alpha^3 = \alpha^0 = 0, \quad (II-38) \]

In this case \( A_\mu^a, (36), \) is the same as \( A_\mu, (8), \) and \( \alpha, (38), \) can be written as

\[ \overrightarrow{\alpha}(x) = \frac{g}{2\pi} \text{grad} \phi(x), \quad (II-39) \]

\[ \phi(x) = \tan^{-1} \frac{y}{x}, \quad (II-40) \]

As clearly shown by this example, \( A_\mu^i, (i \text{ is to stand for } a, b \text{ hereafter}) \) has its singularity on a string \( y_\mu^i \text{ outside of } R_1. \)

In this example, \( \overrightarrow{y}^a \) coincides with the \(-z\) axis and \( \overrightarrow{y}^b \) with the \(+z\) axis. A variation of \( y_\mu^i \) gives rise to a kind of gauge transformation on \( A_\mu^i, (\text{cf. (16)}), 13; \)

\[ \delta A_\mu^i = \delta A_\mu^i \quad (II-41) \]

where

\[ A_\mu^{i+\delta}(x) = -g\epsilon_{\lambda\rho\sigma} \int d\epsilon \left( \epsilon^\alpha D^\lambda (x-y_\epsilon^\lambda) \frac{\partial y_\epsilon^i}{\partial \epsilon} \frac{\partial y_\epsilon^i}{\partial \sigma} \right), \quad (II-42) \]
which, we should note, is regular in $R_1$.

Furthermore, $A^i_\mu$ here can have more than one string, quite contrary to the Dirac formalism. For example, instead of $A^i_\alpha(36)$ we may use the following expression:

\[
A^i_\alpha = \sum_{k=1}^{N} A^k_\alpha,
\]

\[
A^k_\alpha = a_k \frac{\hat{g}}{4\pi r} \frac{\hat{n}_k \times \hat{r}}{r - \hat{n}_k \cdot \hat{r}},
\]

\[
\sum_{k=1}^{N} a_k = 1,
\]

where $\hat{n}_k$ is a unit vector pointing towards the outside of $R_\alpha$.

In this case the gauge transformation (42) can be generalized in a trivial manner. It is worth noting that we can unite these strings to one by means of the generalized gauge transformation discussed above. Thus, without loss of generality, we may limit ourselves to the case in which $A^i_\mu$ has only one string.

Let us now turn to the action formalism. The action of Wu and Yang, 7) $I^{W-Y}$, is given by

\[
I^{W-Y} = I_1 + I_2 + I_3^{W-Y},
\]

\[
I_3^{W-Y} = -e \int A_\mu(\hat{3}) \alpha \hat{3}^\mu,
\]

where $I_1$ and $I_2$ are the same as (20) and (21), respectively,
provided that $F_{\mu \nu}$ is defined by (30) and (31); the definition of $I_{3}^{W-Y}$ is a little complicated. For the purpose of the present paper, however, we need only to discuss some simple cases.*

a) If the charge world line is entirely within $R_{a}$, then $\int$ is defined as the usual integral with $A_{\mu}^{a}$ being the integrand.

b) If the charge world line $z^{\mu}(s)$ is firstly in $R_{a}$ for $-\infty < s < s_{0}$, then in $R_{b}$ for $s_{0} < s \leq s_{1}$ and lastly in $R_{a}$ again for $s_{1} < s < \infty$, then $I_{3}^{W-Y}$ is defined as

\[
I_{3}^{W-Y} = -e \int_{S_{1}}^{\infty} A_{\mu}^{a}(\beta) \alpha \beta^{\mu} - e \beta(S_{1}) - e \int_{S_{0}}^{S_{1}} A_{\mu}^{b}(\beta) \alpha \beta^{\mu}
\]

\[+ e \beta(S_{0}) - e \int_{-\infty}^{S_{0}} A_{\mu}^{a}(\beta) \alpha \beta^{\mu}, \quad \text{(II-48)} \]

where $\beta(s)$ is an abbreviation of $\beta(z^{\mu}(s))$ and $\beta(x)$ is defined in $R_{ab}$ by

\[
[\beta, \mu(x)] = \alpha_{\mu}(x). \quad \text{(II-49)}
\]

Given $\alpha_{\mu}$, such a $\beta$ exists as a multiple-valued function, because of (33). In view of (35), the multiple values are different from each other by an integer multiple of $g$. Wu and Yang have shown that the action is definable only modulo eg. Before passing to the action principle, it may be appropriate to see that the action $I_{3}^{W-Y}$ cannot be properly defined in such a case that a charge and a pole pass through each other. Thus we must impose

*) For more complete discussions, see Ref.7.
Rosenbaum's veto (26) before carrying out the variation principle.

Let us now discuss the variation principle. It is easy to show that the stability of \( I^{W-Y} \) against variations \( \delta z^u(s) \) gives rise to (I-6), exactly as in the usual case without a monopole. As for variations of \( A_\mu \), we first consider \( \delta A^a_\mu \) and \( \delta A^b_\mu \) in the subregions \( R_a \) and \( R_b \) outside of \( R_{ab} \). Stability of \( I^{W-Y} \) leads then to (I-4) in these subregions. Next consider \( \delta A^a_\mu \) and \( \delta A^b_\mu \) in \( R_{ab} \), but with \( \delta A^a_\mu - \delta A^b_\mu = 0 \). Stability of \( I^{W-Y} \) against such variations leads to (I-4) in \( R_{ab} \). Notice that (I-5) is a kinematical equation in the Wu-Yang theory also.

Variation of \( Z_\mu(\tau) \) is still more cumbersome to study since it necessitates a change of the regions \( R_a \) and \( R_b \). Wu and Yang have circumvented this complication by investigating the dual action integral.

First we define a new kind of (pseudo) vector potential \( B_\mu \) as follows:

\[
\tilde{H}_{\mu \nu}(x) = B^i_{\mu, \nu}(x) - B^i_{\nu, \mu}(x) \quad \text{in } \bar{R}_i, \quad (II-50)
\]

where \( B^i_{\mu}(i = a \text{ or } b) \) is defined only in \( \bar{R}_i \); \( \bar{R}_i \) is defined in the same way as \( R_i \) were defined in (27)(28), except that the spherical coordinates are defined with respect to the charge. Magnetic counterparts of (32)-(35) are also assumed. Then we define the dual action \( \bar{I}^{W-Y} \) by

\[
\bar{I}^{W-Y} = -m \int ds - M \int d\tau - \frac{i}{4} \int d^4x \tilde{H}_{\mu \nu}(x) \tilde{H}^{\mu \nu}(x)
\]

\[
- \theta \oint B_\mu(z) d\bar{z}^\mu. \quad (II-51)
\]
For fixed world lines of the charge and the pole, consider the quantities

\[
\frac{\bar{I}_o^{W-Y}(\delta, Z)}{I_o^{W-Y}} = \text{extremum of } I_o^{W-Y} \text{ with respect to } \delta B_\mu, \quad (\text{II-52})
\]

and

\[
\frac{\bar{I}_o^{W-Y}(\delta, Z)}{I_o^{W-Y}} = \text{extremum of } I_o^{W-Y} \text{ with respect to } \delta A_\mu. \quad (\text{II-53})
\]

Wu and Yang have shown the following:

\[
\frac{\bar{I}_o^{W-Y} - I_o^{W-Y}}{I_o^{W-Y}} = \text{integrals at infinity.} \quad (\text{II-54})
\]

Since such integrals at infinity play no role in the action principle, it follows from (54) that the stability of \( I_o^{W-Y} \) against \( \delta Z_\mu \) leads to (I-7); this is because (54) implies \( \delta I_o^{W-Y} = \delta \bar{I}_o^{W-Y} \), and \( \delta I_o^{W-Y} \) against \( \delta Z_\mu \) implies (I-7) just as \( \delta I_0^{W-Y} \) against \( \delta Z_\mu \) implies (I-6). Thus the Maxwell-Lorentz equations are derived from \( I_o^{W-Y} \) by means of the variational principle.

The above discussion, though ingenious, has several defects.

1) First of all it is not clear whether the discussion is compatible with the concept of independent variations of independent variables. It is easy to see that \( A^a_\mu \) and \( A^b_\mu \) are partly, but not wholly, dependent on \( Z_\mu \). 2) Related to this is the fact that it is not certain whether the order of taking variations \( \delta A_\mu \) and \( \delta Z_\mu \) is commutable. 3) Wu-Yang's method cannot be applied to the dyon model. Variations of the world
lines of dyons are cumbersome not only for $I^W_Y$ but also for $I^{W-Y}$. From the practical point of view, this formulation is so complicated that we cannot calculate, for example, the conjugate momentum $P_\mu$ of $Z_\mu$.

In the next chapter we will construct a formalism which is free from such defects.

III. Lagrangian formalism*

In this chapter we construct a slightly modified version of the Dirac theory, such that the action is defined as a certain limit of the Wu-Yang theory.

§1. Action

It is certainly true that in avoiding Dirac's veto the Wu-Yang theory is very successful. However, it has several defects which may possibly make the theory practically useless in some cases. On the contrary, it is interesting to see that the Dirac theory has no problem where the Wu-Yang theory does so. It is expected therefore that by putting both the theories together we may be able to construct a theory which has no such difficulties without recourse to Dirac's veto. To this end let us now begin by comparing the two.

As mentioned in Chap.II, §3, even in the Wu-Yang theory the vector potential $A_\mu$ has a string-like singularity outside of $R_i$. Thus we can compare both the theories by extending either $R_a$ or $R_b$ as much as possible untill $F_{\mu\nu}$ becomes expressible in terms

*) This chapter is based on Ref.14.
only of $A^a_\mu$ or $A^b_\mu$ except on the string.*

In spite of this, we extend both $R_a$ and $R_b$ for the later convenience. Let us consider two strings $y_\mu(\tau, \sigma)$ and $y'_\mu(\tau, \sigma)$, which do not pass through each other. We then define two regions $R$ and $R'$ at each instant of time $t$ as follows:

$$R = \{ \vec{x} \mid \vec{x} \neq \vec{y}(\tau, \sigma), \text{for all } \tau \text{ and } \sigma \text{ such as } y_0(\tau, \sigma) = t \}.$$  

$$R' = \{ \vec{x} \mid \vec{x} \neq \vec{y}'(\tau, \sigma), \text{for all } \tau \text{ and } \sigma \text{ such as } y'_0(\tau, \sigma) = t \}.$$  

In the case that $\vec{y}$ is on the $-z$ axis while $\vec{y}'$ is on the $+z$ axis, (1) and (2) coincide with $R_a$(II-27) and $R_b$(II-28) with $\delta = \pi/2$, respectively (cf. Fig.III).

Fig.III

Let $A^\prime_\mu(x)$ be a vector potential defined in $R(R')$ such that

$$F_{\mu \nu}(x) = A_{\mu, \nu}(x) - A_{\nu, \mu}(x) \quad \text{in } R,$$  

$$F'_{\mu \nu}(x) = A'^{\prime}_{\mu, \nu}(x) - A'^{\prime}_{\nu, \mu}(x) \quad \text{in } R'.$$  

*) It appears that Brandt and Primack were the first to put forward this idea in the literature.\textsuperscript{15} However, the present author had to know it through a private communication with Dr. K. Shiozaki before he came across the work of Brandt and Primack. We shall discuss the work of Brandt and Primack later in this chapter.
Then it follows that there is a function $\alpha_\mu$ in $\mathbb{R} \cap \mathbb{R}'$ such that

$$ A_\mu(x) - A'_\mu(x) = \alpha_\mu(x) \quad \text{(III-5)} $$

$$ \alpha_{\mu,\nu}(x) - \alpha_{\nu,\mu}(x) = 0 \quad \text{(III-6)} $$

All this is a natural extension of the Wu-Yang theory.

In order to make $F_{\mu \nu}$ expressible in terms only of one vector potential, say $A_\mu$ for example, we must make a departure from the Wu-Yang theory. Now, only on the string $y_\mu$ has $A_\mu$ a singularity, where, however, $A'_\mu$ is a regular function, and so is $A_\mu - \alpha_\mu$. Thus $A_\mu - \alpha_\mu$ and its derivatives can be extended uniquely in $\mathbb{R}'$, especially on $y_\mu$ without any ambiguity. Thus in $\mathbb{R}'$ we have

$$ F_{\mu \nu}(x) = A_{\mu,\nu}(x) - A_{\nu,\mu}(x) - \alpha_{\mu,\nu}(x) + \alpha_{\nu,\mu}(x) \quad \text{(III-7)} $$

In the Dirac theory, on the other hand, $F_{\mu \nu}$ is expressed as (II-2) everywhere. Then comparison of (7) with (II-2) gives

$$ \alpha_{\mu,\nu}(x) - \alpha_{\nu,\mu}(x) = \tilde{G}_{\mu \nu}(x) \quad \text{in } \mathbb{R}' \quad \text{(III-8)} $$

In the same manner we have also

$$ \alpha_{\mu,\nu}(x) - \alpha_{\nu,\mu}(x) = - \tilde{G}'_{\mu \nu}(x) \quad \text{in } \mathbb{R} \quad \text{(III-9)} $$

where $G'_{\mu \nu}$ is defined by the same expression as (II-3) but with $y_\mu$ being replaced by $y'_\mu$. For the sake of compactness we define $Y^\mu(\tau, \sigma)$ as
Then we have

$$\alpha_{\mu, \nu}(x) - \alpha_{\nu, \mu}(x) = \tilde{G}_{\mu \nu}(x), \quad (III-11)$$

$$G_{\mu \nu}^r(x) = \frac{g}{\pi} \int_{-\infty}^{\infty} dx \left\{ \frac{\partial Y_\mu}{\partial \tau} \frac{\partial Y_\nu}{\partial \sigma} - \frac{\partial Y_\nu}{\partial \tau} \frac{\partial Y_\mu}{\partial \sigma} \right\}. \quad (III-12)$$

The solution of (11) can be easily obtained uniquely up to homogeneous terms as

$$\alpha_\mu(x) = \int d^4x' D^+(x-x') \frac{\partial}{\partial x'_\nu} \tilde{G}_{\mu \nu}^r(x'), \quad (III-13)$$

where $D^+(x)$ is the retarded Green's function (II-7). We do not necessarily have to restrict $D^+(x)$ in this way, of course, since there is no physical boundary condition to be imposed on $\alpha_\mu$. Thus the use of the retarded Greens' function is merely for the sake of definiteness. Later, in Chap.IV, we will use another Green's function instead of $D^+$. In the above we have implicitly assumed that $\alpha_\mu$ obeys

$$\partial \mu \alpha^\mu(x) = 0, \quad (III-14)$$

i.e., we always work in the Landau gauge. In a case that $\vec{\gamma}$ is the z-axis, (13) coincides with the example (II-38).
It is also easy to see that (13) satisfies (II-33) and (II-35), where $R_{ab}$ is replaced by $R_0 R^{-1}$. Thus $a_\mu$, (13), may be used in the Wu-Yang theory. In this way the correspondence between Dirac's and Wu-Yang's theories becomes clear as far as the expression of $F_{\mu\nu}$ in terms of the vector potential is concerned; in $R$ we have

$$F_{\mu\nu}(x) = A_{\mu,\nu}(x) - A_{\nu,\mu}(x)$$

(III-15)

for both the theories, while in $R'$, especially on $\vec{y}$, we have

$$F_{\mu\nu}(x) = A_{\mu,\nu}(x) - A_{\nu,\mu}(x) - \tilde{Q}_{\mu\nu}(x)$$

(II-2)

for the Dirac theory and

$$F_{\mu\nu}(x) = A_{\mu,\nu}(x) - A_{\nu,\mu}(x) - a_{\mu,\nu}(x) + \alpha_{\nu,\mu}(x)$$

(III-7)

for the Wu-Yang theory. Both expressions coincide when (13) is assumed.

Let us now proceed to the action. It is clear that $I^{W-Y}$ (II-46) coincides with $I^D$ (II-19) when the world line of the charge lies entirely in $R_a$, namely, when the string $y_\mu$ does not pass through the charge. In the case that the world line of the charge intersects the string the difference between both the theories becomes clear. Dirac's action $I^D$ becomes meaningless or undefinable in this case, while $I^{W-Y}$ is well defined by (II-48).

It is convenient to write $\beta$ appearing in (II-48) as

$$\beta(x) = \int^x a_\mu(x) \, dx^\mu$$

(III-16)

* See Appendix D.
where the line integral is along any path lying entirely in $R_{ab}$. Using this we can express (II-48) in terms only of $A_{\mu}^{a}$;

$$I_3^{\omega-\gamma} = - e \int_{s_1}^{s_0} A_{\mu}^{a}(\gamma) \alpha \delta \xi^\mu - e \int_{s_0}^{s_1} \left\{ A_{\mu}^{a}(\gamma) - \alpha_{\mu}(\gamma) \right\} \alpha \delta \xi^\mu \nonumber$$

$$- e \int_{s_0}^{\infty} A_{\mu}^{a}(\gamma) \alpha \delta \xi^\mu - e \int_{-\infty}^{s_0} \alpha_{\mu}(\gamma) \alpha \xi^\mu \quad \text{(III-17)}$$

where $\gamma$ is any path extending from $z_{\mu}(s_0)$ to $z_{\mu}(s_1)$ and lying entirely in $R_{ab}$. This expression can be used in our following discussions.

We can now write down our action $I$ as follows:

$$I = I_1 + I_2 + I_3 \quad \text{(III-18)}$$

Here $I_1$ and $I_2$ are the same as (II-20) and (II-21), respectively, and $F_{\mu \nu}$ is given by (II-2). Thus again (I-5) is a kinematical equation. Further, $I_3$ is defined by

$$I_3 = - e \int A_{\mu}(\gamma) \alpha \delta \xi^\mu \quad \text{(III-19)}$$

if the charged particle always remains in $R$, that is, the string $y_{\mu}$ does not pass through the charge, and by
\[ I_3 = -e \int_{s_1}^{\infty} A_\mu(\tilde{z}) \, d\tilde{z}^\mu - e \int_{-\infty}^{s_0} A_\mu(\tilde{z}) \, d\tilde{z}^\mu - e \int_{s_0}^{s_1} \left\{ A_\mu(\tilde{z}) - \alpha_\mu(\tilde{z}) \right\} \, d\tilde{z}^\mu - e \int_0^s \alpha_\mu(x) \, dx^\mu, \]

(III-20)

if the world line of the charge \( z^\mu(s) \) intersects the string at some \( s, s_0 < s < s_1 \), as shown in Fig.IV-a. In (20) \( \alpha_\mu \) is given

Fig.IV

by (13), in contrast with the Wu-Yang theory. These definitions of \( I_3 \) can easily be generalized to such a case that the charge intersects the string many times.

Roughly speaking, our action (18) is a mixture of \( I^D \) and \( I^{W-Y} \), with which, however, nothing essential is lost. This is as tractable as \( I^D \) and as definite as \( I^{W-Y} \).

Lastly we may add some comments. 1) The path \( \gamma \) cannot be specified strictly, if we require that the action (18) be continuous with respect to any variation of the variables. Thus our action can be defined only modulo eg owing to the arbitrariness of the \( \gamma \) (cf.Fig.IV-b and c). 2) We must invoke Rosenbaum's veto (II-26) in defining the action. These characteristics are shared also by the Wu-Yang theory, however.

3) We can formally rewrite (20) as
\[ I_3 = -e \int_{-\infty}^{\infty} A_\mu(\partial) \, d\partial^\mu - e \oint_\gamma \alpha_\mu(x) \, dx^\mu, \]

(III-21)

where \( \Gamma \) is a loop consisting of \( \gamma(s_0 \rightarrow s_1) \) and \(-z^\mu(s_1 \rightarrow s_0)\). If the path of the charge lies entirely in \( R \cap R' \), (21) reduces to (19) up to modulo \( eg \), because \( \alpha_{\mu, \nu} - \alpha_{\nu, \mu} = 0 \) in this case. Thus the continuity of \( I_3 \) (up to modulo \( eg \)) with respect to the variation of \( z^\mu \) is obvious. 4) We cannot use (21) for all the cases because \( \alpha_{\mu} \) is singular on \( y'_\mu \) as well as \( y_\mu \).

§2. Equations of motion

In this section we derive the Maxwell-Lorentz equations from the action (18) by means of the least action principle.

Our action (18) resembles \( I^D \) (II-19) in appearance more than \( I^{W-Y} \) (II-46) does so. Thus we first evaluate \( \delta I \) against \( \delta z_\mu, \delta y_\mu \) and \( \delta A_\mu \). When the string does not pass through the charge, \( I \) coincides with \( I^D \) and the variations give (II-23); all of the Maxwell-Lorentz equations are obtained in this case in the same manner as in the Dirac theory. Therefore we need only to consider (20) or (21).

Before calculating \( \delta I \), it may be useful to observe that \( \alpha_{\mu} \), (13), depends on \( y_\mu \) and that

\[ \alpha_{\mu} \]

*) This is obtained in the same manner as the derivation of (II-16) in Appendix C.
\[\delta Q_\mu(x) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^4} d\sigma \, \delta^4(x-y) \, \delta Y^\lambda \frac{\partial Y^\rho}{\partial \sigma} \frac{\partial Y^\zeta}{\partial \sigma} \]

\(- \int_{-\infty}^{\infty} \int_{\mathbb{R}^4} d\sigma \, \delta Y^\lambda \frac{\partial \delta Y^\rho}{\partial \sigma} \frac{\partial \delta Y^\zeta}{\partial \sigma} \).

(III-22)

The first term of this expression is singular on an infinitesimal loop composed of \(Y_\mu\) and \(Y_\mu + \delta Y_\mu\), while the second term is a regular response against \(\delta Y_\mu\) in the sense that this term is expressed by a gradient of some distribution.

Now we evaluate the variation of \(I\). In the same manner as in the case of Dirac we take \(z^\mu, y^\mu\) and \(A^\mu\) as independent variables. A straightforward calculation gives

\[
\delta I = \int \delta \mu \left\{ m \ddot{\delta}_\mu + e F_{\mu\nu}(z) \dot{\delta}_\nu \right\} \delta \dot{\delta}^\mu \\
+ \int \delta \tau \left\{ M \dddot{\delta}_\mu + \dot{F}_{\mu\nu}(z) \ddot{\delta}_\nu \right\} \delta \ddot{\delta}^\mu \\
- \int \delta \tau \left\{ \delta^\mu(x) - \bar{F}_{\mu\nu}(x) \right\} \delta A^\nu(x) \\
+ \int \delta \tau \int_{\mathbb{R}^4} d\sigma \, \epsilon_{\mu\lambda\rho\sigma} \left\{ \delta^\rho(y) - F^\rho\sigma(y) \right\} \frac{\partial Y^\lambda}{\partial \sigma} \frac{\partial Y^\zeta}{\partial \sigma} \delta Y^\mu.
\]

*) Refer to the discussion given later in this section.

**) See Appendix E.
where use is made of (22) with

\[ \delta Y^{\mu}(\tau, \sigma) = \Theta(\delta) \delta Y^{\mu}(\tau, \sigma). \]  

(III-24)

The last term vanishes because of the single valuedness of the expression within \{ \}. The first and the fourth terms of (23) differ from the Dirac theory (II-23). By equating to zero the coefficients of \( \delta z_{\mu} \), \( \delta z_{\mu} \), \( \delta A_{\mu} \) and \( \delta y_{\mu} \), we obtain the Maxwell-Lorentz equations. Clearly (I-5) is a kinematical equation.

The expression (23) is, however, ambiguous by itself. Independent variations of \( y_{\mu} \) cause I to diverge indefinitely in the same manner as discussed in the Dirac theory, when the string passes through the charge.\(^*\) We can, however, overcome this difficulty by slightly modifying the theory; there are several ways to do this.

1) Non-singular variations of \( A_{\mu} \) cause no problem and enable us to obtain (I-4). We then define, like (II-53) in the Wu-Yang theory,

\[ \mathcal{I}_{0}(\delta, y) = \text{extremum of } I \text{ with respect to } \delta A_{\mu}. \]  

(III-25)

\(^*\) This is pointed out to the author by Dr. T. Sawada.
In $I_0$ (I-4) always holds. Then we evaluate $\delta I_0$ against $\delta y_\mu$. When $y_\mu$ changes by $\delta y_\mu$, $A_\mu$ should change its singularity so that (I-4) holds true. Thus we obtain

$$\delta I_0 = \int \delta S \left\{ m \ddot{\delta} \mu + e F_{\mu \nu} (\delta) \dot{\delta} \nu \right\} \delta \delta^H$$

$$+ \int \delta \zeta \left\{ M \dddot{\delta} \mu + g \tilde{F}_{\mu \nu} (\delta) \dot{\delta} \nu \right\} \delta \delta^K .$$  (III-26)

Like the Wu-Yang theory this formulation has the problem concerning the order of taking variations: first we perform $\delta A_\mu$ and then $\delta y_\mu$, the order being not commutable. This formulation, however, can be applied to the dyon model, contrary to the Wu-Yang theory.

2) The kinematical equation (I-5) always holds true. Thus $A_\mu$ can be written as

$$A_\mu (x) = A_\mu^T (x) + A_\mu^+ (x) ,$$  (III-27)

where $A_\mu^+$ is given by (II-6). If we assume (27) from the beginning, the divergence problem does not arise since the singularity of $A_\mu$ moves according as $y_\mu$ is varied through $A_\mu^+$. We can then take $z_\mu$, $y_\mu$, $A_\mu^T$ as independent variables, to obtain
\[ S I = \int d^3 \{ \varepsilon_{\mu} \dddot{x}^\mu + e F_{\mu\nu}(\dot{x}) \ddot{x}^\nu \} \delta x^\nu + \int d^4 \{ M \dddot{x}^\mu + g \dddot{Z}^\mu \} \delta Z^\mu - \int d^4 x \{ j^\mu(x) - F_{\mu\nu},^\nu(x) \} \delta A_T^\mu \]

\[ - \frac{\alpha}{\mu_0} \varepsilon_{\mu\nu\lambda} \int d^4 \ddot{x} \{ \ddot{j}^\mu(x) - F_{\mu\nu},^\nu(x) \} \delta D^\nu(x - \dot{x}) \delta \dot{Z}^\rho \delta \dot{Z}^\lambda, \]

(III-28)

where use is made of (C-5). By equating to zero the coefficients of \( \delta z_\mu, \delta \dot{z}_\mu, \delta A_T^\mu \) and \( \delta y_\mu \), we obtain the Maxwell-Lorentz equations.

This formulation has no problem concerning the order of taking variations. Needless to say, it is applicable also to the dyon model. There remains however, a problem, which is of the same nature as we encounter in the usual electrodynamics. In the above we have used \( A^g_\mu \) explicitly, so that \( F_{\mu\nu} \) is written as

\[ F_{\mu\nu}(x) = F_{\mu\nu}^T(x) + F_{\mu\nu}^g(x), \]

(III-29)

\[ F_{\mu\nu}^T(x) = A_{\mu,^T}(x) - A_{\nu,^T}(x), \]

(III-30-a)

\[ F_{\mu\nu}^g(x) = A_{\mu,^g}(x) - A_{\nu,^g}(x) - \ddot{z}_{\mu\nu}(x). \]

(III-30-b)

Thus the contribution of the monopole to \( F_{\mu\nu} \) is considered explicitly. Then the term
diverges, and so does $I_2$ (II-21). As mentioned above, this is
the problem that is familiar in the usual electrodynamics: If
we substitute a solution of the Maxwell equation in the action
or if we take the Coulomb gauge, the action diverges even in
the case of the usual electrodynamics. It is not known at
present how to remove such a difficulty.

In any case we have been able to derive the proper Maxwell-­
Lorentz equations from the action (18). We may therefore say
that a satisfying action has been found, which is simple and
which need not employ Dirac's veto.

§3. Some comments

In the present section we shall exhibit the essential
features of our theory by summarizing them as a number of comments.

1) First of all it is interesting to see that our action (18)
is invariant under the singular gauge transformation (II-13)—
(II-16). This may be seen from (23) by substituting (II-16)
in $\delta A_\mu$ or directly from (28). This is quite contrary to the
Dirac Theory.

2) We have used the string $y'_\mu$ implicitly in (13). This variable
is only a subsidiary one, and it can easily be seen that the
action (18) is invariant under any variations of $y'_\mu$ with
$\delta y'_\mu(\tau,0) = 0$. Thus only the endpoint of $y'_\mu$ and $y'_\mu$ is a
physical variable.

3) The correspondence between the present theory and the Wu-Yang theory is clear. We can reproduce the Wu-Yang theory as follows; we define

\[ A^q_\mu(x) = A_\mu(x) \quad \text{in } R_a, \quad (III-32) \]

\[ A^b_\mu(x) = A_\mu(x) - \alpha_\mu(x) \quad \text{in } R_b, \quad (III-33) \]

then these \( A^a_\mu \) and \( A^b_\mu \) satisfy all the conditions demanded by Wu and Yang. In their theory no vector potential is defined at the location of the monopole. If \( A^b_\mu \) is extended there, we can obtain (I-5) in a straightforward way.

4) The present theory owes its essential part to the discussions given by Brandt and Primack\(^{15}\) which were very suggestive, if not complete. Instead of (20) they used the action \( I_{3-P}^{B-P} \) defined by

\[
I_{3-P}^{B-P} = \lim_{s_0, s_1 \to s} \left\{ -e \int_{s_1}^{\infty} A_\mu(\partial) d\partial^\mu - e \int_{s_0}^{-\infty} A_\mu(\partial) d\partial^\mu - e \int_{s}^{s_1} A'_\mu(\partial) d\partial^\mu \right\}.
\]

Here the last term vanishes since \( A'_\mu \) is non-singular on \( \Gamma \). Accordingly they arrive at
\[ I^{B-P}_3 = -e \int_{\Gamma'} A_\mu (\delta) \, d\tilde{\gamma}^\mu, \]  

(III-35)

where \( \Gamma' \) is the world line of the charge, except that the path going through the string \( y_\mu \) is replaced by an infinitesimal contour going around \( \overrightarrow{y} \). However, this formulation works well only when the variation is taken with respect to \( z_\mu \). In this formulation the limit is to be taken in the end of the calculation. Then for the variation of \( \delta A_\mu (x) \), the support of which is localized in a small region around \( z_\mu (s) \), the response of \( I^{B-P} \) to \( \delta A_\mu \) depends on whether or not the support is larger than the contour; if not, the variation principle gives \( F^{\mu \nu}_{\tilde{\nu} \nu} (y) = 0 \). Furthermore it is not clear how to take variations of \( y_\mu \), especially of \( z_\mu \). In view of all this, we may say that the Brandt-Primack theory is incomplete.

5) Finally we have a brief discussion about the Hamiltonian formalism. Tu, Wu and Yang\(^{16}\) discussed the Hamiltonian in the framework of the Wu-Yang theory. Their work, however, does not start from the Lagrangian. Furthermore their work contains a number of points\(^*\) which are ununderstandable to the

* In (4.11) of their paper, the contribution of \( A_{\xi I} \) is missing. This actually turns out to be \( \tilde{z}_i \cdot \tilde{\gamma}_i A_{\xi I} \), and therefore cannot be dropped. For this reason their formalism does not reproduce the Maxwell equations, except in the case that the monopole is fixed somewhere.
The Hamiltonian formalism of the present theory is now in progress, and we shall give here only a brief discussion thereof.

It is convenient to write \( y_\mu \) as

\[
\mathcal{Y}_\mu(\tau, \sigma) = Z_\mu(\tau) + \omega_\mu(\sigma),
\]

(III-36)

and we take \( z_\mu, Z_\mu \) and \( \Lambda_\mu \) as independent variables. We can easily show that the action \( I \), (18), is invariant under any variations of \( \omega_\mu \), when noting that

\[
\omega_\mu(\sigma) = 0,
\]

(III-37)

and thus

\[
\delta \omega_\mu(\sigma) = 0 \quad \text{at} \quad \sigma = 0.
\]

(III-38)

For a special Lorentz frame, such that

\[
\tilde{g}^0(s) = \tilde{Z}^0(\tau) = \tau^0 = t,
\]

(III-39)

the action (18) is written as

\[
I = -m\int \left[ 1 - \frac{\tilde{g}^2}{2} \right] d\tau - M\int \left[ 1 - \frac{\tilde{Z}^2}{2} \right] d\tau
- \frac{1}{4}\int \text{d}^3x \text{d}t \, \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu}
- e\int \tilde{A}(\tau) d\tau + e\int \tilde{A}(\tau) \tilde{\mathcal{A}} d\tau
\]

(III-40)

provided the string does not pass through the charge. In this case, the conjugate momenta are obtained as follows:
\[ p^i(t) = \frac{\partial I}{\partial \dot{z}^i(t)} = m \frac{\dot{z}^i}{\sqrt{1 - \dot{z}^2}} + e A^i(\hat{z}), \quad (\text{III-41}) \]

\[ \dot{p}^i(t) = \frac{\partial I}{\partial \dot{z}^i(t)} = M \frac{\ddot{z}^i}{\sqrt{1 - \dot{z}^2}} + g B^i(\hat{z}), \quad (\text{III-42}) \]

\[ \eta^i(\hat{x},t) = \frac{\partial I}{\partial A^i(\hat{x},t)} = R^i(\hat{x},t) = E^i(\hat{x},t), \quad (\text{III-43}) \]

where

\[ B^i(\hat{x}) = \int_{-\infty}^{\infty} \tilde{F}^i(\lambda) \frac{d\omega}{\omega}. \quad (\text{III-44}) \]

The Hamiltonian \( H \) is then given in the usual way as

\[ H = p^i \dot{z}^i + \dot{p}^i \dot{z}^i - \int d^3x \, E^i \dot{A}^i + m \sqrt{1 - \dot{z}^2} + M \sqrt{1 - \dot{z}^2} \]

\[ + e A^0(\hat{z}) - e A^i(\hat{z}) \dot{\hat{z}}^i. \quad (\text{III-45}) \]

*) Here use is made of the equation \( F^{\mu \nu},_\nu = j^\mu \). This is related to the last term of (28). We may circumvent this problem by defining \( F^i \delta I_0 / \delta \hat{z}^i \), taking account of the discussion concerning (25) and (26). Clearly the problem remains to be investigated further.
From (45) we can eliminate $z^i$, $\dot{z}^i$ and $\dot{A}^i$ and obtain

$$H = \sqrt{(P - eA(\tilde{\alpha})^2 + m^2 + \sqrt{(P - gB(\tilde{\alpha}))^2 + M^2}}$$

$$+ \int d\vec{x} \frac{\vec{E}^2 + \vec{H}^2}{2} + \int d\vec{x} \left( \dot{\vec{E}} - \nabla \cdot \vec{E} \right) A(\vec{x}, t). \quad (III-46)$$

We can show that the Hamilton equations are obtained in the usual way from (46) as follows.

$$\frac{\partial H}{\partial P^i} = \frac{(P - eA)^i}{(P - eA)^2 + m^2} = \dot{\tilde{\alpha}}^i, \quad (III-47)$$

$$\frac{\partial H}{\partial \tilde{\alpha}^i} = \frac{(P - eA)^k}{(P - eA)^2 + m^2} \left(-e \frac{\partial A^k}{\partial \tilde{\alpha}^i} + e \frac{\partial A^o}{\partial \tilde{\alpha}^i} \right) = -\dot{P}^i, \quad (III-48)$$

$$\frac{\partial H}{\partial \tilde{Z}^i} = \frac{(P - gB)^i}{(P - gB)^2 + M^2} = \dot{\tilde{Z}}^i, \quad (III-49)$$

$$\frac{\partial H}{\partial \tilde{Z}^i} = \frac{(P - gB)^k}{(P - gB)^2 + M^2} \left(-g \frac{\partial B^k}{\partial \tilde{Z}^i} \right) + \int d\vec{x} \ H^i \frac{\partial H^i}{\partial \tilde{Z}^i} = -\dot{\tilde{Z}}^i, \quad (III-50)$$

$$\frac{\delta H}{\delta E^i} = E^i + \partial_i A^o - \tilde{G}^o i = -\dot{A}^i, \quad (III-51)$$

$$\frac{\delta H}{\delta A^i} = \dot{E}^i. \quad (III-52)$$
These equations are all consistent with the Maxwell-Lorentz equations. In the case that the string passes through the charge, there hold the same equations as above, but with \( \vec{A} \) and \( \vec{B} \) being replaced by \( \vec{A} - \vec{a} \) and \( \vec{B} - \vec{a} \), respectively, where \( \vec{a}_\mu \) is obtained from \( \alpha_\mu \), (13), by means of the duality transformation.

The above discussions are unsatisfactory in several points, and further investigation is needed along this line.

6) In summarizing we have modified the action given by Dirac on the basis of Wu-Yang's idea and found that our action gives all of the Maxwell-Lorentz equations. This solves the problem of Dirac's veto. Our theory is equally applicable to the dyon model, and its Hamiltonian formulation is very promising.

IV. Action-at-a-distance formalism

The theory obtained in the previous chapter can also be formulated in an unconventional way, that is, employing the idea of action at a distance of electromagnetism. This formulation seems to be very suitable for the case of magnetic monopoles.

§1. Action and equations of motion

In the theory of magnetic monopoles the vector potential is always a source of troubles as has been shown in the previous chapters. An attempt to use only physical variables in the usual electrodynamics was first made by Mandelstam,\(^ {17}\) and the formulation was then applied to the monopole theory by Cabbibo and Ferrari.\(^ {18}\) This formulation is, however, applicable only to quantum theory.
Another attempt which is suitable both for classical and quantum treatment may be the action-at-a-distance (AAD) theory of electromagnetism. It has been shown that the theory is equivalent to QED when some additional assumptions are made. In this chapter we formulate the problem along the line of the so called non-instantaneous formalism.

First we must discuss the electromagnetic field generated by a charge and a pole. As mentioned in Chap.II the retarded fields are given by (II-5) and (II-6). For our purpose it is convenient to use, instead of \( D^+(x) \), the half-advanced and half-retarded Green's function

\[
D^0(x) = \frac{1}{4\pi} S(x^2). \tag{IV-1}
\]

Before rewriting (II-5) and (II-6), however, we should better define some related quantities. It is appropriate here to define a pseudovector potential \( B_\mu \) by

\[
\widetilde{H}_{\mu\nu}(x) = B_{\mu,\nu}(x) - B_{\nu,\mu}(x) + H_{\mu\nu}(x), \tag{IV-2}
\]

where \( H_{\mu\nu} \), a dual counterpart of \( G_{\mu\nu} \), vanishes everywhere except on a string \( v_\mu(s,\sigma) \) which extends outward from the charge;

\[
v_\mu(s,\sigma) = \delta_\mu(S). \tag{IV-3}
\]

Note that (2) implies (I-4) kinematically. Hereafter we put \( v_\mu \) in the form
\[ y_\mu(\tau, \sigma) = \bar{z}_\mu(\tau) + \omega_\mu(\sigma), \]  

(IV-4)

with

\[ \omega_\mu(\sigma) = 0, \]  

(IV-5)

which were also used in Chap. III. Accordingly we can write

\[ v_\mu \]

in the form

\[ v_\mu(S, \sigma) = \dot{z}_\mu(S) - \omega_\mu(\sigma), \]  

(IV-6)

the procedure being very useful and even essential; imagine

that at some \( z_\mu \) intersects \( y_\mu \) at a point \((\tau, \sigma)\), then at \( z_\mu \)

intersects \( v_\mu \) at \((s, \sigma)\).

All the quantities concerned are now expressed in a

symmetrical manner:

\[ G_{\mu\nu}(x) = g \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta^4(x - \bar{z} - \omega) \left\{ \dot{z}_\mu \dot{w}_\nu - \dot{z}_\nu \dot{w}_\mu \right\}, \]  

(IV-7)

\[ H_{\mu\nu}(x) = -e \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta^4(x - \ddot{z} + \omega) \left\{ \ddot{z}_\mu \dot{w}_\nu - \ddot{z}_\nu \dot{w}_\mu \right\}, \]  

(IV-8)

and

the vector potential generated by a charge

\[ A^\sigma_{\mu}(x) = e \int dS \ D^0(x - \ddot{z}) \ddot{z}_\mu, \]  

(IV-9)
the pseudovector potential generated by a pole

\[ B_\mu^q (x) = \gamma \int d\sigma \ D^\sigma (x-Z) \hat{\sigma}_\mu, \]  

(IV-10)

the vector potential generated by a pole

\[ A_\mu^q (x) = -\gamma \epsilon_{\mu\rho\sigma} \int d\sigma \int_0^\infty \frac{d\omega}{\omega} D^\sigma (x-Z-\omega) \hat{\omega}^\rho \hat{\omega}^\sigma, \]  

(IV-11)

the pseudovector potential generated by a charge

\[ B_\mu^e (x) = \epsilon \epsilon_{\mu\lambda\rho} \int d\sigma \int d\omega \frac{\partial}{\partial x_\lambda} \ D^\sigma (x-Z+\omega) \hat{\omega}^\rho \hat{\omega}^\sigma. \]  

(IV-12)

Here and hereafter a dot means differentiation with respect to \( s, \tau \) and \( \sigma \) for \( z, z \) and \( \omega \), respectively. Then the electromagnetic fields generated by a charge and a pole are given as follows:

\[ F_{\mu \nu}^e (x) = A_{\mu \nu}^e (x) - A_{\nu \mu}^e (x), \]  

(IV-13)

\[ F_{\mu \nu}^q (x) = A_{\mu \nu}^q (x) - A_{\nu \mu}^q (x) - \tilde{\omega}_{\mu \nu} (x), \]  

(IV-14)

\[ \tilde{\tilde{F}}_{\mu \nu}^q (x) = B_{\mu \nu}^q (x) - B_{\nu \mu}^q (x), \]  

(IV-15)

\[ \tilde{\tilde{F}}_{\mu \nu}^e (x) = B_{\mu \nu}^e (x) - B_{\nu \mu}^e (x) + \tilde{\tilde{H}}_{\mu \nu} (x). \]  

(IV-16)
It is easy to show using Stokes' Theorem that (15) and (16) are really dual tensors of (13) and of (14), respectively. The quantities (14) and (16) do not then depend on the string, for they are dual tensors of (16) and (13).

Any of the quantities thus obtained, $A^\mu (x)$ for example, is generated partly by a past event and partly by a future event, as shown in Fig. V-a. Note that $A^\mu_g$ and $B^\mu_e$ receive contributions also from the strings, from the solid line of Fig. V-b.

Suitably modifying the conventional AAD formulation for a two-charge system, * we can now construct an action of AAD theory for a charge-monopole system. For the sake of brevity let us confine ourselves to a system with one charge and one pole. More general cases including a system of dyons can be treated in the same manner.

In order to find an action for the present case we start with the action (III-18). In this case $F_{\mu \nu}$ may be regarded as

$$F_{\mu \nu} (x) = F^e_{\mu \nu} (x) + F^g_{\mu \nu} (x) .$$

Neglecting self-energy terms, we obtain then

$$F_{\mu \nu} (x) F^{\mu \nu} (x) = F^e_{\mu \nu} (x) F^{\mu \nu}_{g} (x) + F^g_{\mu \nu} (x) F^{\mu \nu}_e (x) .$$

*) This is summarized in Appendix F.
It is a little surprising to find that the contribution of $I_2$ (II-21) vanishes if the integrals at infinity are neglected. This is a situation quite contrary to that of a two-charge system. It is also justified to neglect the self-energy term $A^e_\mu(z)$ of $A^e_\mu(z)$ in $I_3$ (III-19 or 20). The self-energy effects appear also in the equations of motion.

Thus our action $I^{AD}$ is expressed formally as

$$I^{AD} = I_1 + I_3^{AD} + (I_4) \ , \quad (IV-19)$$

where $I_1$ is the same as (II-20) and $I_3^{AD}$ is given by

$$I_3^{AD} = -\epsilon \int A^e_\mu(\delta) d\delta^\mu \ , \quad (IV-20)$$

This term becomes singular when the charge intersects the string, and its singularity is canceled by $I_4$

$$I_4 = -\epsilon \oint \alpha_\mu(x) dx^\mu \ , \quad (IV-21)$$

which itself is singular. The last term $I_4$ is to be taken into account when and only when the string passes through the charge. Following the conventions adopted in this chapter, we define $\alpha_\mu$ appearing in (21) by

$$\alpha_\mu(x) = -\frac{\rho \sigma}{2} \int d^2 q \int_0^\infty \frac{\partial}{\partial q^\lambda} D(q(x - \bar{x}) \dot{x}^\rho \ddot{x}^\sigma \ , \quad (IV-22)$$
\[
W_\mu(\xi) = \begin{cases} 
\omega_\mu(\xi) & \text{for } \xi > 0, \\
\omega_\mu(-\xi) & \text{for } \xi < 0,
\end{cases}
\]

(IV-23)

where \(\omega_\mu\) is another string variable which does not intersect \(\omega_\mu\).

Now applying the least action principle, we can easily obtain the equations of motion*

\[
m\ddot{\vartheta}_\mu + e \, \tilde{F}_{\mu\nu}(\vartheta) \, \dot{\vartheta}^\nu = 0,
\]

(IV-24)

\[
M\ddot{Z}_\mu + \tilde{F}_{\mu\nu}(Z) \, \dot{Z}^\nu = 0.
\]

(IV-25)

The contributions from the future events in \(\tilde{F}_{\mu\nu}^g(Z)\) and \(\tilde{F}_{\mu\nu}^e(Z)\) represent the AAD version of the reaction of fields.

§2. Momentum and energy

From the translational invariance in space-time of the action (19), we can obtain a conserved quantity, that is, four-momentum \(P_\mu\).

In deriving the form of \(P_\mu\), we will temporarily neglect the contribution of \(I_4\), and construct \(P_\mu\), which is not correct when a charge passes through the string. The contribution of \(I_4\) may then be added to this in such a way that \(P_\mu\) thus obtained becomes a conserved quantity for all cases.

Now we must introduce the finite action that is expressed by **

* See Appendix G.

** Here and hereafter the superscript AD of \(I^{AD}\) is omitted for the sake of brevity.
Here, for a given set of two surfaces $\Sigma^*$ and $\Sigma^{**}$ we take $z_\mu(s)$ ($z_\mu(\tau)$) to intersect $\Sigma^*$ and $\Sigma^{**}$ at $s = s^*$ ($\tau = \tau^*$) and $s = s^{**}$ ($\tau = \tau^{**}$), respectively (cf. Fig.VI). Let us now observe

**Fig.VI**

that $I^{**}$ remains invariant under a translation such that

\[
\delta \partial_\mu = \delta Z_\mu = \epsilon_\mu ,
\]

\[(IV-27)\]

\[
\delta \omega_\mu = 0 ,
\]

\[(IV-28)\]

where $\epsilon_\mu$ is an arbitrary infinitesimal constant vector. Thus we obtain in the obvious notation

\[
\delta I^{**} = m \int dS \delta Z_\mu \epsilon^\mu + M \int d\tau \delta \dot{Z}_\mu \epsilon^\mu
\]

\[-m \left[ \delta \partial_\mu \right]^{**} \epsilon^\mu - M \left[ \delta \dot{Z}_\mu \right]^{**} \epsilon^\mu
\]

\[-\epsilon_\mu \left( \delta z_\mu \right)^{**} \int \delta \omega_\rho \delta Z^\lambda \omega_\nu \frac{\partial ^\rho}{\partial \omega_\sigma} \left( \frac{\partial \omega^\sigma}{\partial z^\mu} + \frac{\partial \dot{z}^\mu}{\partial z^\nu} \right) \epsilon^\lambda
\]

\[= 0 .
\]

\[(IV-29)\]
Since $\varepsilon_\mu$ is arbitrary we deduce

$$
O = -m \left[ \dot{\delta}_\mu \right]^{**} - eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{z}^{\lambda} \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**} 
- M \left[ \dot{Z}_\mu \right]^{**} - eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**} 
+ eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{z}^{\lambda} \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**}
+ eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{z}^{\lambda} \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**} 
+ eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{z}^{\lambda} \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**},
$$

(IV-30)

where use is made of the equations of motion (24) and (25). The last two terms are skew-symmetric with respect to $z_\mu$ and $Z_\mu$. We can therefore rewrite (30) as

$$
O = -m \left[ \dot{\delta}_\mu \right]^{**} + eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{z}^{\lambda} \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**}
- M \left[ \dot{Z}_\mu \right]^{**} + eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**} 
- eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{z}^{\lambda} \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**}.
$$

(IV-31)

which leads to the expression

$$
P_\mu = m \dot{\delta}_\mu + e A_\mu (z) + M \dot{Z}_\mu + g B_\mu (Z)
- eg \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \epsilon_{\mu \lambda \rho \sigma} \left[ \dot{z}^{\lambda} \dot{\omega}^{\rho} \frac{\partial D^{0}}{\partial \omega^{\lambda}} \right] \delta^{**}.
$$

(IV-32)
As mentioned above, $P_\mu$ thus obtained is not conserved when the 
charge intersects the string. It is, however, easy to make $P_\mu$
conserved by adding a term which is due to $I_4$. Then the final 
expression for $P_\mu$ is given as follows:

$$P_\mu = m \delta_\mu + e A_\mu(\beta) + M \dot{Z}_\mu + \Theta B_\mu(\zeta)$$

$$- e^{\frac{\gamma}{\theta}} \left[ \int_{s_0}^{s} ds \int_{\tau}^{\tau'} d\tau' - \int_{s_0}^{s} ds' \int_{\tau}^{\tau'} d\tau' \right] \dot{A}_\mu \int_{0}^{\infty} d\phi \int_{-\infty}^{\infty} d\phi' \int_{0}^{\infty} d\phi \int_{-\infty}^{\infty} d\phi' \frac{\partial D^\phi}{\partial \phi} \right]$$

$$- e^{\frac{\gamma}{\theta}} \left[ \int_{s_0}^{s} ds \int_{\tau}^{\tau'} d\tau' - \int_{s_0}^{s} ds' \int_{\tau}^{\tau'} d\tau' \right] \int_{0}^{\infty} d\phi \int_{-\infty}^{\infty} d\phi' \int_{0}^{\infty} d\phi \int_{-\infty}^{\infty} d\phi' \frac{\partial D^\phi}{\partial \phi} \right]$$

where $s_0(\tau_0)$ and $s_1(\tau_1)$ represent some $s(\tau)$ before and after the 
interception, respectively. Clearly $P_\mu$ does not depend on the
choice of these points. Notice that (33) reduces to (32) when
there occurs no interception.

At this point it is useful to notice the following properties;
an arbitrary function $f(s, \tau)$ of $s$ and $\tau$ satisfies relations such as

$$\frac{\partial}{\partial s} \left[ \int_{s_0}^{s} ds' \int_{\tau}^{\tau'} d\tau' \right] f(s', \tau') = - \left[ \int_{-\infty}^{\infty} d\tau' \right] f(s, \tau')$$

$$\frac{\partial}{\partial \tau} \left[ \int_{s_0}^{s} ds' \int_{\tau}^{\tau'} d\tau' \right] f(s', \tau') = \int_{-\infty}^{\infty} ds' \frac{\partial}{\partial s} f(s', \tau)$$

and further

$$\left[ \int_{s_0}^{s} ds' \int_{\tau}^{\tau'} d\tau' \right] \frac{\partial}{\partial \tau'} f(s', \tau') = - \left[ \int_{-\infty}^{\infty} d\tau' \right] f(s, \tau')$$
When \( f(s, \tau) \) vanishes at infinity. Using these properties, we can explicitly show that \( P_{\mu} \) is conserved as follows;

\[
\frac{\partial P_{\mu}}{\partial \tau} = m \dddot{\mathbf{z}}_{\mu} + e \frac{\partial A_{\mu}^{q}(\mathbf{z})}{\partial \mathbf{z}^{\lambda}} \dddot{z}^{\lambda} + e g \int_{-\infty}^{\infty} d\sigma e_{\lambda \rho \sigma} \dddot{z}^{\lambda} \dddot{z}^{\rho} \frac{\partial D_{\sigma}}{\partial \omega_{\delta} \omega_{\mu}}
\]

\[
+ e g \int_{-\infty}^{\infty} d\sigma e_{\lambda \rho \sigma} \dddot{z}^{\lambda} \dddot{z}^{\rho} \frac{\partial D_{\sigma}}{\partial \omega_{\delta} \omega_{\mu}}
\]

\[
= m \dddot{z}_{\mu} + e \tilde{A}_{\mu \nu}(\mathbf{z}) \dddot{z}^{\nu} = 0 \quad (IV-38)
\]

Similarly it can be shown that

\[
\frac{\partial P_{\mu}}{\partial \omega_{\mu}} = m \dddot{z}_{\mu} + g \tilde{A}_{\mu \nu}(\mathbf{z}) \dddot{z}^{\nu} = 0 \quad (IV-39)
\]

Since \( P_{\mu} \) is to be an observable, it should not depend on a choice of \( \omega_{\mu} \). That this in fact is the case can be shown by a straightforward but a little complicated calculation, which will therefore be relegated to Appendix H.

The above results thus enable us to adopt (33) as the final expression for the total four-momentum of the system under consideration.
§3. Angular momentum

We now consider the variation of $I_3$, (19), induced by an infinitesimal Lorentz transformation such as

$$\varepsilon^{\mu}_\nu = \varepsilon^{\mu}_\nu \tilde{z}^\nu, \quad \delta^\mu_z = \varepsilon^{\mu}_\nu \tilde{z}^\nu, \quad \delta^\mu_\omega = \varepsilon^{\mu}_\nu \omega^\nu,$$  \hspace{1cm} (IV-40)

where $\varepsilon_{\mu\nu}$ is a set of six skew-symmetric constants:

$$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}.$$  \hspace{1cm} (IV-41)

First let us establish a simple identity which will prove useful later. Introducing the quantity

$$\Lambda = e g e^{\mu}_{\rho\sigma} \delta^\lambda_z \dot{z}^\rho \dot{\omega}^\sigma \frac{\partial}{\partial \omega^\sigma} D^\sigma(\dot{z} - \dot{z} - \dot{\omega}),$$  \hspace{1cm} (IV-42)

we express $I_3^{AD}$, (20), as

$$I_3^{AD} = -\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} d\sigma \Lambda.$$  \hspace{1cm} (IV-43)

The definition (42) shows that $\Lambda$ is a scalar under the infinitesimal Lorentz transformation (40)

$$\delta \Lambda = \frac{\partial \Lambda}{\partial \dot{z}^\mu} \delta \dot{z}^\mu + \frac{\partial \Lambda}{\partial \dot{\omega}^\mu} \delta \dot{\omega}^\mu + \frac{\partial \Lambda}{\partial \tilde{z}^\mu} \delta \tilde{z}^\mu + \frac{\partial \Lambda}{\partial \tilde{\omega}^\mu} \delta \tilde{\omega}^\mu + \frac{\partial \Lambda}{\partial \omega^\mu} \delta \omega^\mu + \frac{\partial \Lambda}{\partial \omega^\mu} \delta \omega^\mu.$$  \hspace{1cm} (IV-44)

Since $\varepsilon_{\mu\nu}$ is arbitrary, (44) is rewritten as

$$\delta \varepsilon_{\mu\nu} \varepsilon^{\mu}_{\rho\sigma} \delta \dot{z}^\rho \dot{\omega}^\sigma D^\sigma - \varepsilon^{\mu}_{\rho\sigma} \varepsilon^{\nu}_{\lambda\rho} \delta \omega^\sigma \dot{\omega}^\rho \dot{\omega}^\mu \dot{\omega}^\nu \dot{\omega}^\nu.$$
+ \mathbf{Z}_{\mu \nu} \in \mathbb{R} \epsilon \partial \epsilon \partial D^6 + \mathbb{E} \mathbb{E}_{\mu \nu} \partial \epsilon \partial D^6

+ (\mathbf{Z}_{\mu \nu} \in \mathbb{R} \epsilon \partial \epsilon \partial D^6 + \mathbb{E} \mathbb{E}_{\mu \nu} \partial \epsilon \partial D^6)

= 0 \quad \text{(IV-45)}

where, for example, \( A^{[\mu \nu]} = A^{\mu \nu} - A^{\nu \mu} \), and \( \partial \omega \) means a differentiation with respect to \( \omega \). The once-integrated form of (45) is also useful, which is given by

\[
\int_0^\infty \left[ \delta \epsilon \partial \epsilon \partial D^6 + \mathbb{E} \mathbb{E}_{\mu \nu} \partial \epsilon \partial D^6

+ (\mathbf{Z}_{\mu \nu} \in \mathbb{R} \epsilon \partial \epsilon \partial D^6 + \mathbb{E} \mathbb{E}_{\mu \nu} \partial \epsilon \partial D^6)

+ \omega_{\mu \nu} \epsilon \mathbb{R} \epsilon \mathbb{R} \left( A^{\mu \nu} \partial \epsilon \partial D^6 - \delta \partial \epsilon \partial D^6 - \partial \epsilon \partial D^6 \right) \right] \omega^6

= 0 \quad \text{(IV-46)}

Here we have used

\[
\int_0^\infty \left\{ \delta \epsilon \partial \epsilon \partial \omega^6 + \frac{\partial \epsilon \partial \omega^6}{\partial \epsilon \partial \omega^6} \right\}

= e_\mu \epsilon_{\mu \nu \rho \sigma} \int_0^\infty \left[ \delta \epsilon \partial \epsilon \partial \omega^6 + \frac{\partial \epsilon \partial \omega^6}{\partial \epsilon \partial \omega^6} \right] \delta \epsilon \partial \epsilon \partial \omega^6

\text{(IV-47)}
which is derivable in a similar way to the calculation, given in Appendix H, to prove $P_\mu$ being independent of $\omega_\mu$.

The derivation of angular momentum $L_{\mu\nu}$ can be performed in the same manner as in the case of $P_\mu$. Using the equations of motion (24) and (25) we obtain $\delta I_{**}$ for the Lorentz transformation (40) as follows;

$$
\delta I_{**} = - \frac{\partial}{\partial t} \left[ \frac{\partial \omega_\mu}{\partial \omega_\mu} \right]_{**} - \int_{-\infty}^{\infty} \int_{0}^{\infty} \left[ \frac{\partial \omega_\mu}{\partial \omega_\mu} \right]_{**} \delta \omega_\mu \delta \omega_\mu
\delta I_{**} = - \frac{\partial}{\partial t} \left[ \frac{\partial \omega_\mu}{\partial \omega_\mu} \right]_{**} - \int_{-\infty}^{\infty} \int_{0}^{\infty} \left[ \frac{\partial \omega_\mu}{\partial \omega_\mu} \right]_{**} \delta \omega_\mu \delta \omega_\mu
$$

The use of the identity (44) enables us to rewrite last three terms of (48) as

$$
\left[ \int_{-\infty}^{\infty} + \int_{0}^{\infty} \right] \left[ \int_{-\infty}^{\infty} + \int_{0}^{\infty} \right] \int_{0}^{\infty} \left[ \frac{\partial \omega_\mu}{\partial \omega_\mu} \right] \delta \omega_\mu \delta \omega_\mu \delta \omega_\mu \delta \omega_\mu
$$

Using (47) we finally arrive at
\[ \delta t^{**} = -m [ \dot{\mathbf{S}}_\mu \delta \mathbf{A}^\mu ]^{**} - \int_{-\infty}^{\infty} d\sigma \left[ \frac{\partial \Lambda}{\partial \mathbf{A}^\mu} \delta \mathbf{A}^\mu \right] \delta^{**} \]

\[ -M [ \dot{Z}_\mu \delta Z^\mu ]^{**} - \int_{-\infty}^{\infty} d\sigma \int_0^\infty d\xi \left[ \frac{\partial \Lambda}{\partial \dot{Z}^\mu} \delta \dot{Z}^\mu \right] \xi^{**} \]

\[ -\left[ \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\xi \right] \left[ \frac{\partial \Lambda}{\partial \mathbf{A}^\mu} \delta \mathbf{A}^\mu + \frac{\partial \Lambda}{\partial \dot{Z}^\mu} \delta \dot{Z}^\mu \right] \]

\[ + e_g \varepsilon_{\mu\nu\rho} \int_{-\infty}^{\infty} d\sigma \left[ \delta \mathbf{A}^\mu \delta \mathbf{A}^\nu \delta \mathbf{A}^\rho \right] \xi^{**} \delta \omega^\mu \omega^\nu \omega^\rho , \quad (IV-50) \]

where we have assumed for the time being that the charge does not intercept the string. The last relation thus leads us to \( L_{\mu \nu} \) which is conserved only when the interception does not occur:

\[ L_{\mu \nu} = \mathcal{Z}_{\mu \nu} \{ m \dot{\mathbf{S}}_{\nu} + e A^\mu (\partial \mathbf{S})_{\nu} \} + \mathcal{Z}_{\mu \nu} \{ M \dot{Z}_{\nu} + e B^\mu (\partial \mathbf{Z})_{\nu} \} \]

\[ + e_g \varepsilon_{\mu \nu \rho \lambda} \int_{-\infty}^{\infty} d\sigma \int_0^\infty d\xi \left\{ \delta \mathbf{A}^\mu \varepsilon_{\nu \lambda \rho \sigma} \dot{Z}^\lambda \dot{Z}^\rho \dot{Z}^\sigma - e \varepsilon_{\nu \lambda \rho \sigma} \mathbf{A}^\nu \varepsilon_{\mu \lambda \rho \sigma} \mathbf{A}^\rho \varepsilon_{\mu \lambda \rho \sigma} \mathbf{A}^\sigma \right\} \]

\[ + e_g \varepsilon_{\nu \lambda \rho \sigma} \int_{-\infty}^{\infty} d\sigma \left[ \delta \mathbf{S}^\lambda \varepsilon_{\nu \lambda \rho \sigma} \mathbf{S}^\rho \mathbf{S}^\sigma \right] \delta \omega^\nu \delta \omega^\rho \delta \omega^\sigma , \quad (IV-51) \]

The experience gained in the derivation of \( P_\mu \), (33), suggests the way of how to find the term which is missing in (51) and which should be added to (51) in order to make \( L_{\mu \nu} \) a conserved quantity
even for the presence of the interception. In fact, the final expression for $L_{\mu\nu}$ is given in two ways that in appearance are very different from each other:

$$L_{\mu\nu} = \delta_{\lambda\mu} \left\{ m \delta_{\nu\mu} + e A^\theta(\tilde{\omega}) \right\} + Z_{\lambda\mu} \left\{ M \delta_{\nu\mu} + e B^\theta(\tilde{Z}) \right\}$$

$$+ e g \left[ \int_{s_{0}}^{\infty} dt \right] \left[ \int_{-\infty}^{s_{0}} dt \right] \int d\delta \left\{ 3 \delta_{\mu\nu} \omega \cdot D^\theta - e \gamma_{\lambda\rho} \delta_{\nu\mu} \omega \cdot D^\theta \right\}$$

$$- e g \left[ \int_{-\infty}^{s} dt \right] \left[ \int_{s_{0}}^{\infty} dt \right] \int d\delta \left\{ 3 \delta_{\mu\nu} \omega \cdot D^\theta \right\}$$

$$+ e g \epsilon_{\lambda\mu} \left[ \int_{-\infty}^{\infty} d\omega \right] \omega \cdot D^\theta$$

(IV-52)

and

$$L_{\mu\nu} = \delta_{\lambda\mu} \left\{ m \delta_{\nu\mu} + e A^\theta(\tilde{\omega}) \right\} + Z_{\lambda\mu} \left\{ M \delta_{\nu\mu} + e B^\theta(\tilde{Z}) \right\}$$

$$- e g \left[ \int_{s_{0}}^{\infty} dt \right] \left[ \int_{-\infty}^{s_{0}} dt \right] \int d\delta \left\{ \tilde{Z} \delta_{\nu\mu} \omega \cdot D^\theta + e \gamma_{\lambda\rho} \delta_{\nu\mu} \omega \cdot D^\theta \right\}$$

$$- e g \left[ \int_{-\infty}^{s_{0}} dt \right] \left[ \int_{s_{0}}^{\infty} dt \right] \int d\delta \left\{ \tilde{Z} \delta_{\nu\mu} \omega \cdot D^\theta \right\}$$

$$- e g \epsilon_{\lambda\mu} \left[ \int_{-\infty}^{\infty} d\omega \right] \omega \cdot D^\theta$$

(IV-53)
That (52) is equivalent to (53) is shown in Appendix I.

It is possible to show by use of (52) and (53) that

$$\frac{\partial L_{\mu\nu}}{\partial S} = 0, \text{ and } \frac{\partial L_{\mu\nu}}{\partial \tau} = 0 \quad (IV-54)$$

Further we can show that $L_{\mu\nu}$ is independent of a choice of $s_0$, $s_1$, $\tau_0$ and $\tau_1$, and that it does not depend on $\omega_\mu$. In view of all this we may say that $L_{\mu\nu}$ given by (52) and (53) provides the final expression for the total angular momentum of the system.

In summarizing, we have constructed an action $I^{AD}$ for the AAD theory of a system consisting of one charge and one pole. From $I^{AD}$ the proper equations of motion are derived without recourse to Dirac's veto. And the Poincaré invariance of $I^{AD}$ enables us to construct the four-momentum $P_\mu$ and the angular momentum $L_{\mu\nu}$ that are conserved and independent of the string. These results can apparently be extended to more general cases, such as those containing more than one charge and/or one pole and dyons. Our conclusion that an AAD theory of a charge-monopole system can actually be constructed is quite contrary to the assertion of Tipler.\(^{10}\)

V. Final remarks

It would be appropriate here to summarize the results obtained in the present paper. Considerations given in Chap.II show that there exists a close connection between Dirac's and

*) See Appendix J.
Wu-Yang's theories. Using the idea of Brandt and Primack, we have been able to construct a theory which may be regarded as an unification of both the theories. We may say that our theory inherits a simple and tractable form from Dirac's theory and definiteness and conceptual transparency from Wu-Yang's theory. Among others we obtain, in our theory, the Maxwell-Lorentz equations straightforwardly by applying the least action principle without recourse to Dirac's veto. Moreover our theory enables us to formulate an AAD version of the theory, which appears to be very promising in studying further related problems. This proves the usefulness of our Lagrangian theory.

There remain, however, a number of problems which require further investigations.

1) It should be made clear whether a Hamiltonian formalism can be constructed from our Lagrangian theory.

2) One of the favourable properties of the Wu-Yang theory is that it does not make use of singular quantities. This is not the case with our theory, however. It would be of interest to try to derive the Maxwell-Lorentz equations from Wu-Yang's action without using the dual action integral; in this case it is desirable to do so in the framework of Wu-Yang's theory, that is, without relying on the concept of strings. However, the present author has the impression that this might not be possible.

3) We have limited ourselves to the classical theory in the present paper. The quantum theory may be studied in two ways. One is to apply the conventional method based on the Hamiltonian and
commutation relations. The other is to make use of the path-integral method, starting either from I (III-18) or from $I^{AD}$ (IV-19).

4) To the best of our knowledge our paper is the first to deal with monopoles on the basis of AAD theory. It is thus important to develop the theory further, following the conventional AAD theory of electromagnetism.

In conclusion we have found a new formulation of Dirac's monopoles, which is as useful and well-defined as the usual electrodynamics. Further we have been able to construct an action-at-a-distance theory of Dirac's monopoles.

Acknowledgement

The author would like to thank all members of the division of theoretical physics of the Institute of Physics, University of Tsukuba for their constant encouragement given to him during his graduate course. Thanks are due, especially, to Dr. T. Sawada for introducing the author to the subject of Dirac's monopoles and for many critical discussions, which includes, in particular, his contribution to the present work mentioned in the text. He is also indebted to Professor S. Kamefuchi for introducing the action-at-a-distance theory to him, for careful reading of the manuscript, and above all for guiding him kindly for a long time. The author also wishes to thank Dr. K. Shiozaki for his contribution mentioned in the text.
Appendix A. Notations

The metric is $g^{\mu\nu} = (1, -1, -1, -1)$, so that a contravariant vector is written as $A^\mu = (A^0, \vec{A})$. For an antisymmetric tensor $A^{\alpha\beta}$, its dual tensor is given by $\tilde{A}^{\alpha\beta} = -1/2 \varepsilon^{\alpha\beta\mu\nu} A_{\mu\nu}$, where $\varepsilon^{\alpha\beta\mu\nu}$ is the completely antisymmetric tensor with $\varepsilon^{0123} = -1$. Thus

$$\varepsilon_{\alpha\beta\rho\sigma} \varepsilon^{\alpha\mu\lambda\nu} = -1 \left\{ \delta^{\mu\nu\lambda\rho} + \delta^{\mu\nu\lambda\sigma} + \delta^{\mu\lambda\nu\rho} - \delta^{\mu\lambda\nu\sigma} \right\}, \quad (A-1)$$

$$\varepsilon_{\alpha\beta\rho\sigma} \varepsilon^{\alpha\beta\nu\lambda} = -2 \left\{ \delta^{\nu\lambda\rho} - \delta^{\nu\lambda\sigma} \right\}, \quad (A-2)$$

where $\delta^{\mu\nu\lambda\rho} = \delta^{\mu\nu\lambda} \delta_{\rho\sigma}$ and $\delta^{\nu\lambda\rho} = \delta^{\nu\lambda}_{\rho\sigma}$.

The electromagnetic field strength is denoted by $F^{\mu\nu}$, where $F^{0i} = E^i$ (electric field) and $F^{0i} = H^i$ (magnetic field). We use the rationalized system: $\text{div} \, E = \rho$, where $\rho$ is the charge density. Thus the charge quantization condition of Dirac\(^1\) is written as $eg = \text{(integer)}h$, with $h$ being Planck's constant. Further we always put $c = h = 1$, though we do not introduce $h$ in the text.

Appendix B. Stokes' Theorem

For any two functions $f$ and $g$ of $\tau$ and $\sigma$, there holds the relation

$$\int d\tau \int d\sigma \left( \frac{\partial f}{\partial \tau} \frac{\partial g}{\partial \sigma} - \frac{\partial f}{\partial \sigma} \frac{\partial g}{\partial \tau} \right) = \oint \left( \frac{\partial g}{\partial \tau} d\tau + \frac{\partial g}{\partial \sigma} d\sigma \right), \quad (B-1)$$

where the integration is to be performed along the line shown in Fig.VII; following Dirac\(^2\) (B-1) is referred to as Stokes' Theorem.
In the text the domains of integration are frequently given as $-\infty < \tau < \infty$ and $0 \leq \sigma < \infty$. In case there is any ambiguity concerning the integral at infinity, we always let $\sigma$ go to infinity first. This convention is in accordance with the situation that the string in fact extends to infinity.

Appendix C. Explicit form of a singular gauge transformation

From the definition of $A_\mu^{+g}$, (II-6), we obtain straightforwardly for a small variation of $y_\mu$:

$$A_\mu^{+g}(x) = g \epsilon_{\mu \lambda \rho \sigma} \left( \int_0^\infty d\tau \int_{-\infty}^{\infty} d\sigma \right)$$

$$\left\{ \begin{array}{c}
\frac{\partial^2 D^+(x-y)}{\partial x^\alpha \partial x^\lambda} g_{\alpha \rho} \frac{\partial y^\rho}{\partial \tau} \frac{\partial y^\sigma}{\partial \sigma} \\
- \frac{\partial^2 D^+(x-y)}{\partial x^\alpha \partial x^\lambda} g_{\rho \sigma} \frac{\partial y^\rho}{\partial \tau} \frac{\partial y^\sigma}{\partial \tau} \\
- \frac{\partial^2 D^+(x-y)}{\partial x^\alpha \partial x^\lambda} g_{\rho \sigma} \frac{\partial y^\rho}{\partial \tau} \frac{\partial y^\sigma}{\partial \sigma} \\
- \frac{\partial}{\partial \tau} \left( \frac{\partial D^+(x-y)}{\partial x^\lambda} g_{\rho \sigma} \right) \frac{\partial y^\rho}{\partial \sigma} \\
- \frac{\partial}{\partial \sigma} \left( \frac{\partial D^+(x-y)}{\partial x^\lambda} g_{\rho \sigma} \right) \frac{\partial y^\rho}{\partial \tau} \end{array} \right\}.$$

(C-1)

Applying Stokes' Theorem to the last two terms, we obtain
\[ \mathcal{A}_\mu^\nu(x) = \mathcal{G} \epsilon_{\mu
u\rho\sigma} \int_{-\infty}^{\infty} \! dt \int \! d\delta \frac{e^{2} D^+ (x-y)}{\partial x^\rho \partial x^\lambda} \left( T^{\alpha \beta_\rho \gamma \delta} - T^{\rho \alpha_\beta_\gamma \delta} - T^{\sigma_\alpha \beta_\rho \delta} \right) \]

\[ - \mathcal{G} \epsilon_{\mu
u\rho\sigma} \int_{-\infty}^{\infty} \! dt \left[ \frac{e^{2} D^+ (x-y)}{\partial x^\rho} \right] \frac{\partial y^\rho}{\partial \tau} \frac{\partial y^\delta}{\partial \sigma} \bigg|_{\delta = 0}, \quad (C-2) \]

where

\[ T^{\alpha \rho \delta} = \mathcal{G} y^\alpha \frac{\partial y^\rho}{\partial \tau} \frac{\partial y^\delta}{\partial \sigma}. \quad (C-3) \]

The use of an identity

\[ \epsilon_{\mu
u\rho\sigma} \left\{ T^{\alpha \rho \delta} - T^{\rho \alpha_\beta_\gamma \delta} - T^{\sigma \alpha \beta_\rho \delta} \right\} = -\frac{1}{2} \mathcal{G} \epsilon_{\mu
u\rho\sigma} \epsilon^{\beta_\gamma \theta \delta} T_{\beta_\gamma \theta} \epsilon_{\gamma \alpha \rho \delta}, \quad (C-4) \]

and (A-1), will finally yield

\[ \mathcal{A}_\mu^\nu(x) = \mathcal{G} \epsilon_{\mu
u\rho\sigma} \int_{-\infty}^{\infty} \! dt \int \! d\delta \square \! D^+ (x-y) T^{\alpha \rho \delta} \]

\[ - \mathcal{G} \epsilon_{\mu
u\rho\sigma} \int_{-\infty}^{\infty} \! dt \int \! d\delta \frac{e^{2} D^+ (x-z)}{\partial x^\rho \partial x^\lambda} T^{\alpha \rho \delta} \]

\[ - \mathcal{G} \epsilon_{\mu
u\rho\sigma} \int_{-\infty}^{\infty} \! dt \frac{e^{2} D^+ (x-z)}{\partial x^\rho} \delta^\rho_\delta \frac{\partial z^\rho}{\partial \tau} \bigg|_{\delta = 0}, \quad (C-5) \]

where use is made of (I-9).

In the case of a singular gauge transformation, the last
term vanishes because of (II-14). Thus we obtain (II-16).

It is interesting to see that as far as (II-14) is assumed

\[ \delta A_\mu \text{ does not depend on the choice of the Green's function up to the non-singular gauge transformation (II-11).} \]

Appendix D. A loop integral of \( \alpha_\mu \)

That \( \alpha_\mu \), (III-13), satisfies

\[ \alpha_{\mu,\nu}(x) - \alpha_{\nu,\mu}(x) = 0 \quad \text{in } R\cap R', \quad (D-1) \]

is clear from the fact that (III-13) is a solution of (III-11). Thus we have only to prove

\[ \oint C \alpha_\mu(x) \, dx^\nu = 0, \quad (D-2) \]

That is, (II-35) with \( R_{ab} \) being replaced by \( R\cap R' \).

Let us first consider the variation of the left-hand side of (2) against a displacement of \( Y_\mu \). It is easy to see by use of (III-22) that the integral remains unchanged as far as \( Y_\mu \) is displaced in such a way that \( \vec{Y} \) does not cross the path of integration of (2).

We can now choose \( \vec{Y} \) to be a straight line passing through the monopole. If this line is chosen to be the z-axis, \( \alpha_\mu \), in a rest frame of the monopole, takes the form given in (II-38).

From (II-39) we then find that (2) holds true.

Appendix E. Derivation of \( \delta I \) (III-23)

Derivation of the coefficients of \( \delta Z_\mu \) and \( \delta A_\mu \) can be made
in the same manner as in the Dirac theory, and may therefore be omitted.

Taking (III-20) and (III-7) into account, we obtain the coefficient of $\delta z_\mu$ quite easily; following the calculations of Appendix G, we can derive it more explicitly.

Let us now consider the coefficient of $\delta y_\mu$, we have to derive those terms which are absent in (II-23). Thus we evaluate the variation of the last term of (III-21) against $\delta y_\mu$, thereby obtaining formally

$$S I_3 = -e \int \gamma \, \alpha_\mu(x) \, dx^\mu$$

$$= -e g \epsilon_{\mu \nu \rho \sigma} \int \gamma \, dx^\mu \int_{-\infty}^{\infty} \, ds \int_{-\infty}^{\infty} \, \delta^4(x-y) \, \delta y^\nu \frac{\partial y^\rho}{\partial s} \frac{\partial y^\sigma}{\partial s}$$

$$+ e g \epsilon_{\nu \lambda \rho \sigma} \int \gamma \, dx^\nu \int_{-\infty}^{\infty} \, ds \int_{-\infty}^{\infty} \, \delta^4(x-y) \, \delta x^\rho \frac{\partial x^\sigma}{\partial s} \frac{\partial y^\nu}{\partial s} \frac{\partial y^\sigma}{\partial s}, \quad (E-1)$$

where we have used (III-22) but with $Y_\mu$ being replaced by $y_\mu$ (here we are concerned only with $\delta y_\mu$). Notice that even when a variation of $Y_\mu$ is taken, the contribution of $\delta y^\nu$ vanishes.

The first term in (1) has no contribution from the path $\gamma$ (cf.Fig. IV). Thus (1) is rewritten as

$$S I_3 = e g \epsilon_{\mu \nu \rho \sigma} \int \gamma \, dx^\mu \int_{-\infty}^{\infty} \, ds \int_{-\infty}^{\infty} \, \delta^4(x-y) \, \delta y^\nu \frac{\partial y^\rho}{\partial s} \frac{\partial y^\sigma}{\partial s}$$

$$+ e g \epsilon_{\nu \lambda \rho \sigma} \int \gamma \, dx^\nu \int_{-\infty}^{\infty} \, ds \int_{-\infty}^{\infty} \, \delta^4(x-y) \, \delta x^\rho \frac{\partial x^\sigma}{\partial s} \frac{\partial y^\nu}{\partial s} \frac{\partial y^\sigma}{\partial s}, \quad (E-2)$$
This gives the terms of (III-23) that were missing in (II-23), where the definition of $j_\mu$ in (I-4) is taken into account.

Appendix F.  Action-at-a-distance formalism for a two-charge system

The conventional classical action $I_{ee}$ is given by

$$I_{ee} = -m_1 \int d\mathbf{s}_1 - m_2 \int d\mathbf{s}_2 - \frac{1}{4} \int d^4x \ F_{\mu\nu}(x) F^{\mu\nu}(x)$$

$$- e_1 \int A_\mu(\partial_\mu) \ 3^\nu_1 \partial_1^\nu \mathbf{s}_1 - e_2 \int A_\mu(\partial_\mu) \ 3^\nu_2 \partial_2^\nu \mathbf{s}_2 .$$  \hspace{1cm} (F-1)

Defining

$$A_1^\mu(\mathbf{x}) = e_1 \int d\mathbf{s}_1 \ D^\mu(\mathbf{x} - \partial_1) \partial_1^\mu ,$$  \hspace{1cm} (F-2)

$$A_2^\mu(\mathbf{x}) = e_2 \int d\mathbf{s}_2 \ D^\mu(\mathbf{x} - \partial_2) \partial_2^\mu ,$$  \hspace{1cm} (F-3)

$$F_{\mu\nu}^i(\mathbf{x}) = A_{\mu,\nu}^i(\mathbf{x}) - A_{\nu,\mu}^i(\mathbf{x}) \ (i = 1, 2) ,$$  \hspace{1cm} (F-4)

and neglecting the self-energy terms, we obtain

$$- \frac{1}{4} \int d^4x \ F_{\mu\nu}(x) F^{\mu\nu}(x) :$$

$$= \frac{1}{2} e_1 \int A_1^\mu(\partial_\mu) \partial_1^\mu + \frac{1}{2} e_2 \int A_2^\mu(\partial_\mu) \partial_2^\mu ,$$  \hspace{1cm} (F-5)

after performing some partial integrations.
Now replacing $A_\mu(z_1)$ in (1) by $A_\mu^{3-1}(z_1)$, we obtain an action $I_{ee}^{AD}$ for AAD theory:

$$I_{ee}^{AD} = -m_1 \int ds_1 - m_2 \int ds_2 - e_1 \epsilon_2 \int ds_1 ds_2 \, \dot{3}_1^\mu \dot{3}_2^\mu \, D^5(\dot{3}_1 - \dot{3}_2).$$  \hspace{1cm} (F-6)

Then equations of motion are given in the usual manner as follows:

$$m_1 \dddot{3}_1^\mu + e_1 \mathcal{F}^{2\mu}_\nu (3_1) \dot{3}_1^\nu = 0, \hspace{1cm} \text{(F-7)}$$

$$m_2 \dddot{3}_2^\mu + e_2 \mathcal{F}^{2\mu}_\nu (3_2) \dot{3}_2^\nu = 0. \hspace{1cm} \text{(F-8)}$$

From the definition of the finite action

$$I_\star^{**} = -m_1 \int ds_1^{**} - m_2 \int ds_2^{**} - e_1 \epsilon_2 \int ds_1^{**} ds_2^{**} \, \dot{3}_1^\lambda \dot{3}_2^\lambda \, D^5(3_1 - 3_2), \hspace{1cm} \text{(F-9)}$$

and the Poincaré invariance of $I_\star^{**}$

$$\delta I_\star^{**} = 0 \text{ for } \delta \dot{3}_i^\mu = \epsilon_i^\mu \text{ or } \delta \dot{3}_i^\nu = \epsilon_i^\nu \dot{3}_i^\nu, \hspace{1cm} \text{(F-10)}$$

where $\epsilon_\mu (\epsilon_{\mu\nu})$ is an arbitrary infinitesimal constant vector (antisymmetric tensor), we obtain the four-momentum $P_\mu$ and the angular momentum $I'_{\mu\nu}$ of the system:

* Hereafter in this appendix $I_{ee}^{AD}$ will be written as $I$. 

\[ P_\mu = m_1 \dot{\theta}_1 \mu + e_1 A^2(\dot{\theta}_1) \mu + m_2 \dot{\theta}_2 \mu + e_2 A^4(\dot{\theta}_2) \mu \]
\[ + e_1 e_2 \left[ \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_2} ds_2 \left\{ \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial \theta_2} D^0(\dot{\theta}_1, \dot{\theta}_2) \right\} \right], \quad (F-11) \]
\[ L_{\mu \nu} = \dot{\theta}_1 \mu \{ m_1 \dot{\theta}_1 \nu + e_1 A^2(\dot{\theta}_1) \nu \} + \dot{\theta}_2 \nu \{ m_2 \dot{\theta}_2 \nu + e_2 A^4(\dot{\theta}_2) \nu \} \]
\[ + e_1 e_2 \left[ \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_2} ds_2 \left\{ 2 D' \dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_1 \mu \dot{\theta}_2 \nu - D^0 \dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_1 \mu \dot{\theta}_2 \nu \right\} \right], \quad (F-12) \]

where \( D_0' \) stands for the derivative of \( D_0 \). In the derivation we have used an identity
\[ \int_{**} ds_1 \left[ \int_{**}^{s_2} ds_2 + \int_{**}^{s_1} ds_1 \right] - \int_{**} ds_2 \left[ \int_{**}^{s_2} ds_1 + \int_{**}^{s_1} ds_1 \right] = - \left[ \int_{s_1}^{s_2} ds_1 \int_{s_1}^{s_2} ds_2 - \int_{s_1}^{s_2} ds_1 \int_{s_2}^{s_1} ds_2 \right] \quad (F-13) \]
(cf. Fig.VIII).

\[ \text{Fig.VIII} \]

It is then easy to see by using equations of motion, (7) and (8), that \( P_\mu \) and \( L_{\mu \nu} \) are conserved.
Appendix G. Derivation of equations of motion from $I^{AD}$

We begin by noting that

$$F_{\mu\nu}(\delta) = \delta \in \mu\nu\beta \int d\sigma \frac{\partial \tilde{\xi}^{\alpha}}{\partial \tilde{\xi}_{\beta}} D^{\alpha}(\tilde{\delta} - \tilde{\xi})$$  \hspace{1cm} (G-1)

$$= -\delta \epsilon_{\mu\nu\rho\sigma} \int d\sigma \int_{\tilde{\sigma}}^{\tilde{\sigma} + \delta} \partial^{\delta} \partial^{\rho} D^{\alpha}(\tilde{\delta} - \tilde{\xi} - \omega) \tilde{\xi}^{\rho} \tilde{\xi}^{\sigma}$$

$$+ \delta \epsilon_{\mu\nu\rho\sigma} \int d\sigma \int_{\tilde{\sigma}}^{\tilde{\sigma} + \delta} \partial^{\delta} \partial^{\rho} \partial^{\mu} D^{\alpha}(\tilde{\delta} - \tilde{\xi} - \omega) \tilde{\xi}^{\rho} \tilde{\xi}^{\sigma}$$

$$+ \delta \epsilon_{\mu\nu\rho\sigma} \int d\sigma \int_{\tilde{\sigma}}^{\tilde{\sigma} + \delta} \partial^{\delta} \partial^{\rho} \delta^{(\tilde{\delta} - \tilde{\xi} - \omega)} \tilde{\xi}^{\rho} \tilde{\xi}^{\sigma}.$$

\hspace{1cm} (G-2)

and

$$\tilde{F}_{\mu\nu}(\tilde{\xi}) = -\epsilon \epsilon_{\mu\nu\rho\sigma} \int d\sigma \frac{\partial \tilde{\xi}^{\alpha}}{\partial \tilde{\xi}_{\beta}} D^{\alpha}(\tilde{\xi} - \tilde{\delta})$$  \hspace{1cm} (G-3)

$$= -\epsilon \epsilon_{\mu\nu\rho\sigma} \int d\sigma \int_{\tilde{\sigma}}^{\tilde{\sigma} + \delta} \partial^{\delta} \partial^{\rho} D^{\alpha}(\tilde{\delta} - \tilde{\xi} + \omega) \tilde{\xi}^{\rho} \tilde{\xi}^{\sigma}$$

$$+ \epsilon \epsilon_{\mu\nu\rho\sigma} \int d\sigma \int_{\tilde{\sigma}}^{\tilde{\sigma} + \delta} \partial^{\delta} \partial^{\rho} \partial^{\mu} D^{\alpha}(\tilde{\delta} - \tilde{\xi} + \omega) \tilde{\xi}^{\rho} \tilde{\xi}^{\sigma}$$

$$+ \epsilon \epsilon_{\mu\nu\rho\sigma} \int d\sigma \int_{\tilde{\sigma}}^{\tilde{\sigma} + \delta} \partial^{\delta} \partial^{\rho} \delta^{(\tilde{\delta} - \tilde{\xi} + \omega)} \tilde{\xi}^{\rho} \tilde{\xi}^{\sigma}.$$ \hspace{1cm} (G-4)

For $\delta I_1$ and $\delta I^{AD}_3$ we obtain in a straightforward manner

$$\delta I_1 = m \int d\sigma \tilde{\xi}_\mu \delta \tilde{\xi}_\mu + M \int d\sigma \tilde{\xi}_\mu \delta \tilde{\xi}_\mu.$$

\hspace{1cm} (G-5)
\[\delta I_3 = e g \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\alpha \left\{ (e_{\alpha \lambda \rho} \delta_{\omega} \delta_{\lambda} \delta_{\rho} D^0 - e_{\lambda \rho \delta} \delta_{\omega} \delta_{\rho} D^0) \delta_{\alpha}^4 \right\} \] 

\[= e \int_{-\infty}^{\infty} \left\{ A_{\alpha, \mu} (z) - A_{\alpha, \mu} (\bar{z}) \right\} \delta_{\alpha}^4 \delta_{\mu}^4 \] 

\[+ g \int_{-\infty}^{\infty} \left\{ B_{\alpha, \mu} (z) - B_{\alpha, \mu} (\bar{z}) \right\} \delta_{\alpha}^4 \delta_{\mu}^4 \],

(G-6)

where \(D^0 = D^0(z, \bar{z}, \omega)\).

The calculation of \(\delta I_4\) can be made in the following way;

\[\delta I_4 = e \int_{s_0}^{s_1} \left\{ \frac{\delta \alpha (\delta)}{\delta \delta^\lambda} \delta_{\delta}^\lambda \delta_{\lambda}^\delta - \frac{\delta \alpha (\bar{\delta})}{\delta \delta^\lambda} \delta_{\lambda}^\delta \right\} \] 

\[+ e g \int_{s_0}^{s_1} \delta_{\alpha}^4 \int_{-\infty}^{\infty} \left\{ e_{\alpha \lambda \rho \delta} e_{\lambda \mu \bar{\delta}} \delta_{\rho} D^0 (z, \bar{z}, \delta - \bar{\delta}) \right\} \] 

\[= e \int_{s_0}^{s_1} \left\{ \delta \lambda \mu (\delta) \delta_{\delta}^\lambda \delta_{\lambda}^\delta \right\} \] 

\[+ e g \int_{-\infty}^{\infty} \left\{ \delta_{\alpha}^4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e_{\alpha \lambda \rho \delta} e_{\lambda \mu \bar{\delta}} \delta_{\rho} D^0 (z, \bar{z}, \delta - \bar{\delta}) \right\} \] 

\[= - e \int_{-\infty}^{\infty} \left\{ \delta \lambda \mu (\delta) \delta_{\delta}^\lambda \delta_{\lambda}^\delta \right\} \] 

\[+ g \int_{-\infty}^{\infty} \left\{ \delta \lambda \mu (\delta) \delta_{\delta}^\lambda \delta_{\lambda}^\delta \right\}, \] 

(G-7)
Here $I_4$ is to be taken into account only when $z-Z-\omega$ becomes zero at some $s(s_0 < s < s_1)$, $\tau$ and $\sigma(\sigma > 0)$.

Putting (5), (6) and (7) together, we obtain equations of motion, (IV-24) and (IV-25).

Appendix H. Proof of independence of $P_\mu$ on the string

For the sake of convenience we devide $P_\mu$ (IV-33) into four parts:

$$P^{(1)}_\mu = m \ddot{x}_\mu + M \ddot{Z}_\mu,$$  \hspace{1cm} (H-1)

$$P^{(2)}_\mu = \epsilon A^g_\mu (\xi) + \chi B^e_\mu (Z),$$  \hspace{1cm} (H-2)

$$P^{(3)}_\mu = -\epsilon \int_{s_0}^{s} \left[ a s \frac{d}{d \tau} - a s \frac{d}{d \tau} \right] d \xi \int_0^\infty d \omega \int_\chi \rho_\xi \frac{\partial^2 \chi (\xi - Z - \omega)}{\partial \omega \partial \omega},$$  \hspace{1cm} (H-3)

$$P^{(4)}_\mu = -\chi \int_{s_0}^{s} \left[ s \frac{d}{d \tau} - s \frac{d}{d \tau} \right] d \xi \int_0^\infty d \omega \int_\chi \rho_\xi \frac{\partial^2 \chi (\xi - Z - \omega)}{\partial \omega \partial \omega},$$  \hspace{1cm} (H-4)

Needless to say, (1) does not depend on $\omega_\mu$. Let us now evaluate the variations of the last three of these against $\delta \omega_\mu$.

Evaluation of (2), (3) or (4) is performed in a similar manner. We shall therefore consider $\delta P^{(3)}_\mu$ in detail, limiting ourselves only to a brief remark for the case of $\delta P^{(2)}_\mu$ and $\delta P^{(4)}_\mu$.

It is convenient to define

$$\tau^{\alpha \beta} \left( \tau \right) = \omega^\alpha \delta \omega^\beta - \omega^\beta \delta \omega^\alpha.$$  \hspace{1cm} (H-5)
The evaluation of $\delta P^{(3)}_{\mu}$ is given as follows:

$$\delta P^{(3)}_{\mu} = -e \delta \left[ \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \right] \delta \left[ \frac{\delta^{D}(3-z-\omega)}{\partial \omega_6 \partial \omega_{\mu}} \right] T_{\rho \delta}$$

$$= -e \delta \left[ \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \right] \delta \left[ \frac{\delta^{D}(3-z-\omega)}{\partial \omega_6 \partial \omega_{\mu}} \right] T_{\rho \delta} + \delta \left[ \frac{\delta^{D}(3-z-\omega)}{\partial \omega_6 \partial \omega_{\mu}} \right] T_{\rho \delta}$$

$$= e \delta \left[ \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \right] \delta \left[ \frac{\delta^{D}(3-z-\omega)}{\partial \omega_6 \partial \omega_{\mu}} \right] T_{\rho \delta} + \delta \left[ \frac{\delta^{D}(3-z-\omega)}{\partial \omega_6 \partial \omega_{\mu}} \right] T_{\rho \delta}$$

$$= e \delta \left[ \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \right] \delta \left[ \frac{\delta^{D}(3-z-\omega)}{\partial \omega_6 \partial \omega_{\mu}} \right] T_{\rho \delta} + \delta \left[ \frac{\delta^{D}(3-z-\omega)}{\partial \omega_6 \partial \omega_{\mu}} \right] T_{\rho \delta}$$

where we have used (IV-36), (IV-37) and (A-1). The numbering the expressions in (6) is to exhibit clearly the cancellation of various expressions: Those with the same number cancel each other out.

In the same manner as above we also obtain
Thus it is now obvious that

\[ \mathcal{S}_P \mu = 0 \quad . \]  

Appendix I. Proof of the equivalence between the two expressions of \( L_{\mu \nu} \)

We prove that (IV-52) and (IV-53) are equivalent to each other first by deriving (IV-53) from (IV-52). For later convenience we devide the \( L_{\mu \nu} \) given by (IV-53) into four parts;

\[ L^{(4)}_{\mu \nu} = \delta_{\mu \nu} \left\{ m \delta_{\nu 1} + e A^\theta (\delta)_{\nu 1} \right\} + \bar{Z} [ L_{\mu} \left\{ \mu \bar{Z}_{\nu 1} + \bar{\theta} B^\theta (\bar{\theta})_{\nu 1} \right\} , \quad (I-1) \]

\[ L^{(5)}_{\mu \nu} = \bar{e} g \left\{ \delta_{\mu 1} \left\{ \frac{d Z}{d \tau} + \frac{d S}{d \tau} \right\} \delta_{\nu 1} \right\} \left\{ \frac{d \delta}{d \omega} \delta_{\mu 1} \bar{Z}^\lambda \omega^\delta D^\alpha \right\} \quad , \quad (I-2) \]
\[ L^{(3)}_{\mu\nu} = -e g \left[ \int_{-\infty}^{s} ds \left[ \omega \mu_{\lambda\nu} \rho \right]_{0}^{\infty} \delta \left( \beta - z - \omega \right) \delta \hat{z} \cdot \omega \cdot \delta \right] \right), \quad (I-3) \]

\[ L^{(4)}_{\mu\nu} = e g \varepsilon_{\lambda} \mu \rho \int_{-\infty}^{\infty} d\sigma \left[ \omega \cdot \delta \right] \cdot \rho \cdot \omega \cdot \delta D^0. \quad (I-4) \]

Using the identity (IV-46) we can rewrite \( L^{(2)}_{\mu\nu} \) as follows:

\[ L^{(2)}_{\mu\nu} = -e g \left[ \int_{-\infty}^{s} ds \left[ \omega \mu_{\lambda\nu} \rho \right]_{0}^{\infty} \delta \left( \beta - z - \omega \right) \delta \hat{z} \cdot \omega \cdot \delta \right] \right), \quad (I-5) \]

The above expression, when combined with \( L^{(3)}_{\mu\nu} \) and \( L^{(4)}_{\mu\nu} \), gives

\[ L^{(2)}_{\mu\nu} + L^{(3)}_{\mu\nu} + L^{(4)}_{\mu\nu} = (the \ first \ term \ of \ (5)) \]
\[ + \varepsilon \varphi \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\omega \lambda \mu \epsilon_{\lambda \mu \nu \rho} \sum_{\theta=0}^{3} \hat{\omega}^\theta \frac{\partial D^\sigma}{\partial \omega^\lambda} \]

\[ - \varepsilon \varphi \left[ \int_{s_0}^{s} ds \int_{\tau_0}^{\tau} d\tau - \int_{s_0}^{s} ds \int_{\tau}^{\tau_0} d\tau \right] a_0 d\sigma \]

\[ \times \left\{ (\hat{\beta}-\omega) \epsilon_{\lambda \mu \nu \rho} \beta^4 (\hat{\beta}-\hat{Z}-\omega) \hat{\beta}^4 \hat{\beta}^4 \hat{\omega}^\rho \right\} . \]

(I-6)

Notice that the third term in (5) cancels \( L_{\mu \nu}^{(3)} \) and that the domain of the integration of the last term in (5) can be written in the same form as that of the last term in (6), because the integrand concerned is nonzero only when \( z-Z-\omega \) becomes zero at some \( s(s_0 < s < s_1) \), \( \tau \) and \( \sigma (\sigma > 0) \).

It is then easy to see that \( L_{\mu \nu}^{(1)} \) and (6) give (IV-53). The converse of the above can be proved in a similar way. Thus we have confirmed that (IV-52) and (IV-53) are equivalent to each other.

Appendix J. Proof of independence of \( L_{\mu \nu} \) on the string

We employ the notation given by (I-1) - (I-4) and (H-5). The proof is given in a way similar to that given in Appendix H.

A straightforward calculation gives

\[ \delta L_{\mu \nu}^{(1)} = \varepsilon \varphi \Delta_{\mu \nu} \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\omega \left\{ \hat{\beta} \varphi \partial \omega \varphi D^\sigma \tilde{\gamma}^\alpha \varphi + \hat{\beta} \varphi \epsilon_{\sigma \tau \rho \omega} \beta^4 (\hat{\beta}-\hat{Z}-\omega) \tilde{\gamma}^\alpha \varphi \right\} . \]

(J-1)
\[ s \mathbb{L}^{(2)}_{\mu \nu} = e g \left[ \int_{-\infty}^{s} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{s} ds \int_{-\infty}^{s} \frac{d\sigma}{d\theta} \right) \right] \frac{d\sigma}{d\theta} \]

\[ \times \left\{ -\delta_{\mu \nu} \left( \hat{z}^{\lambda} \partial \nu \delta \hat{D}^{\mu} \right) \hat{T}^{\lambda}_{\nu} + \hat{z}^{\mu} \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} \right\} \]

\[ + \delta_{\mu \nu} \left[ \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} + \hat{z}^{\mu} \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} \right] \]

\[ + e g \left[ \int_{-\infty}^{s} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{s} ds \int_{-\infty}^{s} \frac{d\sigma}{d\theta} \right) \right] \frac{d\sigma}{d\theta} \]

\[ \left\{ \delta_{\mu \nu} \left( \hat{z}^{\lambda} \partial \nu \delta \hat{D}^{
u} \right) \hat{T}^{\lambda}_{\nu} + \hat{z}^{\mu} \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} \right\} \]

\[ + e g \left[ \int_{-\infty}^{s} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{s} ds \int_{-\infty}^{s} \frac{d\sigma}{d\theta} \right) \right] \frac{d\sigma}{d\theta} \]

\[ \left\{ \delta_{\mu \nu} \left( \hat{z}^{\lambda} \partial \nu \delta \hat{D}^{
u} \right) \hat{T}^{\lambda}_{\nu} + \hat{z}^{\mu} \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} \right\} \]

\[ , \quad (J-2) \]

\[ s \mathbb{L}^{(3)}_{\mu \nu} = - e g \left[ \int_{-\infty}^{s} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{s} ds \int_{-\infty}^{s} \frac{d\sigma}{d\theta} \right) \right] \frac{d\sigma}{d\theta} \]

\[ \times \left\{ \delta_{\mu \nu} \left( \hat{z}^{\lambda} \partial \nu \delta \hat{D}^{
u} \right) \hat{T}^{\lambda}_{\nu} + \hat{z}^{\mu} \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} \right\} \]

\[ - \delta_{\mu \nu} \left[ \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} + \hat{z}^{\mu} \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} \right] \]

\[ - e g \left[ \int_{-\infty}^{s} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{s} ds \int_{-\infty}^{s} \frac{d\sigma}{d\theta} \right) \right] \frac{d\sigma}{d\theta} \]

\[ \left\{ \delta_{\mu \nu} \left( \hat{z}^{\lambda} \partial \nu \delta \hat{D}^{
u} \right) \hat{T}^{\lambda}_{\nu} + \hat{z}^{\mu} \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} \right\} \]

\[ , \quad (J-3) \]

\[ s \mathbb{L}^{(4)}_{\mu \nu} = e g \left[ \int_{-\infty}^{s} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{s} ds \int_{-\infty}^{s} \frac{d\sigma}{d\theta} \right) \right] \frac{d\sigma}{d\theta} \]

\[ \times \left\{ \delta_{\mu \nu} \left( \hat{z}^{\lambda} \partial \nu \delta \hat{D}^{
u} \right) \hat{T}^{\lambda}_{\nu} + \hat{z}^{\mu} \hat{S}^{(z-\omega)} \hat{T}^{\lambda}_{\nu} \right\} \]

\[ + \hat{z}^{\mu} \partial \nu \delta \hat{D}^{
u} \hat{T}^{\lambda}_{\nu} \]

\[ + \hat{T}^{\lambda}_{\mu \nu} \delta \nu \delta \hat{D}^{
u} \]

\[ , \quad (J-4) \]
Here we have numbered the terms in such a way that those with the same number cancel each other out. Some terms remain, however, after the cancellation.

It is easy to see that the expression [10] can be cancelled by the sum of [4] and [11]. When use is made of the identity

\[
\delta_{\gamma \mu} \frac{\partial D^\alpha}{\partial \gamma^\nu} + Z_{\gamma \mu} \frac{\partial D^\alpha}{\partial Z^\nu} + \omega_{\gamma \mu} \frac{\partial D^\alpha}{\partial \omega^\nu} = 0, \tag{J-5}
\]

which is derived from the fact that \( D^0 \) is a function of \( z-Z-\omega \), the term [3] can be rewritten as

\[
-\varepsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \delta_{\gamma \mu} \delta^\beta \omega^\nu \bar{D}^\alpha \tilde{T}_{\alpha \beta} - \delta^\beta \omega_{\gamma \mu} \omega^\nu \bar{D}^\alpha \tilde{T}_{\alpha \beta} - \delta^\alpha \omega_{\gamma \mu} \omega^\nu \bar{D}^\alpha \tilde{T}_{\alpha \beta} \right\}, \tag{J-6}
\]

which is then cancelled by [9], [12] and [13].

Thus all the terms cancelled each other out, and this proves that \( L_{\mu \nu} \) is independent of \( \omega_\mu \).
References

2) P.A.M. Dirac, Phys. Rev. 74, 817 (1948).
19) J.A. Wheeler and R.P. Feynman, Rev. Mod. Phys. 17, 157 (1945), and references quoted therein.
Figures

Fig. I. Division of the space by Wu and Yang.

Fig. II. The path of the integration (II-34). (a) The path with winding number 1. (b) The path with winding number n.
Fig.III. Division of the space.

Fig.IV. (a) The world line of the charge intersecting the string at $s$.
     (b) The path $\gamma$. (c) Another possible type of $\gamma$. 
Fig.V. (a) Past and future events generating $A^e_\mu(x)$. (b) Events on the world sheet of the string $v_\mu$ which generate $B^e_\mu(x)$.

Fig.VI. An illustration of the world lines of the charge and the pole intersecting surfaces $\Sigma^*$ and $\Sigma^{**}$. 
Fig. VII. The path of integration for Stokes' Theorem (B-1).

Fig. VIII. An illustration of the identity (F-13).