Relative Microlinearity
- Towards the General Theory of Fiber Bundles in Infinite-Dimensionl Differential Geometry -

Hirokazu Nishimura
Institute of Mathematics, University of Tsukuba
Tsukuba, Ibaraki, 305-8571, Japan

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Abstract

In our previous papers [Far East Journal of Mathematical Sciences, **35** (2009), 211-223] and [International Journal of Pure and Applied Mathematics, **60** (2010), 15-24] we have developed the theory of Weil prolongation, Weil exponentiability and microlinearity for Frölicher spaces. In this paper we will relativize it so as to obtain the theory of fiber bundles for Frölicher spaces. It is shown that any Weil functor naturally gives rise to a fiber bundle. We will see that the category of fiber bundles over a fixed Frölicher space M and their smooth mappings over M is cartesian closed. We will see also that the category of vector bundles over M and their smooth linear mappings over M is cartesian closed. It is also shown that the tangent bundle functor naturally yields a vector bundle.

1 Introduction

Smooth manifolds have been the central object of study in *finite-dimensional* differential geometry. In our previous paper [20], we proposed that *microlinear* Frölicher spaces will be the central object of study in *infinite-dimensional* (or rather *dimensionless*) differential geometry. The paper was followed by [21] and [22], which dealt with vector fields and differential forms respectively. It is to be followed further by papers which are to be concerned with connections, curvature, jet bundles, and so on. The basic ideas are simple enough. What has been established in synthetic differential geometry, which forces us to work within a well-adapted model so as to make nilpotent infinitesimals visible, can be externalized by using Weil functors. The simplest and best-known Weil functor is the tangent bundle functor, which corresponds to the infinitesimal object of first-order infinitesimals (i.e., real numbers whose squares vanish). Even an ordinary teener of our modern times knows well that such real numbers must be exclusively 0, but, curiously enough, there appeared many such nilpotent infinitesimals to Newton, Leibniz, Euler and other outstanding mathematicians of the 17th and 18th centuries. It is in the 19th century, in the midst of the Industrial Revolution, that nilpotent infinitesimals were ostracized as anathema and replaced by seemingly rigorous ε-δ arguments, which are, fortunately or unfortunately, normal nowadays.

The next most central object in differential geometry has been *fiber bundles*. The principal objective in this paper is to show how to deal with fiber bundles without any reference to local trivializations at all, just as we have replaced smooth manifolds by microlinear Frölicher spaces. Undoubtedly, fiber bundles should be more than a mere smooth mapping $\pi_M^E : E \to M$ of Frölicher spaces. The missing concept is a sort of relativization of microlinearity as well as Weil exponentiability. Given a fixed Frölicher space M, we will develop the theory of M-Weil exponentiability and M-microlinearity on the lines of [19] and [20]. It is to be shown that any Weil functor naturally gives rise to a fiber bundle.We will finally show that the category of fiber bundles over M and their smooth mappings over M as well as the category of vector bundles over M and their smooth linear mappings over M is cartesian closed.

2 Preliminaries

2.1 Frölicher Spaces

Frölicher and his followers have vigorously and consistently developed a general theory of smooth spaces, often called *Frölicher spaces* for his celebrity, which were intended to be the *underlying set theory* for infinite-dimensional differential geometry in a certain sense. A Frölicher space is usually depicted as an underlying set endowed with a class of real-valued functions on it (simply called *structure functions*) and a class of mappings from the set $\mathbb{R}$ of real numbers to the underlying set (called *structure curves*) subject to the condition that structure curves and structure functions should compose so as to yield smooth mappings from $\mathbb{R}$ to itself. It is required that the class of structure functions and that of structure curves should determine each other so that each of the two classes is maximal with respect to the other as far as they abide by the above condition. For a standard reference on Frölicher spaces, the reader is referred to [6]. What is most important among many nice properties about the category **FS** of Frölicher spaces and smooth mappings is that

Theorem 1 *The category* **FS** *is cartesian closed.*

Notation 2 *We use* $\cdot \times \cdot$ *and* $[\cdot, \cdot]$ *for product and exponentiation in the category* **FS***.*

Recall that a Frölicher space endowed with a compatible linear structure is called a *preconvenient vector space*. It is well known that

Theorem 3 *The category* **preCon** *of preconvenient vector spaces and their smooth linear mappings is cartesian closed.*

Notation 4 *We use* $\cdot \times \cdot$ *and* $[\cdot,\cdot]^{\mathrm{Lin}}$ *for product and exponentiation in the category* **preCon**.

These two results can easily be relativized.

Theorem 5 *Let* M *be a Frölicher space. The slice category* $\mathbf{FS}/M$ *is cartesian closed.*

Notation 6 *We use* $\cdot \times_M \cdot$ *and* $[\cdot,\cdot]_M$ *for product and exponentiation in the category* $\mathbf{FS}/M$.

Theorem 7 *The category* $\mathbf{preVect}_M$ *consisting of vector spaces in the category* $\mathbf{FS}/M$ *and their linear morphisms in the category* $\mathbf{FS}/M$ *is cartesian closed.*

Notation 8 *We use* $\cdot \times_M \cdot$ *and* $[\cdot,\cdot]_M^{\mathrm{Lin}}$ *for product and exponentiation in the category* $\mathbf{preVect}_M$.

2.2 Weil Algebras

The notion of a *Weil algebra* was introduced by Weil himself in [25]. We denote by $\mathbf{W}$ the category of Weil algebras and their $\mathbb{R}$-algebra homomorphisms preserving maximal ideals. It is well known that the category $\mathbf{W}$ is left exact. Roughly speaking, each Weil algebra W corresponds to an infinitesimal object $\mathcal{D}_W$ in the shade. By way of example, the Weil algebra $\mathbb{R}[X]/(X^2)$ (=the quotient ring of the polynomial ring $\mathbb{R}[X]$ of an indeterminate X modulo the ideal (X^2) generated by X^2) corresponds to the infinitesimal object of first-order nilpotent infinitesimals, while the Weil algebra $\mathbb{R}[X]/(X^3)$ corresponds to the infinitesimal object of second-order nilpotent infinitesimals. Although an infinitesimal object is undoubtedly imaginary in the real world, as has harassed both mathematicians and philosophers of the 17th and the 18th centuries because mathematicians at that time preferred to talk infinitesimal objects as if they were real entities, each Weil algebra yields its corresponding *Weil functor* on the category of smooth manifolds of some kind to itself, which is no doubt a real entity. Intuitively speaking, the Weil functor corresponding to a Weil algebra stands for the exponentiation by the infinitesimal object corresponding to the Weil algebra at issue. For Weil functors on the category of finite-dimensional smooth manifolds, the reader is referred to §35 of [12], while the reader can find a readable treatment of Weil functors on the category of smooth manifolds modelled on convenient vector spaces in §31 of [13].

Synthetic differential geometry (usually abbreviated to SDG), which is a kind of differential geometry with a cornucopia of nilpotent infinitesimals, was forced to invent its models, in which nilpotent infinitesimals were visible. For a standard textbook on SDG, the reader is referred to [14], while he or she is referred to [11] for the model theory of SDG, which was vigorously constructed by Dubuc [2] and others. Although we do not get involved in SDG herein, we will exploit locutions in terms of infinitesimal objects so as to make the paper highly readable. Thus we prefer to write $\mathcal{W}_D$ and $\mathcal{W}_{D_2}$ in place of $\mathbb{R}[X]/(X^2)$ and

$\mathbb{R}[X]/(X^3)$ respectively, where D stands for the infinitesimal object of first-order nilpotent infinitesimals, and D_2 stands for the infinitesimal object of second-order nilpotent infinitesimals. To Newton and Leibniz, D stood for

$$\{d \in \mathbb{R} \mid d^2 = 0\}$$

while D_2 stood for

$$\{d \in \mathbb{R} \mid d^3 = 0\}$$

We will write $\mathcal{W}_{d\in D_2 \mapsto d^2 \in D}$ for the homomorphim of Weil algebras $\mathbb{R}[X]/(X^2) \to \mathbb{R}[X]/(X^3)$ induced by the homomorphism $X \to X^2$ of the polynomial ring $\mathbb{R}[X]$ to itself. Such locutions are justifiable, because the category $\mathbf{W}$ in the real world and the category of infinitesimal objects in the shade are dual to each other in a sense, being interconnected by the contravariant functors $\mathcal{W}$ (from the category of infinitesimal objects to the category of Weil algebras) and $\mathcal{D}$ (from the category of Weil algebras to the category of infinitesimal objects). To familiarize himself or herself with such locutions, the reader is strongly encouraged to read the first two chapters of [14], even if he or she is not interested in SDG at all.

2.3 Microlinearity

In [19] we have discussed how to assign, to each pair (M, W) of a Frölicher space M and a Weil algebra W, another Frölicher space $M \otimes W$ called the *Weil prolongation of* M *with respect to* W, which is naturally extended to a bifunctor $\mathbf{FS} \times \mathbf{W} \to \mathbf{FS}$, and then to show that the functor $\cdot \otimes W : \mathbf{FS} \to \mathbf{FS}$ is product-preserving for any Weil algebra W. Weil prolongations are well-known as *Weil functors* for finite-dimensional and infinite-dimensional smooth manifolds in orthodox differential geometry, as we have already touched upon in the preceding subsection. We note in passing that

Lemma 9 *For any Frölicher spaces* M *and* N *and for any Weil algebra* W*, we have*

$$[N, M] \otimes W = [N, M \otimes W]$$

The central object of study in SDG is *microlinear* spaces. Although the notion of a manifold (=a pasting of copies of a certain linear space) is defined on the local level, the notion of microlinearity is defined on the genuinely infinitesimal level. For the historical account of microlinearity, the reader is referred to §§2.4 of [14] or Appendix D of [11]. To get an adequately restricted cartesian closed subcategory of Frölicher spaces, we have emancipated microlinearity from within a well-adapted model of SDG to Frölicher spaces within the real world in [20]. Recall that a Frölicher space M is called *microlinear* providing that any finite limit diagram $\mathbb{D}$ in $\mathbf{W}$ yields a limit diagram $M \otimes \mathbb{D}$ in $\mathbf{FS}$, where $M \otimes \mathbb{D}$ is obtained from $\mathbb{D}$ by putting $M\otimes$ to the left of every object in $\mathbb{D}$ and putting $\mathrm{id}_M\otimes$ to the left of every morphism in $\mathbb{D}$. As we have discussed there, all convenient vector spaces are microlinear, so that all C^∞-manifolds in the sense of [13] (cf. Section 27) are also microlinear.

We have no reason to hold that all Frölicher spaces credit Weil prolongations as exponentiation by infinitesimal objects in the shade. Therefore we need a notion which distinguishes Frölicher spaces that do so from those that do not. Here we slighly modify the notion of Weil exponentiability introduced in [19]. A Frölicher space M is called *Weil exponentiable* if

$$M \otimes (W_1 \otimes_\infty W_2) = (M \otimes W_1) \otimes W_2 \tag{1}$$

holds naturally for any Weil algebras W_1 and W_2.

Proposition 10 *If a Frölicher space M is Weil exponentiable, then so is $[N, M]$ for any Frölicher space N.*

Proof. For any Weil algebras W_1 and W_2, we have

$$
\begin{aligned}
&[N, M] \otimes (W_1 \otimes_\infty W_2) \\
&= [N, M \otimes (W_1 \otimes_\infty W_2)] \\
&\text{[by Lemma 9]} \\
&= [N, (M \otimes W_1) \otimes W_2] \\
&= [N, M \otimes W_1] \otimes W_2 \\
&\text{[by Lemma 9]} \\
&= ([N, M] \otimes W_1) \otimes W_2
\end{aligned}
$$

■

Proposition 11 *If a Frölicher space M is Weil exponentiable, then so is $M \otimes W$ for any Weil algebra W.*

Proof. For any Weil algebras W_1 and W_2, we have

$$
\begin{aligned}
&(M \otimes W) \otimes (W_1 \otimes_\infty W_2) \\
&= M \otimes (W \otimes_\infty (W_1 \otimes_\infty W_2)) \\
&= M \otimes ((W \otimes_\infty W_1) \otimes_\infty W_2) \\
&= (M \otimes (W \otimes_\infty W_1)) \otimes W_2 \\
&= ((M \otimes W) \otimes W_1) \otimes W_2
\end{aligned}
$$

■

Proposition 12 *If Frölicher spaces M and N are Weil exponentiable, then so is $M \times N$.*

Proof. For any Weil algebras W_1 and W_2, we have

$$
\begin{aligned}
& (M \times N) \otimes (W_1 \otimes_\infty W_2) \\
& = (M \otimes (W_1 \otimes_\infty W_2)) \times (N \otimes (W_1 \otimes_\infty W_2)) \\
\text{[since the functor } \cdot \otimes (W_1 \otimes_\infty W_2) &: \mathbf{FS} \to \mathbf{FS} \text{ is product-preserving]} \\
& = ((M \otimes W_1) \otimes W_2) \times ((N \otimes W_1) \otimes W_2) \\
& = ((M \otimes W_1) \times (N \otimes W_1)) \otimes W_2 \\
\text{[since the functor } \cdot \otimes W_2 &: \mathbf{FS} \to \mathbf{FS} \text{ is product-preserving]} \\
& = ((M \times N) \otimes W_1) \otimes W_2
\end{aligned}
$$

■

Our present notion of Weil exponentiability is essentially the same as the one in [19], as the following proposition shows.

Proposition 13 *A Frölicher space M is Weil exponentiable iff*

$$
[N, M \otimes (W_1 \otimes_\infty W_2)] = [N, M \otimes W_1] \otimes W_2 \tag{2}
$$

holds naturally for any Frölicher space N and any Weil algebras W_1 and W_2.

Proof. By taking $N = 1$ in (2), we can see that (2) implies (1). To see the converse, it suffices to note that

$$
\begin{aligned}
& [N, M \otimes (W_1 \otimes_\infty W_2)] \\
& = [N, M] \otimes (W_1 \otimes_\infty W_2) \\
& \text{[by Lemma 9]} \\
& = ([N, M] \otimes W_1) \otimes W_2 \\
& \text{[by Proposition 10]} \\
& = [N, M \otimes W_1] \otimes W_2 \\
& \text{[by Lemma 9]}
\end{aligned}
$$

■

Theorem 14 *The category $\mathbf{FS}_{\mathrm{WE}}$ of Weil exponentiable Frölicher spaces and their smooth mappings ($\mathbf{FS}_{\mathrm{ML}}$ of microlinear Frölicher spaces and their smooth mappings, $\mathbf{FS}_{\mathrm{WE,ML}}$ of Weil exponentiable and microlinear Frölicher spaces and their smooth mappings, respectively) is cartesian closed.*

Proof. The case of $\mathbf{FS}_{\mathrm{WE}}$ follows from Theorem 1 and Propositions 10 and 12. The remaining two cases can be dealt with similarly. ■

3 Relativized Weil Prolongation

Now we would like to relativize the notion of Weil prolongation. Since our discussion is parallel to Section 3 of [19], we can be brief. Let $\pi_M^E : E \to M$ be a smooth mapping of Frölicher spaces. Given a Weil Algebra $W = \mathcal{C}^\infty(\mathbb{R}^n)/I$, we will construct the *Weil prolongation* $\pi_M^{E\otimes_M W} = \pi_M^E \overrightarrow{\otimes} W : E \otimes_M W \to M$ *of* the mapping $\pi_M^E : E \to M$ *with respect to* W. We will first define $E \otimes_M W$ set-theoretically. We define an equivalence relation $\equiv \mod I$ on $\mathcal{C}^\infty(\mathbb{R}^n, E)_M = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^n, E) \mid \pi_M^E \circ f \text{ is constant} \right\}$ to be

$$\begin{gathered} f \equiv g \mod I \\ \text{iff} \\ f(0,...,0) = g(0,...,0) \text{ and} \\ \chi \circ f - \chi \circ g \in I \text{ for every } \chi \in \mathcal{C}^\infty(E, \mathbb{R}) \end{gathered}$$

for any $f, g \in \mathcal{C}^\infty(\mathbb{R}^n, E)_M$. The totality of equivalence classes with respect to the equivaleence relation $\equiv \mod I$ is denoted by $E \otimes_M W$, which has the canonical projection $\pi_M^{E\otimes_M W} : E \otimes_M W \to M$. This construction of $E \otimes_M W$ can naturally be extended to a functor $\cdot \overrightarrow{\otimes} W : \mathbf{FS}^\to \to \mathbf{Sets}^\to$, where the category $\mathbf{FS}^\to$ is the category of diagrams in $\mathbf{FS}$ over the underlying category

$$\cdot \to \cdot$$

and similarly for $\mathbf{Sets}^\to$. In other words, given a morphism

$$\begin{array}{ccccc} & & \varphi & & \\ & E & \to & F & \\ \pi_M^E & \downarrow & & \downarrow & \pi_N^F \\ & M & \to & N & \\ & & \underline{\varphi} & & \end{array}$$

in the category $\mathbf{FS}^\to$, we have its induced morphism

$$\begin{array}{ccccc} & & \varphi \overrightarrow{\otimes} W & & \\ & E \otimes_M W & \to & F \otimes_N W & \\ \pi_M^{E\otimes_M W} & \downarrow & & \downarrow & \pi_N^{F\otimes_N W} \\ & M & \to & N & \\ & & \underline{\varphi} & & \end{array}$$

in the category $\mathbf{Sets}^\to$. We note that if $M = 1$, then $E \otimes_M W$ is no other than $E \otimes W$, whose construction has already been discussed in Section 3 of [19]. We endow $E \otimes_M W$ with the initial smooth structure with respect to the mappings

$$X \otimes_M W \overset{\chi \overrightarrow{\otimes} W}{\to} \mathbb{R} \otimes W$$

where χ ranges over $\mathcal{C}^\infty(X, \mathbb{R})$, and $\mathbb{R}$ with the canonical smooth structure is also regarded as an object $\mathbb{R} \to 1$ in the category $\mathbf{FS}^\to$. By doing so, we can lift the functor $\cdot \overrightarrow{\otimes} W : \mathbf{FS}^\to \to \mathbf{Sets}^\to$ to the functor $\mathbf{FS}^\to \to \mathbf{FS}^\to$. As in Proposition 5 of [19], we finally have the bifunctor $\overrightarrow{\otimes} : \mathbf{FS}^\to \times \mathbf{W} \to \mathbf{FS}^\to$, which gives rise to a bifunctor $\otimes_M : \mathbf{FS}/M \times \mathbf{W} \to \mathbf{FS}/M$ by restriction of the category $\mathbf{FS}^\to$ to its faithful subcategory $\mathbf{FS}/M$ (the slice category of $\mathbf{FS}$ over M).

As in Theorem 10 of [19], we have

Theorem 15 *Given a Weil algebra W, the functor $\cdot \otimes_M W : \mathbf{FS}/M \to \mathbf{FS}/M$ is product-preserving.*

From now on through the end of this section, M shall be a Weil exponentiable and microlinear Frölicher space. Now we are going to determine the Weil prolongation $\pi_M^{(M\otimes W_1)\otimes_M W_2} : (M \otimes W_1) \otimes_M W_2 \to M$ of the canonical projection $\pi_M^{M\otimes W_1} : M \otimes W_1 \to M$ with respect to W_2, where W_1 and W_2 are arbitrary Weil algebras.

Notation 16 *Given Weil algebras W_1 and W_2, the equalizer of $\mathcal{W}_{d\in\mathcal{D}_{W_2}\mapsto(0,d)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}} : W_1 \otimes_\infty W_2 \to W_2$ and $\mathcal{W}_{d\in\mathcal{D}_{W_2}\mapsto(0,0)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}} : W_1 \otimes_\infty W_2 \to W_2$ in the category $\mathbf{W}$ is denoted by $W_1 \widetilde{\otimes}_\infty W_2$.*

Now we are going to determine $\mathcal{W}_D \widetilde{\otimes}_\infty \mathcal{W}_D$ by way of example. The following lemma should be obvious.

Lemma 17 *The diagram*

$$\mathcal{W}_{D(2)} \xrightarrow{\mathcal{W}_{(d_1,d_2)\in D\times D\mapsto(d_1,d_1d_2)\in D(2)}} \mathcal{W}_{D\times D} \underset{\mathcal{W}_{d\in D\mapsto(0,0)\in D\times D}}{\overset{\mathcal{W}_{d\in D\mapsto(0,d)\in D\times D}}{\rightrightarrows}} \mathcal{W}_D$$

is an equalizer diagram in $\mathbf{W}$.

Therefore, we have

Proposition 18 *$\mathcal{W}_D \widetilde{\otimes}_\infty \mathcal{W}_D$ is no other than $\mathcal{W}_{D(2)}$.*

Theorem 19 *Given two Weil algebras W_1 and W_2, the Weil prolongation $\pi_M^{(M\otimes W_1)\otimes_M W_2} : (M \otimes W_1) \otimes_M W_2 \to M$ of the canonical projection $\pi_M^{M\otimes W_1} : M \otimes W_1 \to M$ with respect to W_2 is no other than $\pi_M^{M\otimes\left(W_1\widetilde{\otimes}_\infty W_2\right)} : M \otimes \left(W_1 \widetilde{\otimes}_\infty W_2\right) \to M$.*

Remark 20 *It is for this theorem that we need the notion of $W_1 \widetilde{\otimes}_\infty W_2$.*

Proof. It is easy to see that $(M \otimes W_1) \otimes_M W_2$ is no other than the equalizer of

$$(M \otimes W_1) \otimes W_2 = M \otimes (W_1 \otimes_\infty W_2) \underset{\mathrm{id}_M \otimes \mathcal{W}_{d\in\mathcal{D}_{W_2}\mapsto(0,0)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}}}{\overset{\mathrm{id}_M \otimes \mathcal{W}_{d\in\mathcal{D}_{W_2}\mapsto(0,d)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}}}{\rightrightarrows}} M \otimes W_2$$

However, since M is microlinear, the equalizer diagram

$$W_1 \widetilde{\otimes}_\infty W_2 \to W_1 \otimes_\infty W_2 \underset{\mathcal{W}_{d\in\mathcal{D}_{W_2}\mapsto(0,0)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}}}{\overset{\mathcal{W}_{d\in\mathcal{D}_{W_2}\mapsto(0,d)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}}}{\rightrightarrows}} W_2$$

in the category $\mathbf{W}$ naturally gives rise to the equalizer diagram

$$M\otimes\left(W_1\widetilde{\otimes}_\infty W_2\right)\rightarrow M\otimes(W_1\otimes_\infty W_2)\quad \begin{array}{c}\mathrm{id}_M\otimes\mathcal{W}_{d\in\mathcal{D}_{W_2}\mapsto(0,d)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}}\\ \rightrightarrows\\ \mathrm{id}_M\otimes\mathcal{W}_{d\in\mathcal{D}_{W_2}\mapsto(0,0)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}}\end{array}\quad M\otimes W_2$$

in the category $\mathbf{FS}$. Therefore, we are now sure that the Weil prolongation $\pi_M^{(M\otimes W_1)\otimes_M W_2}:(M\otimes W_1)\otimes_M W_2\rightarrow M$ of the canonical projection $\pi_M^{M\otimes W_1}:M\otimes W_1\rightarrow M$ with respect to W_2 is no other than $\pi_M^{M\otimes\left(W_1\widetilde{\otimes}_\infty W_2\right)}:M\otimes\left(W_1\widetilde{\otimes}_\infty W_2\right)\rightarrow M$. ■

4 Relatived Weil Exponentiability

As in Lemma 9, we have

Lemma 21 *For any smooth mappings $\pi_M^E:E\rightarrow M$ and $\pi_M^F:F\rightarrow M$ and for any Weil algebra W, we have*

$$[E,F]_M\otimes_M W=[E,F\otimes_M W]_M$$

Now we relativize the notion of Weil exponentiability.

Definition 22 *A smooth mapping $\pi_M^E:E\rightarrow M$ of Frölicher spaces over M is called Weil-exponentiable with respect to M or, more briefly, M-Weil-exponentiable if*

$$E\otimes_M(W_1\otimes_\infty W_2)=(E\otimes_M W_1)\otimes_M W_2$$

holds naturally for any Weil algebras W_1 and W_2.

As in Proposition 10, we have

Proposition 23 *If a smooth mapping $\pi_M^E:E\rightarrow M$ of Frölicher spaces is M-Weil-exponentiable, then so is $\pi_M^{[F,E]_M}:[F,E]_M\rightarrow M$ for any smooth mapping $\pi_M^F:F\rightarrow M$ of Frölicher spaces.*

As in Proposition 11, we have

Proposition 24 *If a smooth mapping $\pi_M^E:E\rightarrow M$ of Frölicher spaces is M-Weil-exponentiable, then so is its M-Weil prolongation $\pi_M^{E\otimes_M W}:E\otimes_M W\rightarrow M$ with respect to any Weil algebra W.*

As in Proposition 12, we have

Proposition 25 *If $\pi_M^E:E\rightarrow M$ and $\pi_M^F:F\rightarrow M$ are M-Weil-exponentiable smooth mappings of Frölicher spaces, then so is $\pi_M^{E\times_M F}:E\times_M F\rightarrow M$.*

Theorem 26 *The full subcategory* $(\mathbf{FS}/M)_{\mathrm{WE}}$ *of all* M*-Weil-exponentiable smooth mappings of Frölicher spaces of the category* $\mathbf{FS}/M$ *is Cartesian closed.*

We conclude this section by noting that the canonical projection $\pi_M^{M\otimes W}$: $M\otimes W\rightarrow M$ is M-Weil exponentiable for any Weil algebra W, where M is a Weil exponentiable and microlinear Frölicher space. To this end, we need

Proposition 27 *Given Weil algebras* W_1*,* W_2 *and* W_3*, we have*

$$\left(W_1\widetilde{\otimes}_\infty W_2\right)\widetilde{\otimes}_\infty W_3=W_1\widetilde{\otimes}_\infty\left(W_2\otimes_\infty W_3\right)$$

Proof. Since the functor $\cdot\otimes_\infty W:\mathbf{W}\rightarrow\mathbf{W}$ preserves limits, particularly equalizers, $\left(W_1\widetilde{\otimes}_\infty W_2\right)\otimes_\infty W_3$ is no other than the equalizer of

$$W_1\otimes_\infty W_2\otimes_\infty W_3\quad\overset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,d_2,d_3)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\underset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,0,d_3)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\rightrightarrows}}\quad W_2\otimes_\infty W_3$$

Since the canonical injection $W_3\rightarrow W_2\otimes_\infty W_3$ is represented by

$$W_3\quad\overset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto d_3\in\mathcal{D}_{W_3}}}{\rightarrow}\quad W_2\otimes_\infty W_3$$

the equalizer of

$$\left(W_1\widetilde{\otimes}_\infty W_2\right)\otimes_\infty W_3\quad\overset{\mathcal{W}_{d_3\in\mathcal{D}_{W_3}\mapsto(0,d_3)\in\mathcal{D}_{W_1\widetilde{\otimes}_\infty W_2}\times\mathcal{D}_{W_3}}}{\underset{\mathcal{W}_{d_3\in\mathcal{D}_{W_3}\mapsto(0,0)\in\mathcal{D}_{W_1\widetilde{\otimes}_\infty W_2}\times\mathcal{D}_{W_3}}}{\rightrightarrows}}\quad W_3$$

is no other than the intersection of $\left(W_1\widetilde{\otimes}_\infty W_2\right)\otimes_\infty W_3$ with the equalizer of

$$W_1\otimes_\infty W_2\otimes_\infty W_3\quad\overset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,0,d_3)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\underset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,0,0)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\rightrightarrows}}\quad W_2\otimes_\infty W_3$$

Therefore, we are sure that $\left(W_1\widetilde{\otimes}_\infty W_2\right)\widetilde{\otimes}_\infty W_3$ is the equalizer of

$$W_1\otimes_\infty W_2\otimes_\infty W_3\quad\begin{array}{c}\overset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,d_2,d_3)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\rightarrow}\\ \overset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,0,d_3)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\rightarrow}\\ \overset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,0,0)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\rightarrow}\end{array}\quad W_2\otimes_\infty W_3$$

which is no other than the equalizer of

$$W_1\otimes_\infty W_2\otimes_\infty W_3\quad\overset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,d_2,d_3)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\underset{\mathcal{W}_{(d_2,d_3)\in\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}\mapsto(0,0,0)\in\mathcal{D}_{W_1}\times\mathcal{D}_{W_2}\times\mathcal{D}_{W_3}}}{\rightrightarrows}}\quad W_2\otimes_\infty W_3$$

This implies the desired result, completing the proof. ■

Theorem 28 *For any Weil algebra W, the canonical projection $\pi_M^{M\otimes W}: M\otimes W\to M$ is M-Weil exponentiable.*

Proof. We have

$$
\begin{aligned}
&((M\otimes W)\otimes_M W_1)\otimes_M W_2\\
&=\left(M\otimes\left(W\widetilde{\otimes}_\infty W_1\right)\right)\otimes_M W_2\\
&\text{[by Theorem 19]}\\
&=M\otimes\left(\left(W\widetilde{\otimes}_\infty W_1\right)\widetilde{\otimes}_\infty W_2\right)\\
&\text{[by Theorem 19]}\\
&=M\otimes\left(W\widetilde{\otimes}_\infty(W_1\otimes_\infty W_2)\right)\\
&\text{[by Proposition 27]}\\
&=(M\otimes W)\otimes_M(W_1\otimes_\infty W_2)\\
&\text{[by Theorem 19]}
\end{aligned}
$$

■

5 Relativized Microlinearity

Definition 29 *A smooth mapping $\pi_M^E: E\to M$ of Frölicher spaces is called microlinear with respect to M or, more briefly, M-microlinear providing that any finite limit diagram $\mathbb{D}$ in W yields a limit diagram $E\otimes_M\mathbb{D}$ in the category* **FS**, *where $E\otimes_M\mathbb{D}$ is obtained from $\mathbb{D}$ by putting $E\otimes_M$ to the left of every object in $\mathbb{D}$ and putting $\mathrm{id}_E\otimes$ to the left of every morphism in $\mathbb{D}$.*

As in Proposition 14 of [20], we have

Proposition 30 *If $\pi_M^E: E\to M$ is a M-Weil exponentiable and M-microlinear smooth mapping of Frölicher spaces, then so is $\pi_M^{[F,E]_M}:[F,E]_M\to M$ for any smooth mapping $\pi_M^F: F\to M$ of Frölicher spaces.*

As in Proposition 12 of [20], we have

Proposition 31 *If $\pi_M^E: E\to M$ is a M-Weil exponentiable and M-microlinear smooth mapping of Frölicher spaces, then so is $\pi_M^{E\otimes_M W}: E\otimes_M W\to M$ for any Weil algebra W.*

As in Proposition 13 of [20], we have

Proposition 32 *If $\pi_M^E: E\to M$ and $\pi_M^F: F\to M$ are M-microlinear smooth mappings of Frölicher spaces, then so is $\pi_M^{E\times_M F}: E\times_M F\to M$.*

Theorem 33 *The full subcategory $(\mathbf{FS}/M)_{\mathrm{WE,ML}}$ of all M-Weil exponentiable and M-microlinear smooth mappings of Frölicher spaces ($(\mathbf{FS}/M)_{\mathrm{ML}}$ of all M-microlinear smooth mappings of Frölicher spaces, resp.) of the category $\mathbf{FS}/M$ is cartesian closed.*

Proof. This follows from Theorem 5 and Propositions 23, 25, 30 and 32. ■

Now it is appropriate to introduce the following definition.

Definition 34 *A smooth mapping $\pi_M^E : E \to M$ of Frölicher spaces is called a fiber bundle over M provided that it is M-Weil exponentiable and M-microlinear.*

Notation 35 *The category $(\mathbf{FS}/M)_{\mathrm{WE,ML}}$ is also denoted by $\mathbf{Fib}_M$.*

Now we are going to show that

Theorem 36 *Let M be a Weil exponentiable and microlinear Frölicher space. The canonical projection $\pi_M^{M\otimes W} : M \otimes W \to M$ is M-microlinear for any Weil algebra W.*

To this end, we need

Lemma 37 *Given a Weil algebra W and a finite diagram $\mathbb{D}$ of Weil algebras, we have*

$$\mathrm{Lim}\left(W\widetilde{\otimes}_\infty\mathbb{D}\right) = W\widetilde{\otimes}_\infty\mathrm{Lim}\,\mathbb{D}$$

Proof. This follows simply from the well-known two facts that the functor $\cdot \otimes_\infty W : \mathbf{W} \to \mathbf{W}$ is left-exact and that double limits commute. ■

Proof. *(of Theorem 33)* For any finite diagram $\mathbb{D}$ of Weil algebras, we have

$$\begin{aligned}
&\mathrm{Lim}\ ((M\otimes W)\otimes_M \mathbb{D}) \\
&= \mathrm{Lim}\ \left(M\otimes\left(W\widetilde{\otimes}_\infty\mathbb{D}\right)\right) \\
&[\text{by Theorem 19}] \\
&= M\otimes\mathrm{Lim}\ \left(W\widetilde{\otimes}_\infty\mathbb{D}\right) \\
&[\text{since } M \text{ is microlinear}] \\
&= M\otimes\left(W\widetilde{\otimes}_\infty\mathrm{Lim}\,\mathbb{D}\right) \\
&[\text{by Lemma 36}] \\
&= (M\otimes W)\otimes_M \mathrm{Lim}\,\mathbb{D}
\end{aligned}$$

It may be meaningful to state here that ■

Theorem 38 *Let M be a Weil exponentiable and microlinear Frölicher space. The canonical projection $\pi_M^{M\otimes W} : M \otimes W \to M$ is a fiber bundle for any Weil algebra W.*

Proof. This follows from Theorems 28 and 36. ■

6 Vector Bundles

Let us begin this section with two definitions.

Definition 39 *A smooth mapping $\pi_M^E : E \to M$ of Frölicher spaces is called a prevector bundle over M providing that it is endowed with a linear structure in the category $\mathbf{FS}/M$.*

Definition 40 *A prevector bundle $\pi_M^E : E \to M$ is called a vector bundle over M provided that it is M-Euclidean in the sense that*

$$E \otimes_M \mathcal{W}_D = E \times_M E$$

holds naturally.

We restate Theorem 7 in the following way.

Theorem 41 *The totality $\mathbf{preVect}_M$ of prevector bundles over M and their M-linear smooth mappings over M forms a cartesian closed category.*

As in Lemma 9, we have

Lemma 42 *For any objects $\pi_M^E : E \to M$ and $\pi_M^F : F \to M$ in the category $\mathbf{preVect}_M$ and for any Weil algebra W, we have*

$$[E, F]_M^{\mathrm{Lin}} \otimes_M W = [E, F \otimes_M W]_M^{\mathrm{Lin}}$$

As in Proposition 2 and Proposition 10 of [20], we have

Proposition 43 *A prevector bundle $\pi_M^E : E \to M$ is M-Weil exponentiable and M-microlinear, so that it is a fiber bundle over M.*

Now we are going to show that the category $\mathbf{Vect}_M$ of vector bundles over M and their M-linear smooth mappings over M is cartesian closed. To this end, we have to establish the following two propositions.

Proposition 44 *If both $\pi_M^E : E \to M$ and $\pi_M^F : F \to M$ are vector bundles, then so is their product $\pi_M^{E \times_M F} : E \times_M F \to M$.*

Proof. We have to show that $\pi_M^{E \times_M F} : E \times_M F \to M$ is M-Euclidean, for which we have

$$\begin{aligned}
& (E \times_M F) \otimes_M \mathcal{W}_D \\
& = (E \otimes_M \mathcal{W}_D) \times_M (F \otimes_M \mathcal{W}_D) \\
& \text{[by Theorem 15]} \\
& = (E \times_M E) \times_M (F \times_M F) \\
& = (E \times_M F) \times_M (E \times_M F)
\end{aligned}$$

This completes the proof. ■

Proposition 45 *If both $\pi_M^E : E \to M$ and $\pi_M^F : F \to M$ are vector bundles, then so is their exponential $\pi_M^{[E,F]_M^{\mathrm{Lin}}} : [E, F]_M^{\mathrm{Lin}} \to M$.*

Proof. We have to show that $\pi_M^{[E,F]_M^{\mathrm{Lin}}}:[E,F]_M^{\mathrm{Lin}}\rightarrow M$ is M-Euclidean, for which we have

$$\begin{aligned}
&[E,F]_M^{\mathrm{Lin}}\otimes_M\mathcal{W}_D\\
&=[E,F\otimes_M\mathcal{W}_D]_M^{\mathrm{Lin}}\\
&\text{[by Lemma 42]}\\
&=[E,F\times_M F]_M^{\mathrm{Lin}}\\
&=[E,F]_M^{\mathrm{Lin}}\times_M[E,F]_M^{\mathrm{Lin}}\\
&\text{[since the functor }[E,\cdot]_M^{\mathrm{Lin}}:\mathbf{preVect}_M\rightarrow\mathbf{preVect}_M\text{ is product-preserving]}
\end{aligned}$$

■

Theorem 46 *The category* $\mathbf{Vect}_M$ *is cartesian closed.*

Proof. This follows from Theorem 41 and Propositions 44 and 45. ■

To conclude this section, we would like to establish that

Theorem 47 *Let M be a Weil exponentiable and microlinear Frölicher space. The canonical projection $\pi_M^{M\otimes\mathcal{W}_D}:M\otimes\mathcal{W}_D\rightarrow M$ is naturally a vector bundle.*

Proof. We have already explained in detail in Theorem 3 of [21] how the object $\pi_M^{M\otimes\mathcal{W}_D}:M\otimes\mathcal{W}_D\rightarrow M$ in the category $\mathbf{FS}/M$ is naturally endowed with a linear structure. We have to show that it is M-Euclidean, for which we have

$$\begin{aligned}
&(M\otimes\mathcal{W}_D)\otimes_M\mathcal{W}_D\\
&=M\otimes\left(\mathcal{W}_D\widetilde{\otimes}_\infty\mathcal{W}_D\right)\\
&\text{[by Theorem 19]}\\
&=M\otimes\mathcal{W}_{D(2)}\\
&\text{[by Proposition 18]}\\
&=(M\otimes\mathcal{W}_D)\times_M(M\otimes\mathcal{W}_D)
\end{aligned}$$

■

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