ON A NEW ALGORITHM FOR INHOMOGENEOUS DIOPHANTINE APPROXIMATION

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Abstract. The inhomogeneous Diophantine approximation algorithm of Nishioka et al., \((X, T_2, c(x), d(x, y))\), was shown by Komatsu to be efficient for inhomogeneous Diophantine approximation, but lacks a properly founded natural extension and not all periodic points about the approximation are determined. A new algorithm, \((X, T, a(x), b(x, y))\), is proposed in this paper as a modification of \((X, T_2, c(x), d(x, y))\), and is shown to be efficient for inhomogeneous Diophantine approximation similar to \((X, T_2, c(x), d(x, y))\) but also to have a natural extension, which allows all periodic points about \((X, T, a(x), b(x, y))\) to be determined and gives \(\liminf_{q \to \infty} q |q\alpha - \beta - p|\) for the periodic points \((\alpha, \beta)\).

1. Introduction

It is well known that connections exist between the continued fractions algorithm and the minimization of \(|q\alpha - p|\), where \(q\) is an natural number, \(p\) is an integer, and \(\alpha\) is an irrational number. The problem of minimizing \(|q\alpha - \beta - p|\), where \(\beta\) is a real number, is called the inhomogeneous Diophantine approximation. This problem has been considered by many authors (e.g., [12, 18, 13, 6, 7, 1, 2, 3, 4, 8, 21, 10, 11, 5, 14, 16, 17], and detailed information can be obtained by a review of the literature. Many algorithms related to the problem have been used. For example, Ito and Kasahara [10] defined the following algorithm, which was implicitly introduced by Morimoto [18]. Let \(Z = \{(x, y) | 0 \leq y < 1, -y < x < -y + 1\}\), as shown in Fig. 1.
Then for $(x, y) \in Z$:

$$a'(x, y) = \left\lfloor \frac{1 - y}{x} \right\rfloor - \left\lfloor \frac{-y}{x} \right\rfloor, \quad b'(x, y) = -\left\lfloor \frac{-y}{x} \right\rfloor.$$ 

The algorithm $T_1$ is then defined by the following transformation on $Z$ for $(x, y) \in Z$.

$$T_1(x, y) = \left(1 - a'(x, y), b'(x, y) - \frac{y}{x}\right).$$

This algorithm $(Z, T_1, a'(x, y), b'(x, y))$ gives the best solution to the inhomogeneous Diophantine approximation. Constructing the natural extension of the algorithm, they determined all the periodic points about the algorithm. Ito [9] was the first to subsequently find that a certain natural extension of the Diophantine algorithm is useful for investigating the algorithm. Komatsu studied the following algorithm, which was introduced by Nishioka et al. [19]. With $X = [0, 1]^2$, $T_2$ is defined as the following transformation on $X$ for $(x, y) \in X$.

$$T_2(x, y) = \left(1 - c(x), d(x, y) - \frac{y}{x}\right),$$

where $c(x) = \left\lfloor \frac{1}{x} \right\rfloor$ and $d(x, y) = \left\lceil \frac{y}{x} \right\rceil$. Using this algorithm, $(X, T_2, c(x), d(x, y))$, Komatsu [14] obtained $\lim \inf_{q \to \infty} q|qz - \beta - p|$ in some cases.

In this paper, an algorithm $(X, T, a(x), b(x, y))$ is introduces as a modification of $(X, T_2, c(x), d(x, y))$. The new algorithm also gives the best solution for the inhomogeneous Diophantine approximation as does $(X, T_2, c(x), d(x, y))$. However, a natural extension is constructed for $(X, T, a(x), b(x, y))$, which has not been done for $(X, T_2, c(x), d(x, y))$. Using the natural extension of $(X, T, a(x), b(x, y))$, all purely periodic points about the algorithm are determined, and for the purely periodic point $(x, \beta)$, a relation between $\lim \inf_{q \to \infty} q|qz - \beta - p|$ and the natural extension of $(X, T, a(x), b(x, y))$ is obtained. Although all eventually periodic points have been determined by Komatsu [15], all purely periodic points have not.
2. Definition and Some Properties of Algorithm

We denote \( \mathbb{R}, \mathbb{Q}, \) and \( \mathbb{Z} \) the set of all real numbers, the set of all rational numbers and the set of all integers respectively. For \((x, y) \in X \) with \( x \neq 0 \) we define \( a(x) \) by \( \frac{1}{\lfloor x \rfloor} \) and we define \( b(x, y) \) by

\[
b(x, y) = \begin{cases} 
1 & \text{if } y = 0, \\
\left\lfloor \frac{x}{y} \right\rfloor & \text{if } y > 0 \text{ and } \left\lfloor \frac{1}{x} \right\rfloor > \left\lfloor \frac{y}{x} \right\rfloor \text{ or } \left\lfloor \frac{1}{x} \right\rfloor = \frac{y}{x}, \\
0 & \text{if } \left\lfloor \frac{1}{x} \right\rfloor = \left\lfloor \frac{y}{x} \right\rfloor \text{ and } \left\lfloor \frac{1}{x} \right\rfloor \neq \frac{y}{x}.
\end{cases}
\]

We define a transformation \( T \) as follows; for \((x, y) \in X \) if \( x > 0 \), then

\[
T(x, y) = \begin{cases} 
\left( \frac{1}{x} - a(x), b(x, y) - \frac{y}{x} \right) & \text{if } b(x, y) > 0, \\
\left( \frac{1}{x} - a(x), \frac{1}{x} - \frac{y}{x} \right) & \text{if } b(x, y) = 0,
\end{cases}
\]

and if \( x = 0 \), then \( T(x, y) = (x, y) \).

We define \( a_n(x) = a(T^{n-1}(x, y)) \), \( b_n(x, y) = b(T^{n-1}(x, y)) \) and \((x_n, y_n) = T^{n-1}(x, y) \). It is not difficult to see that if \( x \notin \mathbb{Q} \), then for any integer \( n > 0 \) \( a_n(x) \) and \( b_n(x, y) \) are defined.

Lemma 2.1 follows from the continued fraction theory.

**Lemma 2.1.** Let \((x, y) \in X \) and \( x \notin \mathbb{Q} \). Then, for each integer \( n > 0 \)

1. \( q_n(x)x - p_n(x) = (-1)^n q_1 \cdots q_{n+1} \frac{(-1)^n}{q_{n+1}(x) + q_{n+2}(x)} \),
2. \( |q_{n-1}(x)x - p_{n-1}(x)| = a_{n+1}(x, y)|q_n(x)x - p_n(x, y)| + |q_{n+1}(x, y)x - p_{n+1}(x, y)| \),
3. \( |q_n(x)x - p_n(x, y)| > |q_{n+1}(x, y)x - p_{n+1}(x, y)| \),
4. for any integer \( j, k \) with \( q_n(x) < j < q_{n+1}(x, y) \), \( |q_n(x)x - p_n(x, y)| < \left| jx - k \right| \),

where \( \{p_n(x)\}_{-1 \leq n}, \{q_n(x)\}_{-1 \leq n} \) are defined by

\[
p_{-1}(x) = 1, \quad p_0(x) = 0, \\
q_{-1}(x) = 0, \quad q_0(x) = 1,
\]

for \( n \geq 1 \)

\[
p_n(x) = a_n(x)p_{n-1}(x) + p_{n-2}(x), \\
q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x).
\]
LEMMA 2.2. Let \((x, y) \in X\). Then,

1. \(a_n(x) > 0\) and \(a_n(x) \geq b_n(x, y) \geq 0\),
2. if \(b_n(x, y) = 0\), then \(b_{n+1}(x, y) = 1\).

PROOF. The proof of (1) is easy. Let us prove (2). We suppose that \(b_n(x, y) = 0\). Then, we see that \(v(x_n) = \frac{2n}{a_n(x)}\) and \(a(x_n) < \frac{2n}{X_n}\). Since \(x_{n+1} = \frac{1}{x_n} - a(x_n)\) and \(y_{n+1} = \frac{1}{x_n} - \frac{2n}{X_n}\), we have \(x_{n+1} > y_{n+1}\). Thus, we obtain \(b(x_{n+1}, y_{n+1}) = 1\).

Let \((x, y) \in X\) and \(x \not\in \mathbb{Q}\). Let us define integers \(A_n(x, y)\), \(B_n(x, y)\) as follows:

\[
A_1(x, y) = \begin{cases} 
0 & \text{if } b(x, y) > 0, \\
-1 & \text{if } b(x, y) = 0.
\end{cases} \\
B_1(x, y) = \begin{cases} 
b_1(x, y) & \text{if } b(x, y) > 0, \\
0 & \text{if } b(x, y) = 0,
\end{cases}
\]

For \(n > 1\)

\[
A_n(x, y) = \begin{cases} 
A_{n-1}(x, y) + b_n(x, y)p_{n-1}(x) & \text{if } b(x, y) > 0, \\
A_{n-1}(x, y) - p_{n-2}(x) & \text{if } b(x, y) = 0,
\end{cases} \\
B_n(x, y) = \begin{cases} 
B_{n-1}(x, y) + b_n(x, y)q_{n-1}(x) & \text{if } b(x, y) > 0, \\
B_{n-1}(x, y) - q_{n-2}(x) & \text{if } b(x, y) = 0.
\end{cases}
\]

We remark that \(\{B_n(x, y)\}_{n=1, 2, \ldots}\) and \(\{A_n(x, y)\}_{n=1, 2, \ldots}\) are not increasing sequences generally as \(n \to \infty\).

LEMMA 2.3. Let \((x, y) \in X\) and \(x \not\in \mathbb{Q}\). Then, for any \(n > 0\)

\[
y = B_n(x, y)x - A_n(x, y) + (-1)^n y_{n+1}x_1 \cdots x_n. \tag{1}
\]

PROOF. We prove the lemma by the induction on \(n\). Let \(n = 1\). First, let \(b_1(x, y) > 0\). Then, we see \(y_2 = b_1(x, y) - \frac{x_1}{x_1}\). Therefore, we have \(y_1 = b_1(x, y)x_1 - y_2x_1 = B_1(x, y)x - A_1(x, y) - y_2x_1\). Next, let \(b_1(x, y) = 0\). Then, we see \(y_2 = \frac{1}{x_1} - \frac{x_1}{x_1}\). Therefore, we have \(y_1 = 1 - y_2x_1 = B_1(x, y)x - A_1(x, y) - y_2x_1\). Hence, (1) holds for \(n = 1\). Secondly, we suppose that (1) holds for \(n = k\), that is, \(y = B_k(x, y)x - A_k(x, y) + (-1)^k y_{k+1}x_1 \cdots x_k\). Let \(b_{k+1}(x, y) > 0\). Then, we have \(y_{k+2} = b_{k+1}(x, y) - \frac{x_1}{x_k+1} \cdot x_{k+1}\) which implies \(y_{k+1} = b_{k+1}(x, y)x_{k+1} - x_{k+1}y_{k+2}\). Therefore, using \(x_1 \cdots x_{k+1} = (-1)^k (q_kx - p_k)\), we see

\[
y = B_k(x, y)x - A_k(x, y) + (-1)^k y_{k+1}x_1 \cdots x_k,
\]

\[
= B_k(x, y)x - A_k(x, y) + (-1)^k b_{k+1}(x, y)x_1 \cdots x_{k+1}(-1)^{k+1} y_{k+1}x_1 \cdots x_{k+1},
\]

\[
= B_{k+1}(x, y)x - A_{k+1}(x, y) + (-1)^{k+1} y_{k+2}x_1 \cdots x_{k+1}.
\]
Let $b_{k+1}(x, y) = 0$. Then, we have $y_{k+2} = \frac{1}{x_{k+1}} - \frac{y_{k+1}}{x_{k+1}}$, which implies $y_{k+1} = 1 - x_{k+1}y_{k+2}$. Using $x_1 \cdots x_k = (-1)^{k+1}(q_{k-1}x - p_{k-1})$, we have

$$y = B_k(x, y)x - A_k(x, y) + (-1)^k y_{k+1}x_1 \cdots x_k,$$

$$= B_k(x, y)x - A_k(x, y) + (-1)^k x_1 \cdots x_k + (-1)^{k+1} y_{k+2}x_1 \cdots x_{k+1},$$

$$= B_{k+1}(x, y)x - A_{k+1}(x, y) + (-1)^{k+1} y_{k+2}x_1 \cdots x_{k+1}.$$ 

Therefore, (1) holds for $n = k + 1$. Thus, we have Lemma. \hfill \Box

**Lemma 2.4.** Let $(x, y) \in X$ and $x \notin Q$. Then, $\lim_{n \to \infty} (B_n(x, y)x - A_n(x, y)) = y$.

**Proof.** By Lemma 2.3 $|y - B_n(x, y)x + A_n(x, y)| = y_{n+1}x_1 \cdots x_n$. By Lemma 2.1 we have $x_1 \cdots x_n = |q_{n-1}x - p_{n-1}| < \frac{1}{q_n}$. Thus, we have Lemma. \hfill \Box

We define $\Psi = \{ (x, y) \in \mathbb{R}^2 | x \notin Q \text{ and } y \neq mx + n \text{ for any } m, n \in \mathbb{Z} \}$.

**Lemma 2.5.** Let $(x, y), (z, w) \in X$ and $x, z \notin Q$. If $a_n(x) = a_n(z)$ and $b_n(x, y) = b_n(z, w)$, for any integer $n > 0$, then $(x, y) = (z, w)$.

**Proof.** By continued fraction theory we obtain $x = z$. From Lemma 2.4 we have $y = w$. \hfill \Box

**Lemma 2.6.** Let $(x, y) \in X \cap \Psi$. Then, if $b_n(x, y) = 0$ for some integer $n > 0$, then there exists an integer $k > 0$ such that $b_{n+2k}(x, y) > 0$.

**Proof.** We suppose that there exists an integer $m$ such that for any $k \geq 0$ $b_{m+2k}(x, y) = 0$. Then, from Lemma 2.2 we have $b_{n+2k+1}(x, y) = 1$ for any $k \geq 0$. Let $(u, v) = T^{m-1}(x, y)$. Then, $b_{2k}(u, v) = 0$ and $b_{2k+1}(u, v) = 1$ for any $k \geq 0$. We see easily that $b_n(u, 1) = b_n(u, v)$ for any integer $n \geq 1$. From Lemma 2.5 we have $v = 1$. Then, we see $(x, y) \notin \Psi$. But it is a contradiction. Therefore, we have Lemma. \hfill \Box

**Lemma 2.7.** Let $(x, y) \in X \cap \Psi$. Then, if $a_n(x) = b_n(x, y)$ for some integer $n > 0$, then there exists an integer $k > n$ such that $a_k(x) \neq b_k(x, y)$.

**Proof.** We suppose that there exists an integer $m$ such that for any $k \geq m$ $a_k(x) = b_k(x, y)$. Let $(u, v) = T^{m-1}(x, y)$. It is not difficult to see that $b_j(u, 1 - u) = b_j(u, v)$ for any integer $j \geq 1$. From Lemma 2.5 we have $v = 1 - u$. \hfill \Box
Then, by using the equation \((u, v) = T^{m-1}(x, y)\) we see easily \((x, y) \notin \Psi\). But it is a contradiction. Therefore, we have Lemma. \(\square\)

Lemma 2.8. Let \((x, y) \in X\) and \(x \notin Q\). We suppose that there exist integers \(e, f\) such that \(y = ex + f\). If \(e \geq 0\), then there exists an integer \(n \geq 0\) such that \(y_n = 0\). If \(e < 0\), then there exists an integer \(n \geq 0\) such that \(y_n = 1 - x_n\).

Proof. Let \(e \geq 0\). Since \(0 \leq ex + f \leq 1\), we see that \(-e < f \leq 0\) for \(e > 0\) and \(f = 0, 1\) for \(e = 0\) respectively. If \(b_1(x, y) > 0\), then we have

\[
y_2 = b_1(x, y) - \frac{y}{x} = -f \left( \frac{1}{x} - a_1(x) \right) - fa_1(x) + b_1(x, y) - e
\]

If \(b_1(x, y) = 0\), then we have \(y_2 = \frac{1}{x} - \frac{y}{x} = (1 - f)\left( \frac{1}{x} - a_1(x) \right) + (1 - f)a_1(x) - e\). Therefore, by the induction for each integer \(n > 0\) there exists integers \(r_n \) and \(s_n\) such that \(y_n = r_nx_n + s_n\), \(r_n \geq 0\) and \(r_n \geq r_{n+1}\) for \(r_n > 0\). We see also that if \(r_n > 0\) and \(b_1(x, y) > 0\), then \(r_n > r_{n+1}\). Since from Lemma 2.2 we see \(b_n(x, y) > 0\) for infinitely many \(n\), there exists a integer \(m > 0\) such that \(r_m = 0\). Therefore, \(y_m = 0\) or \(y_m = 1\). If \(y_m = 1\), then we have \(y_{m+1} = 0\). Thus, we have Lemma.

Let \(e < 0\). Since \(0 \leq ex + f \leq 1\), we see that \(0 < f \leq |e|\). We suppose that \(b_1(x, y) > 0\). Then, we have \(y_2 = -fx_2 - fa_1(x) + b_1(x, y) - e\). We see easily that if \(f = -e = 1\), then we have \(-fa_1(x) + b_1(x, y) - e = 1\) if \(f = -e > 1\), then we have \(-fa_1(x) + b_1(x, y) - e < f\). Next, we suppose that \(b_1(x, y) = 0\). Since the fact that \(f = 1\) implies \(b_1(x, y) > 0\), we see \(f < 1\). Then, \(y_2 = (1 - f) \cdot \left( \frac{1}{x} - a_1(x) \right) + (1 - f)a_1(x) - e\). Therefore, by the induction we see that for each integer \(n > 0\) there exists integers \(r_n \) and \(s_n\) such that \(y_n = r_nx_n + s_n\), \(r_n < 0\) and \(|r_n| \geq |r_{n+1}|\). We see also that if \(|r_n| = |r_{n+1}|\) and \(|r_n| > 1\), then \(|r_{n+2}| > |r_{n+1}|. Therefore, there exists an integer \(m > 0\) such that \(r_m = -1\) and \(s_m = 1\). \(\square\)

Lemma 2.9. Let \((x, y) \in X\), \(x \notin Q\) and \((x, y) \notin \Psi\). Then, following (1) or (2) holds:

1. there exists integer \(m > 0\) such that for any integer \(k \geq 0\) \(b_{m+2k}(x, y) = 0\);
2. there exists integer \(m > 0\) such that for any integer \(n \geq m\) \(a_n(x) = b_n(x, y)\).

Proof. From Lemma 2.8 there exists an integer \(m\) such that \(y_m = 0\) or \(y_m = 1 - x_m\). We suppose \(y_m = 0\). Then, we see that for each integer \(k \geq 0\) \(b_{m+1+2k}(x, y) = 0\). Next, we suppose \(y_m = 1 - x_m\). Then, we see that for each integer \(n \geq m\) \(a_n(x) = b_n(x, y)\). \(\square\)
LEMMA 2.10. Let \( \{a_n\}_{n=1,2,...} \) and \( \{b_n\}_{n=1,2,...} \) be integral sequences such that for any integer \( n > 0 \)
1. \( a_n > 0 \) and \( a_n \geq b_n \geq 0 \),
2. if \( b_n = 0 \), then \( b_{n+1} = 1 \),
3. if \( a_n = 0 \), then there exists an integer \( k > 0 \) such that \( b_{n+2k} = 0 \),
4. if \( a_n = b_n \), then there exists an integer \( k > 0 \) such that \( a_{n+k} \neq b_{n+k} \).

Then, there exists \( (x, y) \in X \cap \Psi \) such that \( a_n = a_n(x) \) and \( b_n = b_n(x, y) \).

PROOF. We define \( \Delta_{m,n} \) for integers \( m \) and \( n \) with \( m > 0 \) and \( m \geq n \geq 0 \) as follows:
\[
\pi_{m,n} = \begin{cases} 
\{(x, y) \in [0, 1]^2 \mid \frac{1}{m+1} \leq x \leq \frac{1}{m}, (n-1)x \leq y \leq nx\} & \text{if } n \geq 1, \\
\{(x, y) \in [0, 1]^2 \mid \frac{1}{m} \leq x \leq \frac{1}{m}, y \geq mx\} & \text{if } m \geq n \text{ and } n = 0.
\end{cases}
\]

We define transformation \( T(a, b) \) on \( \mathbb{R}^2 \) for integers \( a, b \) with \( a > 0 \) and \( a \geq b \geq 0 \) as follows:
\[
T(a, b)(x, y) = \begin{cases} 
\left(\frac{1}{x} - a, b - \frac{y}{x}\right) & \text{if } b > 0, \\
\left(\frac{1}{x} - a, \frac{1}{x}\right) & \text{if } b = 0.
\end{cases}
\]

Similarly, we define transformation \( F(a, b) \) on \( \mathbb{R}^2 \) for integers \( a, b \) with \( a > 0 \) and \( a \geq b \geq 0 \) as follows:
Lemma 2.9. We see that for large $T$, we have $T^{m-1} = T^{m-1}$. Further, we have $T^{m-1} = T^{m-1}(u_n, v_n) = (a_m', b_m')$ for $n > 0$ and $a_m' = a_m' + 1 = b_m'$. Since $T^{m-1} = T^{m-1}(u_n, v_n) = (a_m', b_m')$, we obtain $T^{m-1} = T^{m-1}(u_n, v_n) = (a_m' + 1, 1)$ as $i \to \infty$. Then, we have $b_m' = 1$. By the induction, we see that $b_m' = 1$ for any $m > 0$. But it contradicts the condition of $\{b_n\}_{n=1,2,...}$. Secondly, we suppose that $b_m' = 0$. Since $T^{m-1}(u_n, v_n) = (a_m', b_m'a_m')$, we see that $T^{m-1}(u_n, v_n) = (a_m' + 1, a_m')$ and $b_m' = 1$. Then, we see easily that $T^{m-1}(u_n, v_n) = (a_m' + 2, 0)$ as $i \to \infty$. By the induction, we see that $b_m' = 1$ for any $m > 0$ and $a_m' + 2 = 0$ for any odd $j > 0$. But it contradicts the condition of $\{b_n\}_{n=1,2,...}$. Therefore, $b_m' = b_n$ for any integer $n > 0$. From Lemma 2.9 we see $(\alpha, \beta) \in \Pi$. Thus, we have Lemma.

\[F(a,b)(x,y) = \begin{cases} \frac{b-y}{x+a}, & \text{if } b > 0, \\ \frac{1-y}{x+a}, & \text{if } b = 0. \end{cases}\]

We can easily check $F(a,b) \circ T(a,b) = T(a,b) \circ F(a,b) = \text{identity map.}$

We define $Y = \{(x,y) \in X \mid y \leq x\}. \quad$ Then, we see that if $b > 0$, then $\pi_{a,b} = F(a,b)(X)$ and $F(a,b) : X \to \pi_{a,b}$ is bijective and if $b = 0$, then $\pi_{a,b} = F(a,b)(Y)$ and $F(a,b) : Y \to \pi_{a,b}$ is bijective. Noting that $F(a,1)(X) \subset Y$, we see that if $b > 0$, then $F(a_{b_1}) \cdots F(a_{b_{-1}}, b_{n-1}) F(a_{b_n}) X$ is included in $X$ and it become a quadrangle with inner points. Similarly, we get that if $b = 0$, then $F(a_{b_1}) \cdots F(a_{b_{-1}}, b_{n-1}) F(a_{b_n}) Y$ is included in $X$ and it become a triangle with inner points. If $b > 0$, let $(u_n, v_n)$ be an inner point in $F(a_{b_1}) \cdots F(a_{b_{-1}}, b_{n-1}) F(a_{b_n}) X$ and $F(a_{b_1}) \cdots F(a_{b_{-1}}, b_{n-1}) F(a_{b_n}) Y$. It is not difficult to see that $a_k(u_n) = a_k$ and $b_k(u_n, v_n) = b_k$ for $k = 1, 2, \ldots, n$. Since $X$ is compact, there exist an increasing integral sequence $\{n_i\}$ and $(\alpha, \beta) \in X$ such that $(u_n, v_n) \to (\alpha, \beta)$ as $i \to \infty$. Let $(\alpha_n, \beta_n) = T^{n-1}(\alpha, \beta)$. By continued fraction theory $a_k(\alpha) = a_k$ for any integer $k > 0$. We suppose that there exists an integer $m > 0$ such that $b_m(\alpha, \beta) \neq b_m$. Let $m' > 0$ be an integer such that $b_m'(\alpha, \beta) \neq b_m'$. And for any $0 < k < m'$, $b_k(\alpha, \beta) = b_k$. Then, we have $T^{m'-1}(u_n, v_n) \to (\alpha_m', b_m')$ as $i \to \infty$. On the other hand, we see that for large $i$ $T^{m'-1}(u_n, v_n) \in \pi_{a_m', b_m'}$. Therefore, $(\alpha_m', b_m')$ is in the boundary set of $\pi_{a_m', b_m'}$. Therefore, we see easily that $b(\alpha_m', b_m') = b(\alpha_m', b_m') 
eq 0$ (see Figure 2.2). Further more, if $b(\alpha_m', b_m') < a(\alpha_m', b_m')$, then we have $b(\alpha_m', b_m') + 1 = b_m'$ and if $b(\alpha_m', b_m') = a(\alpha_m', b_m')$, then we have $b_m' = 0$. First, we suppose that $b(\alpha_m', b_m') + 1 = b_m'$. Since $T^{m'-1}(u_n, v_n) \to (\alpha_m', b(\alpha_m', b_m') a_m')$, we obtain $T^{m'}(u_n, v_n) \to (a_m', b_m')$ as $i \to \infty$. Then, we have $b_m' = 1$. By the induction we see that $b_m' = 1$ for any even $m > 0$ and $b_m' + 1$ for any odd $m > 0$. But it contradicts the condition of $\{b_n\}_{n=1,2,...}$. Secondly, we suppose that $b_m' = 0$. Since $T^{m'-1}(u_n, v_n) \to (\alpha_m', b_m'a_m')$ as $i \to \infty$, we see that $T^{m'}(u_n, v_n) \to (\alpha_m' + 1, a_m')$ and $b_m' = 1$. Then, we see easily that $T^{m'}(u_n, v_n) \to (\alpha_m' + 2, 0)$ as $i \to \infty$. By the induction, we see that $b_m' = 1$ for any even $m > 0$ and $\alpha_m' + 2 = 0$ for any odd $m > 0$. But it contradicts the condition of $\{b_n\}_{n=1,2,...}$. Therefore, $b_n(\alpha, \beta) = b_n$ for any integer $n > 0$. From Lemma 2.9 we see $(\alpha, \beta) \in \Pi$. Thus, we have Lemma. \[\square\]

Lemma 2.11. Let $(x,y) \in X$ and $x \notin \mathbb{Q}$. Then,
\begin{enumerate}
\item $b_n(x,y) \geq 0$ for any $n > 0$ and $A_n(x,y) \geq 0$ for any $n > 1$,
\end{enumerate}
(2) \( \limsup_{n \to \infty} B_n(x, y) = \infty \) and \( \limsup_{n \to \infty} A_n(x, y) = \infty \).

(3) If \( (x, y) \in \Psi \), then \( \lim_{n \to \infty} B_n(x, y) = \infty \) and \( \lim_{n \to \infty} A_n(x, y) = \infty \).

**Proof of (1).** We suppose that \( B_n(x, y) < 0 \) for some integer \( n > 0 \). Without loss of generality we suppose that \( B_j(x, y) \geq 0 \) for any integer \( 0 < j < n \). \( B_1(x, y) \geq 0 \) implies \( n > 1 \). From the fact that \( B_{n-1}(x, y) \geq 0 \) and \( B_n(x, y) < 0 \) we see \( b_n(x, y) = 0 \). Then, we have \( B_n(x, y) = B_{n-1}(x, y) - q_{n-2}(x) \). By Lemma 2.2 we have \( b_{n-1}(x, y) > 0 \). If \( n - 1 > 1 \), then we have \( B_{n-1}(x, y) - q_{n-2}(x) = B_{n-2}(x, y) + (b_{n-1}(x, y) - 1)q_{n-2}(x) \geq 0 \). But it is a contradiction. If \( n - 1 = 1 \), then we have \( B_{n-1}(x, y) - q_{n-2}(x) = b_1(x, y) - 1 \geq 0 \). But it is a contradiction. Similarly, we see \( A_n(x, y) \geq 0 \) for any \( n > 1 \).

**Proof of (2).** First, we are proving that \( B_{n+2}(x, y) \geq B_n(x, y) \) for any \( n \geq 1 \) and equation holds iff \( b_{n+1}(x, y) = 1 \) and \( b_{n+2}(x, y) = 0 \). If \( b_{n+1}(x, y) > 0 \) and \( b_{n+2}(x, y) > 0 \), then the proof is easy. We suppose that \( b_{n+1}(x, y) = 0 \) and \( b_{n+2}(x, y) = 1 \). Then, we have \( B_{n+1}(x, y) = B_n(x, y) - q_{n-1}(x) \) and \( B_{n+2}(x, y) = B_{n+1}(x, y) + b_{n+2}(x, y)q_{n+1}(x) \). Therefore, we have \( B_{n+2}(x, y) > B_n(x, y) \). Next, we suppose that \( b_{n+1}(x, y) > 0 \) and \( b_{n+2}(x, y) = 0 \). Then, we have \( B_{n+1}(x, y) = B_n(x, y) + b_{n+1}(x, y)q_n(x) \) and \( B_{n+2}(x, y) = B_{n+1}(x, y) - q_n(x) \). Therefore, we see \( B_{n+2}(x, y) - B_n(x, y) = (b_{n+1}(x, y) - 1)q_n(x) \), which implies that \( B_{n+2}(x, y) \geq B_n(x, y) \) and the equation holds iff \( b_{n+1}(x, y) = 1 \). Therefore, we see that \( \lim_{n \to \infty} B_{2n}(x, y) < \infty \) iff there exists some integer \( m > 0 \) such that for any \( n > m \) \( b_{2n}(x, y) = 0 \) and \( b_{2n-1}(x, y) = 1 \). We suppose that for some integer \( m > 0 \) for any \( n > m \) \( b_{2n}(x, y) = 0 \) and \( b_{2n-1}(x, y) = 1 \). Then, we obtain \( \lim_{n \to \infty} B_{2n+1}(x, y) = \infty \). Thus we have the proof of (2).

**Proof of (3).** From the proof of (2) we see that \( \lim_{n \to \infty} B_{2n}(x, y) < \infty \) iff there exists some integer \( m > 0 \) such that for any \( n > m \) \( b_{2n}(x, y) = 1 \) and \( b_{2n-1}(x, y) = 0 \). By Lemma 2.6 we see that \( \lim_{n \to \infty} B_{2n}(x, y) = \infty \). Similarly, we have \( \lim_{n \to \infty} B_{2n+1}(x, y) = \infty \). Thus, we have \( \lim_{n \to \infty} B_n(x, y) = \infty \). Similarly, we have \( \lim_{n \to \infty} A_n(x, y) = \infty \).

**Lemma 2.12.** Let \( (x, y) \in X \cap \Psi \). For any integer \( n \geq 1 \), \( |B_n(x, y)x - A_n(x, y) - y| \geq |B_{n+2}(x, y)x - A_{n+2}(x, y) - y| \). The equation holds if and only if \( b_{n+2}(x, y) = 0 \) and \( b_{n+1}(x, y) = 1 \) \( (B_n(x, y) = B_{n+2}(x, y)) \).

**Proof.** First, we suppose that \( b_{n+1}(x, y) \geq 1 \). We also suppose that \( n \) is odd. From Lemma 2.1 and Lemma 2.3, we have

\[
B_{n+1}(x, y)x - A_{n+1}(x, y) < y < B_{n+1}(x, y)x - A_{n+1}(x, y) - (q_n(x)x - p_n(x)) \\
\leq B_n(x, y)x - A_n(x, y).
\]
We suppose $b_{n+2}(x, y) = 0$. Then, since $B_{n+2}(x, y)x - A_{n+2}(x, y) = B_{n+1}(x, y)x - A_{n+1}(x, y) - (q_n(x) - p_n(x))$, by (2) we get $y < B_{n+2}(x, y)x - A_{n+2}(x, y) \leq B_n(x, y)x - A_n(x, y)$, which follows the lemma. We remark that $B_{n+1}(x, y)x - A_{n+1}(x, y) - (q_n(x) - p_n(x)) = B_n(x, y)x - A_n(x, y)$ if and only if $b_{n+1}(x, y) = 1$. We suppose $b_{n+2}(x, y) > 0$. Then, from Lemma 2.1 and Lemma 2.3, we have $0 < b_{n+2}(x, y)(q_{n+1}(x)x - p_{n+1}(x)) < -(q_n(x) - p_n(x))$. Therefore, we get

$$B_{n+2}(x, y)x - A_{n+2}(x, y) < B_{n+1}(x, y)x - A_{n+1}(x, y) - (q_n(x) - p_n(x))$$

$$\leq B_n(x, y)x - A_n(x, y),$$

which implies Lemma. We can prove similarly in the case of even $n$. Next, we suppose that $b_{n+1}(x, y) = 0$. Then, from Lemma 2.1 and Lemma 2.3, we have

$$B_{n+1}(x, y)x - A_{n+1}(x, y) < y < B_{n+1}(x, y)x - A_{n+1}(x, y) + (q_n(x) - p_n(x))$$

$$= B_n(x, y)x - A_n(x, y).$$

Using $b_{n+2}(x, y) = 1$, we get $B_{n+2}(x, y)x - A_{n+2}(x, y) = B_{n+1}(x, y)x - A_{n+1}(x, y) + (q_{n+1}(x) - p_{n+1}(x)) < B_n(x, y)x - A_n(x, y)$, which implies Lemma. We can prove similarly in the case of even $n$. □

**Lemma 2.13.** Let $(x, y) \in X \cap \Psi$. If $n > 0$ is odd, then $B_n(x, y)x - A_n(x, y) - y > 0$ and for any integers $m, j$ with $0 < m < B_n(x, y)$, if $mx - j - y > 0$, then

$$B_n(x, y)x - A_n(x, y) - y < mx - j - y.$$ 

If $n > 0$ is even, then $B_n(x, y)x - A_n(x, y) - y < 0$ and for any integers $m, j$ with $0 < m < B_n(x, y)$, if $mx - y - j < 0$, then

$$B_n(x, y)x - A_n(x, y) - y > mx - y - j.$$ 

**Proof.** We are proving the lemma by using the induction on $n$. Let $n = 1$. From Lemma 2.3 we have $B_1(x, y)x - A_1(x, y) - y = x_1y_2 > 0$. We suppose that there exist integers $m, k$ with $0 < m < B_1(x, y)$ such that $mx - j - y > 0$ and $B_1(x, y)x - A_1(x, y) - y \geq mx - j - y$. Let $b_1(x, y) = 0$. Then, from the fact $B_1(x, y) = 0$ we have a contradiction. Let $b_1(x, y) > 0$. Then, we have $B_1(x, y) = b_1(x, y)$ and $A_1(x, y) = 0$. We see that $mx - y = B_1(x, y)x - y + (m - B_1(x, y))x = x_1y_2 + (m - B_1(x, y))x < 0$. Therefore, $mx - j - y > 0$ implies $j < 0$. On the other hand, we have $B_1(x, y)x - mx = y + x_1y_2 - mx < 1$. By the assumption, we see $0 < B_1(x, y)x - y - (mx - j - y) = B_1(x, y)x - mx + j$. On the other hand, $B_1(x, y)x - mx < 1$ and $j < 0$ implies $B_1(x, y)x - mx + j < 0$. This is a contradiction. Thus we have the proof for $n = 1$. We suppose that the lemma
holds for any \( n \) with \( 1 \leq n \leq k \). Let \( n = k + 1 \). We suppose that \( k + 1 \) is odd. From Lemma 2.3 we have \( B_{k+1}(x, y)x - A_{k+1}(x, y) - y > 0 \). We suppose that there exist integers \( m, j \) with \( 0 < m < B_{k+1}(x, y) \) such that \( B_{k+1}(x, y)x - A_{k+1}(x, y) - y > mx - j - y > 0 \). We suppose \( b_{k+1}(x, y) > 0 \). First, we suppose \( m \geq B_k(x, y) \). Since \( B_{k+1}(x, y) - m \leq B_{k+1}(x, y) - B_k(x, y) = b_{k+1}(x, y)q_k(x) < q_{k+1}(x) \), from Lemma 2.1 we obtain \( |(B_{k+1}(x, y) - m)x - A_{k+1}(x, y) + j| \geq |q_k(x)x - p_k(x)| \). On the other hand, by using Lemma 2.3 we have

\[
|(B_{k+1}(x, y) - m)x - A_{k+1}(x, y) + j|
= B_{k+1}(x, y)x - A_{k+1}(x, y) - y - (mx - j - y)
< B_{k+1}(x, y)x - A_{k+1}(x, y) - y < |q_k(x)x - p_k(x)|.
\]

But it is a contradiction. Secondly, we suppose \( m < B_k(x, y) \). If \( m \leq B_{k-1}(x, y) \), using Lemma 2.12 we have a contradiction from the assumption of the induction. Therefore, we have \( m > B_{k-1}(x, y) \). We suppose \( b_k(x, y) > 0 \). Since \( B_k(x, y) - m \leq B_k(x, y) - B_{k-1}(x, y) = b_k(x, y)q_{k-1}(x) < q_k(x) \), from Lemma 2.1 we have \( |(B_k(x, y) - m)x - A_k(x, y) + j| \geq |q_{k-1}(x)x - p_{k-1}(x)| \). On the other hand, we obtain

\[
|(B_k(x, y) - m)x - A_k(x, y) + j|
= mx - j - y - (B_k(x, y)x - A_k(x, y) - y)
< B_{k+1}(x, y)x - A_{k+1}(x, y) - y - (B_k(x, y)x - A_k(x, y) - y)
= b_{k+1}(x, y)|q_k(x)x - p_k(x)|.
\]

From Lemma 2.1 we have \( b_{k+1}(x, y)|q_k(x)x - p_k(x)| < |q_{k-1}(x)x - p_{k-1}(x)| \). But it is a contradiction. Next, we suppose \( b_k(x, y) = 0 \). Then, since \( B_{k-1}(x, y) > B_k(x, y) \), the fact \( m > B_{k-1}(x, y) \) contradicts the assumption \( m < B_k(x, y) \). Secondly, we suppose \( b_{k+1}(x, y) = 0 \). If \( m \leq B_{k-1}(x, y) \), then it contradicts the assumption of the induction. Therefore, we have \( m > B_{k-1}(x, y) \) by using Lemma 2.12. Since \( B_{k+1}(x, y) - m < B_{k+1}(x, y) - B_{k-1}(x, y) = (b_k(x, y) - 1)q_{k-1}(x) < q_k(x) \), by using Lemma 2.1 we have \( |(B_{k+1}(x, y) - m)x - A_k(x, y) + j| \geq |q_{k-1}(x)x - p_{k-1}(x)| \). On the other hand, we see

\[
|(B_{k+1}(x, y) - m)x - A_k(x, y) + j|
= B_{k+1}(x, y)x - A_{k+1}(x, y) - y - (mx - j - y)
< B_{k+1}(x, y)x - A_{k+1}(x, y) - y - (B_k(x, y)x - A_k(x, y) - y)
= |q_{k-1}(x)x - p_{k-1}(x)|.
\]
But it is a contradiction. For even \( k + 1 \) we have a proof similarly. Therefore, we have the proof for \( n = k + 1 \). Thus, we obtain the lemma.

\[ \square \]

**Lemma 2.14.** Let \((x, y) \in X \cap \Psi\). Let \( n > 0 \) be an integer. Then, \( B_n(x, y) \leq q_n(x) + q_{n-1}(x) \). If \( b_n(x, y) > 0 \), then \( B_n(x, y) \geq q_{n-1}(x) \). If \( b_n(x, y) = 0 \), then \( B_n(x, y) \leq q_{n-1}(x) \). Furthermore,

\[ \lim_{n \to \infty} \frac{(B_n(x, y) - q_{n-1}(x))}{b_n(x, y)} = \infty. \]

**Proof.** Let \( n > 0 \) be an integer. Using the induction on \( n \) it is not difficult to see that \( B_n(x, y) \leq q_n(x) + q_{n-1}(x) \). We suppose \( b_n(x, y) > 0 \). Then, we have \( B_n(x, y) - q_{n-1}(x) = B_{n-1}(x, y) + (b_n(x, y) - 1)q_{n-1}(x) \geq B_{n-1}(x, y) \). Therefore, using Lemma 2.11, we have \( B_n(x, y) - q_{n-1}(x) \geq 0 \) and

\[ \lim_{n \to \infty} \frac{(B_n(x, y) - q_{n-1}(x))}{b_n(x, y)} = \infty. \]

Let \( n > 0 \) be an integer with \( b_n(x, y) = 0 \). If \( n = 1 \), then we see easily \( B_n(x, y) \leq q_{n-1}(x) \). Let \( n > 1 \). Then, we have \( B_n(x, y) = B_{n-1}(x, y) - q_{n-2}(x) \leq q_{n-1}(x) \).

Following Theorem is a analogous to the result by Komatsu [14].

**Theorem 2.15.** Let \((x, y) \in X \cap \Psi\).

\[ \liminf_{q \to \infty} q\|qx - y\| \]

\[ = \liminf_{n \to \infty} \min_{\tau} \{ B_n(x, y) | B_n(x, y)x - A_n(x, y) - y | \}, \]

\[ \tau(B_n(x, y) - q_{n-1}(x))(B_n(x, y) - q_{n-1}(x))x - (A_n(x, y) - p_{n-1}(x)) - y | \}, \]

where \( q \in \mathbb{Z} \) and for \( z \in \mathbb{R} \) \( \|z\| = \min \{|z - m| \mid m \in \mathbb{Z} \} \) and \( \tau(u) = u \) for \( u > 0 \) and \( \tau(u) = \infty \) for \( u \leq 0 \).

**Proof.** We are proving that for each \( n > 1 \) with \( b_n > 0 \) if for an integer \( q \) \( B_{n-1}(x, y) < q < B_n(x, y) \), then

\[ q\|qx - y\| \]

\[ \geq \min_{j=n, n-1} \{ B_j(x, y) | B_j(x, y)x - A_j(x, y) - y | \}, \]

\[ \tau(B_j(x, y) - q_{j-1}(x))(B_j(x, y) - q_{j-1}(x))x - (A_{j-1}(x, y) - p_j(x)) - y | \}. \]
It follows Theorem 2.15. Let $n > 1$ and $b_n(x, y) > 0$. Let $B_{n-1}(x, y) < q < B_n(x, y)$. We suppose that $n$ is odd. If $qx - q' < B_{n-1}(x, y)x - A_{n-1}(x, y)$ for an integer $q'$, then from Lemma 2.3 we have $|q(qx - q' - y)| > |B_{n-1}(x, y)(B_{n-1}(x, y)x - A_{n-1}(x, y) - y)|$. We suppose that $B_{n-1}(x, y)x - A_{n-1}(x, y) < qx - q' < B_n(x, y)x - A_n(x, y)$ for an integer $q'$. From Lemma 2.13, we have $qx - q' < y$. Since $B_n(x, y)x - A_n(x, y) = B_{n-1}(x, y)(q_{n-1}(x)x - p_{n-1}(x))$, there exists an integer $j$ such that $0 \leq j < b_n(x, y)$ and

$$j(q_{n-1}(x)x - p_{n-1}(x)) \leq qx - q' - (B_{n-1}(x, y)x - A_{n-1}(x, y))$$

$$< (j + 1)(q_{n-1}(x)x - p_{n-1}(x)).$$

Then, we have $|(q - B_{n-1}(x, y) - jq_{n-1}(x))x - q' + A_{n-1}(x, y) + jp_{n-1}(x)| < |B_{n-1}(x, y)x - p_{n-1}(x)|$. On the other hand, we have $|q - B_{n-1}(x, y) - jq_{n-1}(x)| < b_n(x, y)q_{n-1}(x) < q_n(x)$. Using Lemma 2.1 we have $q - B_{n-1}(x, y) - jq_{n-1}(x) = 0$. We see easily that $q' - A_{n-1}(x, y) - jp_{n-1}(x) = 0$. Then, we have

$$q|qx - q' - y| = (B_{n-1}(x, y) + jq_{n-1}(x))(|B_{n-1}(x, y) + jq_{n-1}(x)|x$$

$$- (A_{n-1}(x, y) + jp_{n-1}(x)) - y|$$

$$\geq \min_{0 \leq l \leq b_n(x, y) - 1} \{(B_{n-1}(x, y) + lj_{n-1}(x))(B_{n-1}(x, y) + lj_{n-1}(x))x$$

$$- (A_{n-1}(x, y) + lj_{n-1}(x)) - y|\}.$$
\[ q|qx - q' - y| \geq \min \{ B_{n-1}(x, y)|B_{n-1}(x, y)x - A_{n-1}(x, y) - y|, \]
\[ (B_n(x, y) - q_{n-1}(x))(B_n(x, y) - q_{n-1}(x))x - (A_n(x, y) - p_{n-1}(x)) - y|}. \]

We suppose that \( B_n(x, y)x - A_n(x, y) < qx - q' \) for an integer \( q' \). We consider the case of \( b_{n-1}(x, y) > 0 \). We suppose \( B_n(x, y)x - A_n(x, y) - (q_{n-2}(x)x - p_{n-2}(x)) \leq qx - q' \). Then, we have \( y < B_{n-1}(x, y)x - A_{n-1}(x, y) - (q_{n-2}(x)x - p_{n-2}(x)) \leq qx - q' \). Therefore, noting \( B_{n-1}(x, y) - p_{n-2}(x) \geq 0 \) from Lemma 2.14, we have
\[ q|qx - q' - y| \geq (B_{n-1}(x, y) - q_{n-2}(x)) \times |(B_{n-1}(x, y) - q_{n-2}(x))x - (A_{n-1}(x, y) - p_{n-2}(x)) - y|. \]

Next, we suppose \( B_n(x, y)x - A_n - (q_{n-2}(x)x - p_{n-2}(x)) > qx - q' \). Then, we have \( 0 < qx - q' - (B_n(x, y)x - A_n(x, y)) < -(q_{n-2}(x)x - p_{n-2}(x)) \). Noting \( 0 < B_n(x, y) - q < b_n(x, y)q_{n-1}(x) \), similarly to the previous argument, we see that there exists an integer \( j' \) such that \( 0 \leq j' < b_n(x, y) \) and \( (B_n(x, y)x - A_n(x, y)) - (qx - q') = q_{n-2}(x)x - p_{n-2}(x) + j'(q_{n-1}(x)x - p_{n-1}(x)) \). Therefore, we have
\[ qx - q' = B_n(x, y)x - A_n(x, y) - (q_{n-2}(x)x - p_{n-2}(x)) - j'(q_{n-1}(x)x - p_{n-1}(x)) \]
\[ = B_{n-1}(x, y)x - A_{n-1}(x, y) - (q_{n-2}(x)x - p_{n-2}(x)) \]
\[ + (b_n(x) - j')(q_{n-1}(x)x - p_{n-1}(x)). \]

Using (5) and \( B_{n-1}(x, y)x - A_{n-1}(x, y) - q_{n-2}(x)x - p_{n-2}(x) > y \), we see \( 0 < B_{n-1}(x, y)x - A_{n-1}(x, y) - (q_{n-2}(x)x - p_{n-2}(x)) - y < qx - q' - y \). Therefore,
\[ q|qx - q' - y| > (B_{n-1}(x, y) - q_{n-2}(x)) \times |B_{n-1}(x, y)x - A_{n-1}(x, y) - (q_{n-2}(x)x - p_{n-2}(x)) - y|. \]

We consider the case of \( b_{n-1}(x, y) = 0 \). We suppose that \( B_n(x, y)x - A_n(x, y) - (q_{n-2}(x)x - p_{n-2}(x)) \leq qx - q' \). Since \( B_n(x, y)x - A_n(x, y) - (b_{n-1}(x, y)x - A_{n-1}(x, y)) = q_{n-1}(x)x - p_{n-1}(x) \), we have \( 0 < y - (B_{n-1}(x, y)x - A_{n-1}(x, y)) < q_{n-1}(x)x - p_{n-1}(x) \). On the other hand, we obtain \( qx - q' - y > qx - q' - (B_n(x, y)x - A_n(x, y)) \geq -(q_{n-2}(x)x - p_{n-2}(x)) \). Therefore, \( q|qx - q' - y| > B_{n-1}(x, y)|B_{n-1}(x, y)x - A_{n-1}(x, y) - y| \). Secondly, we suppose \( B_n(x, y)x - A_n(x, y) - (q_{n-2}(x)x - p_{n-2}(x)) > qx - q' \). Then, \( 0 < qx - q' - (B_n(x, y)x - A_n(x, y)) < -(q_{n-2}(x)x - p_{n-2}(x)) \). Using \( 0 < B_n(x, y) - q < q_{n-1}(x) \) and Lemma
2.1, we have a contradiction. Therefore, we have the inequality (4). Thus, we have Lemma.

**Lemma 2.16.** Let \((x, y) \in X \cap \Psi\). For any integer \(n > 0\),

\[
\liminf_{q \to \infty} q\|qx - y\| = \liminf_{q \to \infty} q\|qx_n - y_n\|,
\]

where \((x_n, y_n) = T^{-1}(x, y)\).

**Proof.** We are proving that \(\liminf_{q \to \infty} q\|qx - y\| = \liminf_{q \to \infty} q\|qx_2 - y_2\|\). It follows the lemma. Let \(e = \liminf_{q \to \infty} q\|qx - y\|\) and \(f = \liminf_{q \to \infty} q\|qx_2 - y_2\|\). Then, there exist an increasing positive integral sequences \(\{p'_k\}_{k=1,2,\ldots}\) and \(\{q'_k\}_{k=1,2,\ldots}\) such that \(f = \liminf_{k \to \infty} q'_k|q'_kx_2 - y_2 - p'_k|\). We suppose that \(b_1(x, y) > 0\). Then, for \(k > 0\) we have

\[
q'_k|q'_kx_2 - y_2 - p'_k| = q'_k\left|q'_k\left(\frac{1}{x_1} - a_1(x)\right) - \left(b_1(x, y) - \frac{y_1}{x_1}\right) - p'_k\right|
\]

\[
= q'_k\left|\left(q'_ka_1(x) + p'_k + b_1(x, y)\right)x_1 - y_1 - q'_k\right|
\]

\[
= (q'_ka_1(x) + p'_k + b_1(x, y))(q'_ka_1(x) + p'_k + b_1(x, y))x_1 - y_1 - q'_k
\]

\[
q'_k\left(\frac{q'_k}{x_1} - \frac{q'_k}{x_1}a_1(x) + p'_k + b_1(x, y)\right).
\]

Since \(\frac{p'_k}{q'_k} \to x_2\) as \(k \to \infty\), we see that \(\lim_{k \to \infty} x_1(q'_ka_1(x) + p'_k + b_1(x, y)) = \lim_{k \to \infty} \frac{q'_k}{x_1}a_1(x) + \frac{p'_k}{q'_k} + \frac{b_1(x, y)}{q'_k} = 1\). Thus, \(e \leq f\). If \(b_1(x, y) = 0\), we have \(e \leq f\) by the same manner. Similarly, we have \(e \geq f\). Thus, we have the lemma.

**3. Natural Extension**

\(\mathbb{Z}^+\) denotes the set of all positive integers. We define \(\Omega_1\), \(\Omega_2\), \(\Omega'_1\) and \(\Omega'_2\) as follows:

\[
\Omega_1 = \{(x, y) \in [0, 1]^2 \mid (x, y) \in \Psi, y \leq x\},
\]

\[
\Omega_2 = \{(x, y) \in [0, 1]^2 \mid (x, y) \in \Psi, y > x\},
\]

\[
\Omega'_1 = \{(x, y) \mid (x, y) \in \Psi, y > 1, x \leq -1, y \leq -x + 1\},
\]

\[
\Omega'_2 = \{(x, y) \mid (x, y) \in \Psi, 0 \leq y \leq 1, x \leq -1\}.
\]
Let \( \Omega = \{ \Omega_1 \times (\Omega'_1 \cup \Omega'_2) \} \cup (\Omega_2 \times \Omega'_1) \).

We define a transformation \( \bar{T} \) on \( \Omega \) as follows: for \((x, y, z, w) \in \Omega\)

\[
\bar{T}(x, y, z, w) = \begin{cases} 
\left( \frac{1}{x} - a(x), b(x, y) - \frac{y}{z}, a(x), b(z, w) - \frac{w}{z} \right) & \text{if } b(x, y) > 0, \\
\left( \frac{1}{x} - a(x), \frac{1}{z} - \frac{y}{z}, \frac{1}{z} - a(x), \frac{1}{z} - \frac{w}{z} \right) & \text{if } b(x, y) = 0.
\end{cases}
\]

We see easily that \( \bar{T} \) is well defined.

**Theorem 3.** \( \bar{T} \) is bijective.

**Proof.** We define \( \Delta_{m,n} \) for \( m \in \mathbb{Z}_+ \) and \( n \in \mathbb{Z}_+ \cup \{0\} \) with \( m \geq n \) as follows;

\[
\Delta_{m,n} = \begin{cases} 
\{(x, y) \in X \cap \Psi \mid \frac{1}{m+1} < x < \frac{1}{m}, (n-1)x < y < nx \} & \text{if } n \geq 1, \\
\{(x, y) \in X \cap \Psi \mid \frac{1}{m+1} < x < \frac{1}{m}, y > mx \} & \text{if } m \geq n \text{ and } n = 0.
\end{cases}
\]

Then, we see easily that \( T : \Delta_{m,n} \to X \cap \Psi \) is bijective for \( n > 0 \) and \( T : \Delta_{m,0} \to \Omega_1 \) is bijective. We define \( \Delta'_{m,n} \) for \( m \in \mathbb{Z}_+ \) and \( n \in \mathbb{Z}_+ \cup \{0\} \) with \( m \geq n \) as follows; if \( n = 1 \), then we see \( \Delta'_{m,n} = \{(x, y) \in \Omega'_1 \mid -(m + 1) < x < -m, 1 < y < -x + m + 2 \} \) and if \( n > 1 \), then we see \( \Delta'_{m,n} = \{(x, y) \in \Omega'_1 \mid -(m + 1) < x < -m, -x - m + n < y < -x - m + n + 1 \} \) and if \( n = 0 \), then we see \( \Delta'_{m,n} = \{(x, y) \in \Omega'_2 \mid -(m + 1) < x < -m \} \).

We see that for \( m \in \mathbb{Z}_+ \) and \( n \in \mathbb{Z}_+ \cup \{0\} \) with \( m \geq n \) and \( n \neq 1 \) \( (T_{(m,n)})_{\Omega_1} \Omega'_1 \to \Delta'_{m,n} \) is bijective and \( (T_{(m,1)})_{\Omega_1} \Omega'_1 \cup \Omega'_2 \to \Delta'_{m,1} \) is bijective, where \( T_{(m,n)} \) is defined in Section 2. On the other hand, we have
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\[ \Omega = \bigcup_{(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+, m \geq n, n \neq 1} \Delta_{m,n} \times \Omega_1' \bigcup_{m \in \mathbb{Z}_+} \Delta_{m,1} \times (\Omega_1' \cup \Omega_2') \quad \text{(disjoint)} \]

\[ = \bigcup_{(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+, m \geq n, n \neq 1} (X \cap \Psi) \times \Delta_{m,n}' \bigcup_{m \in \mathbb{Z}_+} (X \cap \Psi) \times (\Delta_{m,1}') \quad \text{(disjoint)} \]

We see that \( \tilde{T}_{\Delta_{m,n} \times \Omega_1} \Delta_{m,n} \times \Omega_1' \rightarrow (X \cap \Psi) \times \Delta_{m,n}' \) is bijective for \( (m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \) with \( n \neq 1 \) and \( \tilde{T}_{\Delta_{m,1} \times (\Omega_1' \cup \Omega_2')} \Delta_{m,1} \times (\Omega_1' \cup \Omega_2') \rightarrow (X \cap \Psi) \times \Delta_{m,1}' \) for \( m \in \mathbb{Z}_+ \) is bijective and \( \tilde{T}_{\Delta_{m,0} \times \Omega_1'} \Delta_{m,0} \times \Omega_1' \rightarrow \Omega_1 \times \Delta_{m,0}' \) is bijective for \( m \in \mathbb{Z}_+ \). Therefore, \( \tilde{T} \) is bijective.

Following Lemma 3.2 is easily proved.

**Lemma 3.2.** Let \( K \) be a real quadratic field over \( \mathbb{Q} \). Let \( (x, y) \in K^2 \cap X \cap \Psi \). Then, if \( (x, y, \bar{x}, \bar{y}) \in \Omega \), then \( (T(x, y), \overline{\mathcal{T}(x, y)}) = \tilde{T}(x, y, \bar{x}, \bar{y}) \), where for \( z \in K \) \( \overline{z} \) is the algebraic conjugate of \( z \) related to \( K/\mathbb{Q} \).

Komatsu [15] determine the all eventually periodic points in \( (X, \mathcal{T}_2) \). Following Lemma is the similar result.
Lemma 3.3. Let \((x, y) \in X \cap \Psi\), \(x\) be a quadratic irrational number and \(y \in \mathbb{Q}(x)\). Then, \((x, y, \bar{x}, \bar{y})\) is an eventually periodic point related to \(\bar{T}\), where for \(z \in \mathbb{Q}(x)\) \(\bar{z}\) is an algebraic conjugate of \(z\) related to \(\mathbb{Q}(x)/\mathbb{Q}\).

Proof. Since \(y \in \mathbb{Q}(x)\), there exist \(r_n, s_n \in \mathbb{Q}\) such that \(y_n = r_n + s_n x_n\). Let \(d_n\) be the denominator of \(r_n, s_n\). By using induction, we see \(d_0 = d_n\) for all \(n\). From the well known fact about continued fraction of quadratic irrational numbers, there exists an integer \(m\) such that \(\{x_m, x_{m+1}, \ldots\}\) is purely periodic. It is known that \(\bar{x}_n < -1\) for each \(n \geq m\). We define a constant \(c_1\) by \(c_1 = \min\{|\bar{x}_n| \mid n \geq m\}\). Let \(c_2 = \max\{a_n(x) \mid n = 1, \ldots\}\). Let \(r = \frac{c_1(c_2 + 1)}{c_1 - 1}\). Then, if \(n > m\) and \(|\bar{y}_n| > r\), we have

\[
|\bar{y}_{n+1}| < c_2 + \frac{\bar{y}_n}{\bar{x}_n} < c_2 + \frac{\bar{y}_n}{c_1} = |\bar{y}_n| - \frac{|\bar{y}_n|(c_1 - 1)}{c_1} + c_2 < |\bar{y}_n| - 1.
\]

Therefore, there exists \(n_1\) such that \(n_1 > m\) and \(|\bar{y}_{n_1}| \leq r\). On the other hand, if \(n > m\) and \(|\bar{y}_n| \leq r\), then we have

\[
|\bar{y}_{n+1}| < c_2 + \frac{\bar{y}_n}{\bar{x}_n} < 2r.
\]

We suppose that \(\limsup_{n \to \infty} |\bar{y}_n| = \infty\). Let \(n_2 = \min\{k \mid k > n_1, |\bar{y}_k| > 3r\}\). We assume \(|\bar{y}_{n_2-1}| > r\). Then, we have \(|\bar{y}_n| < |\bar{y}_{n_2-1}| - 1\). Therefore, we have \(|\bar{y}_{n_2-1}| > 3r\). But it is a contradiction. Next, we assume \(|\bar{y}_{n_2-1}| \leq r\). Then, by using previous argument, we have \(|\bar{y}_{n_2}| \leq 3r\). But it is a contradiction. Thus, there exists \(c > 0\) such that \(|\bar{y}_n| < c\) for all \(n\). From the facts that \(|\bar{y}_n| < c\) and \(|\bar{y}_n| < 1\) for all \(n\), we see that there exits \(c_3\) such that \(|r_n|, |s_n| < c_3\) for all \(n\). Using the fact \(d_0 = d_n\) for all \(n\), we see that \(\{y_n \mid n = 0, 1, \ldots\}\) has finitely many numbers. Thus, \((x, y, \bar{x}, \bar{y})\) is an eventually periodic point related to \(T\). 

Lemma 3.4. Let \((x, y) \in X \cap \Psi\), \(x\) be a quadratic irrational number and \(y \in \mathbb{Q}(x)\), where for \(z \in \mathbb{Q}(x)\) \(\bar{z}\) is an algebraic conjugate of \(z\) related to \(\mathbb{Q}(x)/\mathbb{Q}\). Then, there exists an integer \(n > 0\) such that \((x_n, y_{n+1}, \bar{x}_n, \bar{y}_n) \in \Omega\).

Proof. By Lemma 3.3 \(\{(x_n, y_n) \mid n = 0, 1, \ldots\}\) is eventually periodic. Therefore, there exist integers \(m_1, m_2 > 0\) such that for any \(n \geq m_1\) \((x_{n+m_2}, y_{n+m_2}) = (x_n, y_n)\). We define \(m_1\) as follows. If \(b_n > 0\) for any \(n \geq m_1\), then we set \(m_3 = m_1\). If there exists \(m' \geq m_1\) such that \(b_{m'}(x, y) = 0\), then we set \(m_3 = m'\). If for integers \(a, b\) \(b > 0\) and \(a \geq b\), then it is not difficult to see that \(T(a, b)(c(l(\Omega_1))) \subset \{(x, y) \in \Omega\}\).
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\[ cl(\Omega_1' \backslash \{ -a - 1 \leq x \leq -a \} ) \]

where \( cl(\Omega_1') \) is the closure of \( \Omega_1' \). Therefore, if \( b_n(x, y) > 0 \) for any \( n \geq m_1 \), then we have

\[
T_{(a_{m_3+m_2-1}(x), b_{m_3+m_2-1}(x, y))} \cdots T_{(a_{m_1}, b_{m_1}(x, y))} \eta \subseteq \eta,
\]

where \( \eta = \{ (x, y) \in cl(\Omega_1') \mid -a_{m_3+m_2-1}(x) - 1 \leq x \leq -a_{m_3+m_2-1}(x) \} \). It is not difficult to see that for integers \( a, a' \geq 1 \) \( T_{(a_1, 0)} cl(\Omega_1') \subseteq \{ (x, y) \in cl(\Omega_1') \mid -a - 1 \leq x \leq -a \} \). By lemma 2.2 \( m_2 > 1 \) and \( b_{m_3+m_2-1}(x, y) \neq 0 \). Thus, we have

\[
T_{(a_{m_3+m_2-1}(x), b_{m_3+m_2-1}(x, y))} \cdots T_{(a_{m_1}, b_{m_1}(x, y))} \eta \subseteq \eta.
\]

By Bronwell’s fixed point theorem there exists \( (x', y') \in \{ (x, y) \in cl(\Omega_1') \mid -a_{m_3+m_2-1}(x) - 1 \leq x \leq -a_{m_3+m_2-1}(x) \} \) such that \( T_{(a_{m_3+m_2-1}(x), b_{m_3+m_2-1}(x, y))} \cdots T_{(a_{m_1}, b_{m_1}(x, y))} (x', y') = (x', y') \). We see easily that \( (x', y') = (x_{m_3+m_2}, y_{m_3+m_2}) \). Therefore, we have \( (x_{m_1}, y_{m_1}, x_{m_2}, y_{m_2}) \in \Omega \).

**Lemma 3.5.** Let \( (x, y) \in X \cap \Psi \), \( x \) be a quadratic irrational number and \( y \in Q(x) \). Let \( (x, y, x', y') \in \Omega \), where for \( z \in Q(x) \) \( z \) is an algebraic conjugate of \( z \) related to \( Q(x)/Q \). Then, \( (x, y, x', y') \) is a purely periodic point related to \( \tilde{T} \).

**Proof.** By Lemma 3.3 there exist integers \( m, m_1 \geq 1 \) such that for any integer \( n > m \) \( (x_n, y_n) = (x_{n+m_1}, y_{n+m_1}) \). Since \( (x_1, y_1, x_1, y_1) \in \Omega \), by Lemma 3.2 we have \( (x_n, y_{n}, x_{n}, y_{n}) \in \Omega \) for any integer \( n > 0 \). Since \( \tilde{T} \) is bijective on \( \Omega \), for each integer \( n > m \) we have \( (x_{n-1}, y_{n-1}, x_{n-1}, y_{n-1}) = (x_{n+m_1-1}, y_{n+m_1-1}, x_{n+m_1-1}, y_{n+m_1-1}) \). By using the induction we have \( (x_1, y_1, x_1, y_1) = (x_{1+m_1}, y_{1+m_1}, x_{1+m_1}, y_{1+m_1}) \). Thus, \( (x, y, x', y') \) is a purely periodic point related to \( \tilde{T} \).

**Theorem 3.6.** Let \( (x, y) \in X \cap \Psi \). \( x \) is a quadratic irrational number, \( y \in Q(x) \) and \( (x, y, x', y') \in \Omega \) if and only if \( (x, y) \) is a purely periodic point related to \( T \), where for \( z \in Q(x) \) \( z \) is an algebraic conjugate of \( z \) related to \( Q(x)/Q \).

**Proof.** The necessary condition of the theorem is proved in Lemma 3.5. Let us prove the sufficient condition. We assume that \( (x, y) \in X \cap \Psi \) and \( (x, y) \) is a purely periodic point related to \( T \). Then, it is not difficult to see that \( x \) is a quadratic irrational number and \( y \in Q(x) \). Using Theorem 3.1 and Lemma 3.4, we see that \( (x, y, x', y') \in \Omega \).

Following Lemma 3.7 is a well known result.

**Lemma 3.7 (É. Galois).** Let \( 0 < x < 1 \) be a quadratic irrational number and let \( x \) have purely periodic continued fraction expansion. Then,
\[ \lim_{n \to \infty} \left( \frac{g_n(x)}{q_{n-1}(x)} + \bar{x}_{n+1} \right) = 0, \text{ where for } z \in \mathbb{Q}(x) \bar{z} \text{ is an algebraic conjugate of } z \text{ related to } \mathbb{Q}(x)/\mathbb{Q}. \]

**Proof.** Let \( W = [0, 1] \times (-\infty, -1]. \) We define a transformation \( \rho \) on \( W \) as follows: for \( (x, y) \in W \)

\[
\rho(x, y) = \begin{cases} 
\left( \frac{1}{x} - a(x), \frac{1}{y} - a(x) \right) & \text{if } x \neq 0, \\
(x, y) & \text{if } x = 0.
\end{cases}
\]

We see easily that \( \rho \) is well defined. Since \( x \) is reduced, \( \bar{x} < -1 \) (see [20]). Therefore, \( (x, \bar{x}) \in W. \) We see easily that \( \rho^n(x, \bar{x}) = (x_{n+1}, \bar{x}_{n+1}). \) On the other hand, for each integer \( n > 0 \left( x_{n+1} - \frac{q_n(x)}{q_{n-1}(x)} \right) \in W. \) We see for each integer \( n > 0 \)

\[
\rho \left( x_{n+1} - \frac{q_n(x)}{q_{n-1}(x)} \right) = \left( x_{n+2} - \frac{q_{n-1}(x)}{q_n(x)} - a_{n+1}(x) \right) = \left( x_{n+2} - \frac{q_{n+1}(x)}{q_n(x)} \right).
\]

Therefore, we have \( \rho^{n-1}(x_2, -\frac{q_1(x)}{q_0(x)}) = \left( x_{n+1} - \frac{q_n(x)}{q_{n-1}(x)} \right). \) We denote \( u_n = -\frac{q_n(x)}{q_{n-1}(x)} \) for each integer \( n > 0. \) Then, we have

\[
|x_{n+2} - u_{n+1}| = \frac{|x_{n+1} - u_n|}{|x_{n+1}|} \leq \frac{|x_{n+1} - u_n|}{C},
\]

where \( C = \min\{|x_j| : j = 1, 2, \ldots\}. \) Therefore, we have \( |x_{n+1} - u_n| \leq \frac{|x_j - u_j|}{C^{n-1}} \) for each \( n > 0. \) Since \( C > 1, \) we obtain the lemma.

**Lemma 3.8.** Let \( (x, y) \in X \cap \Psi \) and let \( (x, y) \) be a purely periodic point related to \( T. \) Then, \( \lim_{n \to \infty} \left( \frac{b_n(x, y)}{q_{n-1}(x)} - \bar{y}_{n+1} \right) = 0. \)

**Proof.** We see easily that \( \bar{T} \) is naturally extended to \( \Omega_\# = \{ \Omega_1 \times \text{cl}(\Omega_1') \cup \Omega_2 \times \text{cl}(\Omega_2') \}. \) We also denote it \( \bar{T}. \) For each integer \( k \geq 1 \) \( u_k \) denotes \( -\frac{q_k(x)}{q_{k-1}(x)} \) and \( v_k \) denotes \( \frac{B_k(x, y)}{q_k(x)}. \) First, we show that \( (x_2, y_2, u_1, v_1) \in \Omega_\#. \) and for \( n \geq 1 \) \( \bar{T}^{n-1}(x_2, y_2, u_1, v_1) = (x_{n+1}, y_{n+1}, u_n, v_n). \) We suppose \( b_1(x, y) > 0. \) Then, we see that \( -\frac{q_1(x)}{q_0(x)} = -a_1(x) \) and \( \frac{B_1(x, y)}{q_0(x)} = b_1(x, y). \) Since \( 0 < b_1(x, y) \leq a_1(x, y), \) we have \( \left( x_2, y_2, -\frac{q_1(x)}{q_0(x)}, b_1(x, y) \right) \in \Omega_\#. \) We suppose \( b_1(x, y) = 0. \) Then, we see that \( \frac{B_1(x, y)}{q_0(x)} = 0 \) and \( y_2 = \frac{1}{x_1} - \frac{v_1}{x_1}. \) From the fact that \( a_1 = \left[ \frac{v_1}{x_1} \right], \) we have \( \frac{1}{x_1} - a_1 \geq \frac{1}{x_1} - \frac{v_1}{x_1}. \) Therefore, we have \( \left( x_2, y_2, -\frac{q_1(x)}{q_0(x)}, b_1(x, y) \right) \in \Omega_\#. \) Secondly, we suppose that for an integer \( k > 0 \) \( \bar{T}^{k-1}(x_2, y_2, u_1, v_1) = (x_{k+1}, y_{k+1}, u_k, v_k). \) Then,
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we have \( \frac{1}{u_k} - a_{k+1}(x) = \frac{q_{k-1}(x)}{q_k(x)} - a_{k+1}(x) = u_{k+1} \). We suppose that \( b_{k+1}(x, y) > 0 \). Then, we have \( b_{k+1}(x, y) - \frac{u_k}{u_k} = b_{k+1}(x, y) + \frac{P_k(x, y)}{q_k(x)} = v_{k+1} \). Therefore, we have \( \bar{T}(x_{k+1}, y_{k+1}, u_k, v_k) = (x_{k+2}, y_{k+2}, u_{k+1}, v_{k+1}) \). We suppose that \( b_{k+1}(x, y) = 0 \). Then, we have \( \frac{1}{u_k} = \frac{b_k(x, y) - q_{k-1}(x)}{q_k(x)} = \frac{b_{k+1}(x, y)}{q_k(x)} = v_{k+1} \). Therefore, we have \( \bar{T}(x_{k+1}, y_{k+1}, u_k, v_k) = (x_{k+2}, y_{k+2}, u_{k+1}, v_{k+1}) \). Thus, we have the proof of that for \( n \geq 1 \) \( \bar{T}^{n-1}(x_2, y_2, u_1, v_1) = (x_{n+1}, y_{n+1}, u_n, v_n) \). Since for \( n \geq 1 \) \( \bar{T}^{n-1}(x_2, y_2, \bar{x}_2, \bar{y}_2) = (x_{n+1}, y_{n+1}, \bar{x}_{n+1}, \bar{y}_{n+1}) \). If \( b_{n+1}(x, y) > 0 \), then we obtain

\[
|v_{n+1} - \bar{y}_{n+2}| = \left| \frac{v_n}{u_n} - \frac{\bar{y}_{n+1}}{x_{n+1}} \right| = \left| \frac{v_n}{u_n} - \frac{v_n}{x_{n+1}} + \frac{v_n}{x_{n+1}} - \frac{\bar{y}_{n+1}}{x_{n+1}} \right|
\]

and if \( b_{n+1}(x, y) = 0 \), then we obtain

\[
|v_{n+1} - \bar{y}_{n+2}| = \left| \frac{1}{u_n} - \frac{v_n}{u_n} - \frac{1}{x_{n+1}} + \frac{\bar{y}_{n+1}}{x_{n+1}} \right|
\]

\[\leq \left( 1 + \frac{v_n}{u_n} \right) \left| \frac{x_{n+1} - u_n}{x_{n+1}} \right| + \frac{|v_n - \bar{y}_{n+1}|}{|x_{n+1}|} \tag{6}\]

\[
\text{and if } b_{n+1}(x, y) = 0, \text{ then we obtain}
\]

\[|v_{n+1} - \bar{y}_{n+2}| \leq 3(n - 1) \frac{|x_2 - u_1|}{C_{n-1}} + \frac{|v_1 - \bar{y}_2|}{C_{n-1}}, \tag{7}\]

Since \((u_n, v_n) \in cl(\Omega_1 \cup \Omega_2)^c\), \( \frac{|x_1|}{|u_n|} \leq 2 \) for each integer \( n > 0 \). From the proof of Lemma 3.7, (6) and (7) we see that

\[|v_{n+1} - \bar{y}_{n+2}| \leq 3(n - 1) \frac{|x_2 - u_1|}{C_{n-1}} + \frac{|v_1 - \bar{y}_2|}{C_{n-1}}, \]

where \( C = \min\{|x_j| \mid j = 1, 2, \ldots\} \). Thus, we have the lemma. \( \square \)

**Theorem 3.9.** Let \((x, y) \in [0, 1]^2\) be a periodic point of \( \bar{T} \). Then,

\[\lim_{q \to \infty} q\|qx - y\| = \min\left\{ \frac{y_n\bar{y}_n - \tau(y_n - 1)(1 - y_n)}{x_n - \bar{x}_n} ; n = 0, 1, 2, \ldots \right\},\]

where \( \|x\| = \min\{|m - x| \mid m \in \mathbb{Z}\} \) and \( \tau(u) = u \) for \( u > 0 \) and \( \tau(u) = \infty \) for \( u \leq 0 \).

**Proof.** From Theorem 2.15 we have

\[
\lim_{q \to \infty} q\|qx - y\|
\]

\[= \lim_{n \to \infty} \min\{B_n(x, y) \mid B_n(x, y)x - A_n(x, y) - y(; \tau(B_n(x, y) - q_{n-1}(x))
\]

\[\times \mid(B_n(x, y) - q_{n-1}(x))x - (A_n(x, y) - p_{n-1}(x, y)) - y\mid.\]
Using Lemma 2.1 and Lemma 2.3

\[ B_n(x, y) \left| B_n(x, y)x - A_n(x, y) - y \right| = B_n(x, y)y_{n+1}x_1 \cdots x_n \]

\[ = B_n(x, y)y_{n+1} \left( q_{n-1}(x)x - p_{n-1}(x) \right) \]

\[ = \frac{B_n(x, y)y_{n+1}}{q_{n-1}(x) \left( \frac{q_n(x)}{q_{n-1}(x)} + x_{n+1} \right)} . \]

If \( b_n(x, y) > 0 \), we have similarly

\[ (B_n(x, y) - q_{n-1}(x))(B_n(x, y) - q_{n-1}(x))x - (A_n(x, y) - p_{n-1}(x)) - y \]

\[ = (B_n(x, y) - q_{n-1}(x))(-1)^n y_{n+1}x_1 \cdots x_n - (q_{n-1}(x)x - p_{n-1}(x)) \]

\[ = (B_n(x, y) - q_{n-1}(x))q_{n-1}(x)x - p_{n-1}(x)) \left| 1 - y_{n+1} \right| \]

\[ = \frac{(B_n(x, y) - q_{n-1}(x)) \left| 1 - y_{n+1} \right| \times \frac{1}{q_{n-1}(x) \left( \frac{q_n(x)}{q_{n-1}(x)} + x_{n+1} \right)}} . \]

From Lemma 2.14 we note that if \( b_n(x, y) > 0 \), \( B_n(x, y) - q_{n-1}(x) \leq 0 \) and \( 0 < \frac{1}{y_{n+1}} < 1 \). Using Lemma 3.7 and Lemma 3.8, we have Theorem 3.9. \( \square \)

References


[4] J. W. S. Cassels, Über \( \lim_{x \to +\infty} x(\sqrt{3}x + a - y) \), (German) Math. Ann. 127 (1954), 288–304.


On a New Algorithm for Inhomogeneous Diophantine Approximation


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