UTHEP-596
arXiv:1001.1553

Constant mean curvature surfaces in AdS_3

Kazuhiro Sakai[†],

Research and Education Center for Natural Sciences
and Department of Physics, Keio University
Hiyoshi, Yokohama 223-8521, Japan

Yuji Satoh[‡]

Institute of Physics, University of Tsukuba
Tsukuba, Ibaraki 305-8571, Japan

Abstract

We construct constant mean curvature surfaces of the general finite-gap type in AdS_3. The special case with zero mean curvature gives minimal surfaces relevant for the study of Wilson loops and gluon scattering amplitudes in $\mathcal{N} = 4$ super Yang–Mills. We also analyze properties of the finite-gap solutions including asymptotic behavior and the degenerate (soliton) limit, and discuss possible solutions with null boundaries.

January 2010

[†]sakai@phys-h.keio.ac.jp
[‡]ysatoh@het.ph.tsukuba.ac.jp

1. Introduction

Classical string solutions in anti-de Sitter spaces attract much attention in the study of the AdS/CFT correspondence. Closed strings with a large spin have been of particular interest as they predict the strong coupling limit of the anomalous dimension of long operators in $\mathcal{N} = 4$ super Yang–Mills [1]. Recently open string solutions have also been becoming of importance in the study of gluon scattering amplitudes at strong coupling [2–4]. Besides the physical significance, these classical strings possess a practical advantage that they satisfy integrable equations of motion. This property provides us with a firm technical basis of quantitative investigations.

The classical integrability allows us to construct several kinds of general class of solutions in an explicit form. Among others, one of the most general classes is known as the finite-gap solutions. Finite-gap solutions are expressed in terms of Riemann theta functions and contour integrals associated with an Riemann surface called the spectral curve. The form of general spectral curves was studied in the case of bosonic closed strings in $AdS_3 \times S^1$ [5] and also in the case of superstrings in $AdS_5 \times S^5$ [6] and in $AdS_4 \times \mathbb{CP}^3$ [7]. Beyond the spectral curve, it is possible in some simple cases to express the finite-gap solutions themselves in a fully explicit form. Indeed, explicit finite-gap solutions were constructed in $\mathbb{R}^3, S^3, H^3$ [8], in dS_3 [9] and also in $S^3 \times \mathbb{R}^1$ [10–12].

With applications to string theory in mind, it would certainly be useful to construct general finite-gap solutions in AdS_3 and clarify their general properties. This is the aim of the present paper. We focus on the case with Euclidean world-sheet, since we are mainly motivated by the study of the gluon scattering amplitudes, where relevant solutions are mostly space-like and correspond to Euclidean world-sheet. For the construction of such solutions, see [13–26].

At first sight, desired solutions would seem to be obtained immediately by a slight modification of the above results. Actually, there still remain several nontrivial problems. In order to obtain fully explicit real-valued finite-gap solutions in AdS_3 with a desired world-sheet signature, one has to solve, in addition to the equations of motion, the Virasoro constraints and the reality condition. While the construction of solutions to the equations of motion remains intact, the rest conditions have to be solved case by case. Concerning the Virasoro constraints, it is worth mentioning that the general finite-gap solutions in AdS_3 form a distinct class among those in AdS_3 with extra directions, like $AdS_3 \times S^1$. The Virasoro constraints in the former case are of light-like form, which gives rise to entirely different singularity structures of the meromorphic differentials on the spectral curve. Therefore general construction of finite-gap solutions in AdS_3 without extra directions may as well be discussed separately. The reality condition is intimately related to the signature of the world-sheet. While closed strings are considered with Minkowski world-sheet, we need solutions with Euclidean world-sheet. Taking these respects into account, we need to reconsider the construction of the finite-gap solutions rather than to try to modify the results obtained in the study of closed strings.

In the conformal gauge with the Virasoro constraints imposed, the equations of motion are equivalent to the condition that the immersion of the world-sheet describes a minimal surface, i.e., a surface with vanishing mean curvature. In this paper we slightly relax the condition and consider surfaces with a constant mean curvature. The problem of finding constant mean curvature surfaces is studied in detail by Bobenko [8] with the target space $\mathbb{R}^3$, S^3, H^3. Our discussion in the first half of the present paper serves as an extension of Bobenko's construction to the AdS_3 case. In section 2 we write down the fundamental equations to solve and see that the problem essentially reduces to solving a variant of the elliptic Sinh-Gordon equation. In section 3 we construct the general finite-gap solutions to the above elliptic Sinh-Gordon equation, with special care of the reality condition. In section 4 we solve the rest conditions and construct the general finite-gap constant mean curvature surfaces in AdS_3.

In the latter half of the paper we investigate aspects of the solutions. In section 5 we study the asymptotic behavior of the finite-gap solutions at the boundary of the world-sheet, which depends largely on the world-sheet signature and the reality condition. We also consider the degenerate limit where finite-gap solutions reduce to soliton solutions, and discuss what kind of null boundary solutions are obtained. This provides an extension of the search for the null boundary solutions within the finite-gap solutions of genus one [22]. In section 6 we study the relation between our expression and the Krichever's one constructed for dS_3 [9]. We conclude in section 7.

2. Equations for constant mean curvature surfaces in AdS_3

Let $\vec{Y} = (Y_{-1}, Y_0, Y_1, Y_2)^{\mathrm{T}} \in \mathbb{R}^{2,2}$ denote the global coordinate parametrizing the AdS_3 spacetime. The AdS_3 is given as a hypersurface

$$\vec{Y} \cdot \vec{Y} := -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1 \tag{2.1}$$

in $\mathbb{R}^{2,2}$. Inner product of two vectors $\vec{A}, \vec{B}$ in $\mathbb{R}^{2,2}$ is defined by

$$\vec{A} \cdot \vec{B} = \eta_{ab} A^a B^b \tag{2.2}$$

with the metric

$$\eta_{ab} = \eta^{ab} = \mathrm{diag}(-1, -1, +1, +1). \tag{2.3}$$

Let z be the complex coordinate parametrizing the Euclidean world-sheet. We consider a map $\vec{Y}(z, \bar{z})$ from the complex z-plane to the AdS_3 given by (2.1). We impose the Virasoro constraints

$$\vec{Y}_z^2 = \vec{Y}_{\bar{z}}^2 = 0, \tag{2.4}$$

where $\vec{Y}_z = \partial_z \vec{Y}, \ \vec{Y}_{\bar{z}} = \partial_{\bar{z}} \vec{Y}$. These conditions imply that the surface in consideration is space-like.[1] For space-like surfaces one can introduce the notation

$$2e^{\varphi} := \vec{Y}_z \cdot \vec{Y}_{\bar{z}}\,, \qquad \varphi(z, \bar{z}) \in \mathbb{R}, \tag{2.5}$$

without loss of generality. Let us also introduce a pseudo vector

$$N_a := \frac{1}{2} e^{-\varphi} \epsilon_{abcd} Y^b Y_z^c Y_{\bar{z}}^d, \qquad \epsilon_{(-1)012} = +1, \tag{2.6}$$

which satisfies

$$\vec{N} \cdot \vec{Y} = \vec{N} \cdot \vec{Y}_z = \vec{N} \cdot \vec{Y}_{\bar{z}} = 0, \qquad \vec{N} \cdot \vec{N} = 1. \tag{2.7}$$

The set of vectors $\vec{Y}, \vec{Y}_z, \vec{Y}_{\bar{z}}, \vec{N}$ span a basis of the moving frame in $\mathbb{R}^{2,2}$. When writing them down into the matrix form

$$\varPhi = (\vec{Y}, \vec{Y}_z, \vec{Y}_{\bar{z}}, \vec{N})^{\mathrm{T}}, \tag{2.8}$$

one can check that the following equations hold:

$$\varPhi_z = \mathcal{U} \varPhi, \qquad \varPhi_{\bar{z}} = \mathcal{V} \varPhi, \tag{2.9}$$

with

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \varphi_z & 0 & A \\ 2e^{\varphi} & 0 & 0 & 2iHe^{\varphi} \\ 0 & -iH & -\frac{1}{2}Ae^{-\varphi} & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2e^{\varphi} & 0 & 0 & 2iHe^{\varphi} \\ 0 & 0 & \varphi_{\bar{z}} & -\bar{A} \\ 0 & \frac{1}{2}\bar{A}e^{-\varphi} & -iH & 0 \end{pmatrix}. \tag{2.10}$$

[1]The Virasoro constraints (2.4) imply that $\vec{Y}_s = (\vec{Y}_z + \vec{Y}_{\bar{z}})/2, \ \vec{Y}_t = i(\vec{Y}_z - \vec{Y}_{\bar{z}})/2$ are both space-like or both time-like. As $\vec{Y}$ is time-like and there cannot be more than two time-like directions in $\mathbb{R}^{2,2}$, the surface must be space-like. Then it follows that $\vec{Y}_z \cdot \vec{Y}_{\bar{z}} \geq 0$.

Here A, H are given by

$$A := \vec{Y}_{zz} \cdot \vec{N}, \qquad 2iHe^{\varphi} := \vec{Y}_{z\bar{z}} \cdot \vec{N}. \tag{2.11}$$

H is actually the mean curvature of the surface. Note that

$$\overline{\vec{N}} = -\vec{N}, \qquad \overline{H} = H. \tag{2.12}$$

As mentioned in the beginning, classical string solutions are minimal surfaces and thus correspond to the case

$$H = 0. \tag{2.13}$$

Indeed, (2.9) with $H = 0$ immediately leads to the equations of motion for the classical strings

$$\vec{Y}_{z\bar{z}} - (\vec{Y}_z \cdot \vec{Y}_{\bar{z}})\vec{Y} = 0. \tag{2.14}$$

In this paper, however, we consider slightly more general situation where the mean curvature is constant,

$$H = \text{const.} \tag{2.15}$$

Under this restriction, the compatibility condition

$$[\partial_z - \mathcal{U}, \partial_{\bar{z}} - \mathcal{V}] = 0 \tag{2.16}$$

yields the following equations

$$\varphi_{z\bar{z}} - 2(1 + H^2)e^{\varphi} + \frac{1}{2}A\bar{A}e^{-\varphi} = 0, \tag{2.17}$$

$$A_{\bar{z}} = 0. \tag{2.18}$$

Note that A turns out to be analytic in z. If we make the following change of notations,

$$A(z) = \delta(z)e^{\alpha(z)} = 2\sqrt{1 + H^2}e^{2\alpha(z)}, \tag{2.19}$$

$$\delta(z) = 2\sqrt{1 + H^2}e^{\alpha(z)}, \tag{2.20}$$

$$w = \int^z \delta(z)dz, \tag{2.21}$$

$$\hat{\varphi} = \varphi - \alpha - \bar{\alpha}, \tag{2.22}$$

eq.(2.17) is transformed into a variant[2] of the elliptic Sinh-Gordon equation

$$\hat{\varphi}_{w\bar{w}} - \sinh\hat{\varphi} = 0. \tag{2.23}$$

We stress that $\hat{\varphi}(w,\bar{w})$ has to be real. We are going to study this equation in detail in the next section, where the general real finite-gap solutions are constructed. Those finite-gap solutions are single-valued on the w-plane, and thus the map $w(z)$ must not have any branch points at $|z| < \infty$ in order for the surface $Y(z,\bar{z})$ to be single-valued on the z-plane. This is in contrast with the null boundary solutions of Alday–Maldacena [3] which are single-valued on the z-plane but generically have branch points in the map $w(z)$ when there are more than 4 cusps. Therefore our finite-gap solutions and the multi-cusp solutions generically belong to different categories of solutions. We will discuss some exceptional cases in section 5.

In terms of new parameters, (2.9), (2.10) can be expressed as

$$\varPhi_w = \hat{\mathcal{U}}\varPhi, \qquad \varPhi_{\bar{w}} = \hat{\mathcal{V}}\varPhi, \tag{2.24}$$

with $\hat{\mathcal{U}} := \delta^{-1}\mathcal{U},\ \hat{\mathcal{V}} := \bar{\delta}^{-1}\mathcal{V}$, or explicitly

$$\hat{\mathcal{U}} = \begin{pmatrix} 0 & \delta^{-1} & 0 & 0 \\ 0 & \varphi_w & 0 & e^{\alpha} \\ 2\delta^{-1}e^{\varphi} & 0 & 0 & 2iH\delta^{-1}e^{\varphi} \\ 0 & -iH\delta^{-1} & -\frac{1}{2}e^{-\varphi+\alpha} & 0 \end{pmatrix}, \tag{2.25}$$

$$\hat{\mathcal{V}} = \begin{pmatrix} 0 & 0 & \bar{\delta}^{-1} & 0 \\ 2\bar{\delta}^{-1}e^{\varphi} & 0 & 0 & 2iH\bar{\delta}^{-1}e^{\varphi} \\ 0 & 0 & \varphi_{\bar{w}} & -e^{\bar{\alpha}} \\ 0 & \frac{1}{2}e^{-\varphi+\bar{\alpha}} & -iH\bar{\delta}^{-1} & 0 \end{pmatrix}. \tag{2.26}$$

The structure of the above equations becomes clearer if one performs a suitable gauge transformation of the form

$$\varPhi = g(w,\bar{w})\tilde{\varPhi}. \tag{2.27}$$

In terms of $\tilde{\varPhi}$, equations (2.24) become

$$\tilde{\varPhi}_w = \tilde{\mathcal{U}}\tilde{\varPhi}, \qquad \tilde{\varPhi}_{\bar{w}} = \tilde{\mathcal{V}}\tilde{\varPhi}, \tag{2.28}$$

with

$$\tilde{\mathcal{U}} = g^{-1}\hat{\mathcal{U}}g - g^{-1}g_w, \qquad \tilde{\mathcal{V}} = g^{-1}\hat{\mathcal{V}}g - g^{-1}g_{\bar{w}}. \tag{2.29}$$

[2]Eq. (2.23) differs from the canonical elliptic Sinh-Gordon equation by the sign in front of $\sinh\hat{\varphi}$.

By taking

$$g = \begin{pmatrix} 0 & ie^{ih} & e^{-ih} & 0 \\ 0 & 0 & 0 & 2e^{\alpha} \\ 2ie^{\varphi-\alpha} & 0 & 0 & 0 \\ 0 & -ie^{ih} & e^{-ih} & 0 \end{pmatrix} \tag{2.30}$$

with

$$e^{ih} = \left(\frac{1+iH}{1-iH}\right)^{\frac{1}{4}}, \tag{2.31}$$

one can recast $\tilde{\mathcal{U}}, \tilde{\mathcal{V}}$ into the tensor product form

$$\tilde{\mathcal{U}} = U(\nu) \otimes \mathbf{1} + \mathbf{1} \otimes U(\nu'), \qquad \tilde{\mathcal{V}} = V(\nu) \otimes \mathbf{1} + \mathbf{1} \otimes V(\nu'), \tag{2.32}$$

$$\nu = e^{ih}, \qquad \nu' = ie^{-ih}. \tag{2.33}$$

The components U, V are given by

$$U(\nu) = \frac{1}{2}\begin{pmatrix} -\hat{\varphi}_w & -i\nu \\ -i\nu & \hat{\varphi}_w \end{pmatrix}, \qquad V(\nu) = \frac{i}{2\nu}\begin{pmatrix} 0 & e^{-\hat{\varphi}} \\ e^{\hat{\varphi}} & 0 \end{pmatrix}, \tag{2.34}$$

and $\mathbf{1}$ denotes the 2×2 unit matrix. Tensor product here is understood as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}. \tag{2.35}$$

Due to the structure (2.32), one can construct solutions to the equations (2.28) as a tensor product of more elementary ones. Indeed, (2.28) are solved by

$$\tilde{\Phi} = \Phi_0 M, \qquad \Phi_0 = \Psi(\nu) \otimes \Psi(\nu'), \tag{2.36}$$

where M is a 4×4 Matrix independent of $w, \bar{w}$ and $\Psi(\nu)$ is a 2×2 matrix obeying the set of equations

$$\Psi_w(\nu) = U(\nu)\Psi(\nu), \qquad \Psi_{\bar{w}}(\nu) = V(\nu)\Psi(\nu). \tag{2.37}$$

Note that the compatibility of the above linear equations

$$[\partial_w - U, \partial_{\bar{w}} - V] = 0 \tag{2.38}$$

yields eq. (2.23).

To summarize, once the linear problem (2.37) is solved, one can always construct a solution to the original linear equations (2.9) in the form

$$\varPhi = g\varPhi_0 M = g\varPsi(\nu) \otimes \varPsi(\nu')M. \tag{2.39}$$

With suitably chosen M, the first row of $\varPhi$ gives a real constant mean curvature surface $Y(z, \bar{z})$. We give the explicit construction of $\varPsi(\nu)$ in the next section and then determine M in Section 4.

3. Real finite-gap solutions of the modified elliptic Sinh-Gordon equation

In this section we construct the general real finite-gap solutions to the equation

$$\hat{\varphi}_{w\bar{w}} - \sinh\hat{\varphi} = 0, \qquad \hat{\varphi} \in \mathbb{R}. \tag{3.1}$$

This equation is a real form of the complex Sine-Gordon equation, for which construction of the general finite-gap solutions has been well-studied [27]. The main point here is to select real valued solutions. Such reality condition was studied in detail by Bobenko [8] in the case of the canonical elliptic Sinh-Gordon equation. Since the present case differs from that case purely by the reality condition, one can make full use of the result in the reference. Therefore we first summarize the result of complex valued finite-gap solutions in [8] and then study the reality condition in the present case. We basically follow the notations in [8], but some of them are modified accordingly in order for a better fit to the present case.[3]

In the standard approach to classical integrable equations, one starts with the auxiliary linear problem. As mentioned in the last section, eq.(3.1) can be regarded as the compatibility condition

$$[\partial_w - U, \partial_{\bar{w}} - V] = 0 \tag{3.2}$$

of the auxiliary linear problem (2.37) with U, V given in (2.34). The linear differential operators exhibit the following reduction relations

$$\sigma_3\left(\partial_w - U(\nu)\right)\sigma_3 = \partial_w - U(-\nu), \qquad \sigma_3\left(\partial_{\bar{w}} - V(\nu)\right)\sigma_3 = \partial_{\bar{w}} - V(-\nu), \tag{3.3}$$

$$R^{-1}\left(\partial_w - U(\nu)\right)R = \overline{\partial_{\bar{w}} - V(\bar{\nu}^{-1})}, \qquad R^{-1}\left(\partial_{\bar{w}} - V(\nu)\right)R = \overline{\partial_w - U(\bar{\nu}^{-1})}, \tag{3.4}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & e^{-\hat{\varphi}/2} \\ e^{\hat{\varphi}/2} & 0 \end{pmatrix}. \tag{3.5}$$

[3] $V(\nu), R$ are modified from those in [8].

These reductions induce a holomorphic involution $\nu \to -\nu$ and an antiholomorphic involution $\nu \to \bar{\nu}^{-1}$ of the solutions. In particular, the latter determines the reality condition of the present system, which we will discuss later.

Finite-gap solutions are characterized by the spectral curve. In the present case, it is a Riemann surface of the hyperelliptic curve C defined by

$$C: \quad \tilde{\mu}^2 = \lambda \prod_{l=1}^{2g} (\lambda - \lambda_l). \tag{3.6}$$

Note that it is parametrized by $\lambda := \nu^2$, due to the invariance under the involution $\nu \to -\nu$. Let a_n, b_n, $n = 1, \dots, g$ denote the canonical basis of the 1-cycles and du_n the basis of holomorphic abelian differentials. One can take du_n satisfying the normalization conditions

$$\oint_{a_m} du_n = 2\pi i \delta_{mn}. \tag{3.7}$$

The associated period matrix is given by

$$B_{mn} = \oint_{b_m} du_n. \tag{3.8}$$

This defines the Riemann theta function

$$\theta(\boldsymbol{z}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^g} \exp\left(\frac{1}{2}\langle \boldsymbol{m}, B\boldsymbol{m}\rangle + \langle \boldsymbol{z}, \boldsymbol{m}\rangle\right), \qquad \boldsymbol{z} \in \mathbb{C}^g, \tag{3.9}$$

where $\langle \cdot, \cdot \rangle$ denotes g-dimensional Euclidean inner-product.

In order to express the solutions of (2.37), let us introduce the Riemann surface $\hat{C}$ parametrized by ν. $\hat{C}$ is a double cover of C, consisting of two branches $\nu = \pm\sqrt{\lambda}$. We take a closed path

$$\mathcal{L} \in [a_1 + \cdots + a_g] \tag{3.10}$$

as the branch cut connecting the two branches. With slight abuse of notation we will use the same symbols for contours on C and their lift on $\hat{C}$. We also introduce abelian differentials of the second kind $d\Omega_i$ $(i = 1, 2)$ fixed by the normalization condition

$$\oint_{a_n} d\Omega_i = 0, \qquad i = 1, 2, \qquad n = 1, \dots, g \tag{3.11}$$

and the asymptotic behavior

$$d\Omega_1 \to d\nu, \qquad \nu \to \infty, \tag{3.12}$$

$$d\Omega_2 \to -\frac{d\nu}{\nu^2}, \qquad \nu \to 0. \tag{3.13}$$

Now, the general complex finite-gap solutions of eq.(3.1) can be expressed as

$$\hat{\varphi} = 2\log\frac{\theta(\boldsymbol{X})}{\theta(\boldsymbol{X}+\boldsymbol{\Delta})} \tag{3.14}$$

where

$$\boldsymbol{X} = -\frac{i}{2}(\boldsymbol{U}w - \boldsymbol{V}\bar{w}) + \boldsymbol{D}, \tag{3.15}$$

$$U_n = \oint_{b_n} d\Omega_1, \qquad V_n = \oint_{b_n} d\Omega_2, \tag{3.16}$$

$$\boldsymbol{\Delta} = \pi i(1,\ldots,1), \tag{3.17}$$

and $\boldsymbol{D} \in \mathbb{C}^g$ is a constant. The corresponding solution to eq.(2.37), namely the Baker–Akhiezer function, is given by

$$\Psi(\nu) = \begin{pmatrix} \psi_1(\nu) & \psi_1^*(\nu) \\ \psi_2(\nu) & \psi_2^*(\nu) \end{pmatrix} = \begin{pmatrix} \frac{\theta(\boldsymbol{u}+\boldsymbol{X})\theta(\boldsymbol{D})}{\theta(\boldsymbol{u}+\boldsymbol{D})\theta(\boldsymbol{X})}e^{\omega} & \frac{\theta(-\boldsymbol{u}+\boldsymbol{X})\theta(\boldsymbol{D})}{\theta(-\boldsymbol{u}+\boldsymbol{D})\theta(\boldsymbol{X})}e^{-\omega} \\ \frac{\theta(\boldsymbol{u}+\boldsymbol{X}+\boldsymbol{\Delta})\theta(\boldsymbol{D})}{\theta(\boldsymbol{u}+\boldsymbol{D})\theta(\boldsymbol{X}+\boldsymbol{\Delta})}e^{\omega} & -\frac{\theta(-\boldsymbol{u}+\boldsymbol{X}+\boldsymbol{\Delta})\theta(\boldsymbol{D})}{\theta(-\boldsymbol{u}+\boldsymbol{D})\theta(\boldsymbol{X}+\boldsymbol{\Delta})}e^{-\omega} \end{pmatrix}, \tag{3.18}$$

where

$$\omega = -\frac{i}{2}\left(\Omega_1 w - \Omega_2\bar{w}\right), \tag{3.19}$$

$$\boldsymbol{u}(\nu) = \int_\ell d\boldsymbol{u}, \quad \Omega_i(\nu) = \int_\ell d\Omega_i. \tag{3.20}$$

Here ℓ is a path which connects points ∞ and ν without intersecting $\mathcal{L}$. The above Baker–Akhiezer function transforms under the holomorphic involution as

$$\Psi(-\nu) = \sigma_3\Psi(\nu)\sigma_1, \tag{3.21}$$

which is compatible with the reduction (3.3). The construction so far is identical to that in [8], as we have regarded w and $\bar{w}$ as independent variables.[4]

We are now in a position to consider the reality condition. From now on we set w and $\bar{w}$ complex conjugate to each other. The reality of $\hat{\varphi}$, combined with the reduction (3.4), induces an antiholomorphic involution $\nu \to \bar{\nu}^{-1}$. It appears on the spectral curve C as

$$\tau: \quad \lambda \to \overline{\lambda}^{-1}. \tag{3.22}$$

In order for C to have this property, we consider C with all the branch points λ_i divided into pairs

$$\overline{\lambda}_{2n-1} = \lambda_{2n}^{-1}, \quad n = 1,\ldots,g, \qquad |\lambda_i| \neq 1. \tag{3.23}$$

[4] As far as complex conjugation does not concern, eq.(3.1) is identified with the canonical elliptic Sinh-Gordon equation by the transformation $(w,\bar{w}) \mapsto (w,-\bar{w})$.

The canonical basis a_n, b_n can be chosen so that they transform under the involution as

$$\tau a_n = -a_n, \qquad \tau b_n = b_n - a_n + \sum_{i=1}^{g} a_i\,. \tag{3.24}$$

With this basis, the period matrix has the form

$$B_{mn} = B^{\mathrm{R}}_{mn} + \pi i(1-\delta_{mn}), \qquad B^{\mathrm{R}}_{mn} < 0 \tag{3.25}$$

and the theta function exhibits the following simple conjugation property

$$\overline{\theta(\boldsymbol{z})} = \theta(\overline{\boldsymbol{z}}). \tag{3.26}$$

If we fix $\mathcal{L}$ so that

$$\tau\mathcal{L} = \mathcal{L}, \tag{3.27}$$

the anti-holomorphic involution τ^* on $\hat{C}$ is realized as

$$\overline{\tau^*\nu} = \nu^{-1}. \tag{3.28}$$

With these choice of cycles and branches, we see that

$$\tau^* d\Omega_1 = \overline{d\Omega_2} \tag{3.29}$$

and thus

$$\overline{\boldsymbol{U}} = \boldsymbol{V}. \tag{3.30}$$

One can also show that

$$\overline{\boldsymbol{u}(\nu)} = \boldsymbol{u}(\overline{\nu}^{-1}) - \boldsymbol{\Delta} \quad \mathrm{mod}\ \ 2\pi i\mathbb{Z}^g. \tag{3.31}$$

Now observe that

$$\overline{\hat{\varphi}} = \overline{2\log\frac{\theta(\boldsymbol{X})}{\theta(\boldsymbol{X}+\boldsymbol{\Delta})}} = 2\log\frac{\theta(\overline{\boldsymbol{X}})}{\theta(\overline{\boldsymbol{X}}-\boldsymbol{\Delta})} = 2\log\frac{\theta(\overline{\boldsymbol{X}})}{\theta(\overline{\boldsymbol{X}}+\boldsymbol{\Delta})}. \tag{3.32}$$

Taking account of the periodicity $\theta(\boldsymbol{z}+2\pi i\boldsymbol{m}) = \theta(\boldsymbol{z})$, $\boldsymbol{m}\in\mathbb{Z}^g$, one finds that $\hat{\varphi}$ is real if $\boldsymbol{X}$ is real mod πi. Since $\overline{\boldsymbol{U}} = \boldsymbol{V}$, this condition is equivalent to the condition that $\boldsymbol{D}$ is real mod πi

$$\boldsymbol{D}\in\mathbb{R}^g \ \ \mathrm{mod}\ \ \pi i\mathbb{Z}^g. \tag{3.33}$$

Next let us see how the antiholomorphic involution acts on the Baker–Akhiezer function. Let us introduce the following quantities

$$a_\pm(\nu) := \frac{\theta(\boldsymbol{u}(\nu) \pm \boldsymbol{D})}{\theta(\boldsymbol{u}(\nu) \pm \boldsymbol{D} + \boldsymbol{\Delta})}, \tag{3.34}$$

$$d := \psi_1\psi_2^* - \psi_1^*\psi_2 = -2\frac{\theta^2(\boldsymbol{D})}{\theta(\boldsymbol{0})\theta(\boldsymbol{\Delta})} \cdot \frac{\theta(\boldsymbol{u})\theta(\boldsymbol{u}+\boldsymbol{\Delta})}{\theta(\boldsymbol{u}-\boldsymbol{D})\theta(\boldsymbol{u}+\boldsymbol{D})}. \tag{3.35}$$

One can verify that

$$\overline{a_\pm(\nu)} = a_\pm^{-1}(\bar\nu^{-1}), \tag{3.36}$$

$$\overline{d(\nu)} = d(\bar\nu^{-1})a_+(\bar\nu^{-1})a_-(\bar\nu^{-1}). \tag{3.37}$$

The Baker–Akhiezer function satisfies the following conjugation relation

$$\overline{\Psi(\nu)} = R\Psi(\bar\nu^{-1})\begin{pmatrix} a_+(\bar\nu^{-1}) & 0 \\ 0 & -a_-(\bar\nu^{-1}) \end{pmatrix}. \tag{3.38}$$

This is compatible with the reduction relation (3.4).

4. The general finite-gap solutions in $\boldsymbol{AdS_3}$

We now have the general finite-gap solutions to the set of linear equations (2.9). In order to have real constant mean curvature surfaces, Φ has to satisfy in addition the correct normalization condition and the reality condition. Knowing the explicit form of $\Psi(\nu)$, one can evaluate these conditions and determine the form of M in (2.39).

We use abbreviated notation $a_\pm = a_\pm(\nu)$, $a'_\pm = a_\pm(\nu')$, $d = d(\nu)$, $d' = d(\nu')$ below. It it is helpful to note that $|\nu| = |\nu'| = |a_\pm| = |a'_\pm| = 1$.

By definition, Φ has to satisfy the following normalization condition

$$\Phi\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\Phi^{\mathrm{T}} = \begin{pmatrix} -1 & & & \\ & 0 & 2e^\varphi & \\ & 2e^\varphi & 0 & \\ & & & 1 \end{pmatrix}. \tag{4.1}$$

It can be translated into the condition for M as

$$M\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}M^{\mathrm{T}} = \frac{i}{2dd'}\begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}. \tag{4.2}$$

Next let us see the reality condition

$$\overline{\Phi} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & -1 \end{pmatrix}\Phi. \tag{4.3}$$

With the knowledge of conjugation properties studied in the last section, it can be written in terms of M as

$$\overline{M} = i \begin{pmatrix} \left(a_+a'_+\right)^{-1} & & & \\ & -\left(a_+a'_-\right)^{-1} & & \\ & & -\left(a_-a'_+\right)^{-1} & \\ & & & \left(a_-a'_-\right)^{-1} \end{pmatrix} M. \tag{4.4}$$

The matrix satisfying the equations (4.2), (4.4) is found to be

$$\begin{aligned} M = \frac{1}{2|b|^{\frac{1}{2}}} & \begin{pmatrix} e^{-\frac{\pi i}{4}}\left(a_+a'_+\right)^{\frac{1}{2}} & & & \\ & e^{\frac{\pi i}{4}}\left(a_+a'_-\right)^{\frac{1}{2}} & & \\ & & e^{\frac{\pi i}{4}}\left(a_-a'_+\right)^{\frac{1}{2}} & \\ & & & e^{-\frac{\pi i}{4}}\left(a_-a'_-\right)^{\frac{1}{2}} \end{pmatrix} \\ & \times \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & -\sigma & \sigma & \\ -\sigma & & & \sigma \end{pmatrix} \Lambda, \end{aligned} \tag{4.5}$$

where

$$b(\nu, \nu') = dd' \left(a_+a_-a'_+a'_-\right)^{\frac{1}{2}} \in \mathbb{R}, \qquad \sigma = \operatorname{sign}(b), \tag{4.6}$$

and $\Lambda \in \mathrm{O}(2,2)$ is an arbitrary constant matrix. With this M and $\Psi(\nu)$ obtained in section 3, (2.39) gives the general finite-gap solutions for the real constant mean curvature surfaces in AdS_3.

Finally let us see the case $h = 0$, where the surfaces become minimal and give the string solutions. For simplicity we set $\Lambda = 1$. Collecting the results so far, the immersion of the surfaces is given by

$$\begin{aligned} Y_{-1} + Y_2 &= c_{\boldsymbol{u},\boldsymbol{u}'}\, I_{\boldsymbol{u},\boldsymbol{u}'}\, e^{\omega+\omega'}, & Y_{-1} - Y_2 &= \sigma c_{-\boldsymbol{u},-\boldsymbol{u}'}\, I_{-\boldsymbol{u},-\boldsymbol{u}'}\, e^{-(\omega+\omega')}, \\ Y_0 + Y_1 &= c_{-\boldsymbol{u}',\boldsymbol{u}}\, I_{-\boldsymbol{u}',\boldsymbol{u}}\, e^{\omega-\omega'}, & Y_0 - Y_1 &= \sigma c_{\boldsymbol{u}',-\boldsymbol{u}}\, I_{\boldsymbol{u}',-\boldsymbol{u}}\, e^{-(\omega-\omega')}, \end{aligned} \tag{4.7}$$

where

$$c_{\boldsymbol{u},\boldsymbol{u}'} = \left(\frac{1}{|b|} \frac{\theta^4(\boldsymbol{D})}{\theta(\boldsymbol{u}+\boldsymbol{D})\theta(\boldsymbol{u}+\boldsymbol{D}+\boldsymbol{\Delta})\theta(\boldsymbol{u}'+\boldsymbol{D})\theta(\boldsymbol{u}'+\boldsymbol{D}+\boldsymbol{\Delta})} \right)^{\frac{1}{2}}, \tag{4.8}$$

$$I_{\boldsymbol{u},\boldsymbol{u}'} = \frac{e^{\frac{\pi i}{4}}\theta(\boldsymbol{u}+\boldsymbol{X})\theta(\boldsymbol{u}'+\boldsymbol{X}+\boldsymbol{\Delta}) + e^{-\frac{\pi i}{4}}\theta(\boldsymbol{u}+\boldsymbol{X}+\boldsymbol{\Delta})\theta(\boldsymbol{u}'+\boldsymbol{X})}{\theta(\boldsymbol{X})\theta(\boldsymbol{X}+\boldsymbol{\Delta})}. \tag{4.9}$$

Here $\boldsymbol{u}, \boldsymbol{u}'$ as well as ω, ω' are the quantities evaluated at $\nu = 1, \nu = i$, respectively.

5. General properties and the degenerate limit

In this section we discuss properties of the solutions. First, we note that the coordinates Y_a given in the form (4.7) are oscillating with sign changes: Since $\theta(\boldsymbol{z}+B\boldsymbol{n}) = \exp\big[-\frac{1}{2}\langle\boldsymbol{n},B\boldsymbol{n}\rangle - \langle\boldsymbol{z},\boldsymbol{n}\rangle\big]\theta(\boldsymbol{z})$ with $\boldsymbol{n}\in\mathbb{Z}^g$, $I_{\boldsymbol{u},\boldsymbol{u}'}$ obtains a factor $e^{-\langle\boldsymbol{n},\boldsymbol{u}+\boldsymbol{u}'\rangle}$ under $\boldsymbol{X}\to\boldsymbol{X}+B\boldsymbol{n}$. Note that the imaginary part of $\boldsymbol{u}+\boldsymbol{u}'$ is $\boldsymbol{\Delta}$ (mod $2\pi i\mathbb{Z}^g$). Then, when the spectral curve has an odd genus and each component of $\boldsymbol{n}$ is, e.g., ±1, $B\boldsymbol{n}$ is real (mod $2\pi i\mathbb{Z}^g$) and $I_{\boldsymbol{u},\boldsymbol{u}'}$ changes the sign. This implies that Y_a change the sign as w and hence $\boldsymbol{X}$ vary. By analyzing the zeros of the theta function, one can also observe the sign change in the case of even genus. In the above discussion, it is important that $\mathrm{Im}(\boldsymbol{u}+\boldsymbol{u}')=\boldsymbol{\Delta}/i$, $\boldsymbol{X}$ is real (mod $\pi i\mathbb{Z}^g$) and B has a particular form given in (3.25), which is essential for the reality of the surfaces.

For the application to the gluon scattering amplitudes, string solutions with null boundaries without such oscillations are needed. Since the oscillation may disappear in the limit where the period matrix B diverges, we discuss such a limit in the following.

A way to realize such a limit is to make the spectral curve degenerate. As the simplest one, we consider the limit achieved by shrinking every branch cut to a point on the unit circle:

$$\lambda_{2n-1},\lambda_{2n}(=\bar{\lambda}_{2n-1}^{-1})\to e^{2i\phi_n}:=\nu_n^2 \quad (n=1,...,g). \tag{5.1}$$

In general, this type of degeneration gives soliton solutions [27].[5] In this limit, the geometric quantities describing the solutions are simplified. The abelian differentials become

$$du_n=\frac{\nu_n d\lambda}{\sqrt{\lambda}(\lambda-\nu_n^2)}, \tag{5.2}$$

$$d\Omega_1=d\nu, \quad d\Omega_2=d\nu^{-1}, \tag{5.3}$$

and hence

$$u_n=\log\frac{\nu-\nu_n}{\nu+\nu_n}, \tag{5.4}$$

$$\Omega_1=\nu, \quad \Omega_2=\frac{1}{\nu}, \tag{5.5}$$

$$U_n=\overline{V}_n=2\nu_n. \tag{5.6}$$

[5] For applications of soliton solutions to AdS strings, see [28] and references therein.

The period matrix is also given by

$$B_{mn} = \log\Big(\frac{\nu_m - \nu_n}{\nu_m + \nu_n}\Big)^2. \tag{5.7}$$

The diagonal elements become divergent: $B_{nn} \to -\infty$. As for $\boldsymbol{D}$, we set

$$D_n = -\frac{1}{2}B_{nn} + \eta_n, \tag{5.8}$$

with real (mod πi) and finite η_n. The theta function then reduces to a finite sum:

$$\theta(\boldsymbol{z} + \boldsymbol{D}) \to \sum_{m_k=0,1} \exp\Big[\sum_{k>l} m_k B_{kl} m_l + \sum_k m_k (z+\eta)_k\Big]. \tag{5.9}$$

Now we are ready to discuss the string solutions in the degenerate limit. First, let us consider the case where the original spectral curve is of genus one. To analyze the solutions, it is useful to note that, for example, $I_{u,u'}$ factorizes as $I_{u,u'} \sim (1 - e^{u+u'+\rho})/(1-e^{\rho})$ for u, u' given in (5.4) and thus

$$Y_{-1} + Y_2 \sim \frac{\theta(\boldsymbol{u} + \boldsymbol{u}' + \boldsymbol{X} + \boldsymbol{\Delta})}{\theta(\boldsymbol{X} + \boldsymbol{\Delta})} e^{\frac{1}{2}(s+t)} \tag{5.10}$$

up to a numerical factor. Here,

$$\rho_n = X_n + \frac{1}{2}B_{nn}, \tag{5.11}$$

and we have omitted the subscript n. One then finds that the solutions reduce to those obtained in [22], so that $Y_{-1} + Y_2 \sim [\cos\theta - \frac{1}{\sqrt{2}} \tanh\frac{\rho+\pi i}{2}] e^{\frac{1}{2}(s+t)}$ with $w = \frac{1}{2}(s+it)$ and θ a constant. In the solutions, the unwanted oscillation is in fact absent. When $\mathrm{Im}(\rho + \pi i) = 0$, the solutions describe the surfaces with six null boundaries extending to the boundary of AdS_3. When $\mathrm{Im}\,\rho = 0$,[6] the zeros of $\cosh\frac{\rho+\pi i}{2}$ are mapped to the AdS boundary. Thus the surfaces have non-null boundaries in addition to the null boundaries which are the image of the world-sheet boundary $|w| \to \infty$.

Let us move on to the case where the spectral curve has $g = 2$. Similarly to the $g = 1$ case, $I_{\boldsymbol{u},\boldsymbol{u}'}$ factorizes and the solutions take the form (5.10). The surfaces then have null boundaries coming from the world-sheet boundary. However, since

$$\theta(\boldsymbol{X} + \boldsymbol{\Delta}) = 1 - e^{\rho_1} - e^{\rho_2} + e^{B_{12}+\rho_1+\rho_2}, \tag{5.12}$$

and $e^{B_{12}} < 0$, the denominator of $Y_{-1} + Y_2$ vanishes at internal points of the world-sheet. These zeros give non-null boundaries, as in the case of $g = 1$ with $\mathrm{Im}\,\rho = 0$.

For $g \geq 3$, because of the factors $e^{B_{mn}} < 0$ $(m \neq n)$, the surfaces similarly have both null and non-null boundaries. The number of the null boundaries can be increased according to g. We find that the case of $g = 1$ is special in that the degenerate limit (5.1) gives the string solutions with only null boundaries.

[6]This case is obtained by shifting B in [22] by $\pi i/2$.

6. Expression respecting the vector representation

In general, an auxiliary linear problem to integrable string equations of motion admits several representations. In the case of a constant Ricci curvature target space, it can be expressed either in the spinor representation or in the vector representation of the orthogonal isometry group [29]. The construction in this paper is based on the spinor representation. This results in the tensor product structure of the solutions. On the other hand, it should certainly be possible to express the solution in a form respecting the vector representation. Indeed, Krichever's construction for the dS_3 case [9] was of this form. In this section we will see how our solutions are related to the Krichever's. For simplicity we restrict ourselves to the genus-one case, but the discussion can be generalized to the case of arbitrary genus.

At $g = 1$, the Riemann theta function (3.9) reduces to the elliptic theta function

$$\theta(z) = \theta_3(z|B), \tag{6.1}$$

with $B < 0$. In this section we adopt a slightly modified convention for the Jacobi theta functions, denoted by $\theta_j(z|B)$, $j = 1, 2, 3, 4$. The canonical Jacobi theta functions ϑ_j are expressed in terms of θ_j as

$$\vartheta_j(z|\tau) = \theta_j(2\pi i z|2\pi i\tau), \qquad j = 1, 2, 3, 4. \tag{6.2}$$

Below we abbreviate $\theta_j(z|B)$ as $\theta_j(z)$.

The spectral curve $\hat{C}$ compatible with the reality condition takes the form

$$\tilde{\mu}^2 = \nu^2(\nu^2 - \nu_1^2)(\nu^2 - \overline{\nu}_1^{-2}). \tag{6.3}$$

This curve can be uniformized as

$$\nu = ie^{i\gamma}\frac{\theta_2(u)}{\theta_1(u)}, \qquad \nu_1 = e^{i\gamma}\frac{\theta_4(0)}{\theta_3(0)}, \tag{6.4}$$

$$\tilde{\mu} = ie^{3i\gamma}\frac{\theta_2(0)^2\theta_2(u)\theta_3(u)\theta_4(u)}{\theta_3(0)\theta_4(0)\theta_1(u)^3}. \tag{6.5}$$

Here $\gamma \in \mathbb{R}$ denotes the phase of ν_1. (6.4) inversely expresses the relation (3.20). Below we let u, u' denote $u(\nu)$ evaluated at $\nu = 1, \nu = i$, respectively. It then follows from (6.4) that

$$i\frac{\theta_2(u)}{\theta_1(u)} = \frac{\theta_2(u')}{\theta_1(u')}. \tag{6.6}$$

Now one can show that our solutions can be transformed into the Krichever's form. The essential difference of the two expressions is the world-sheet coordinate dependence through theta functions, which is collected in $I_{\boldsymbol{u},\boldsymbol{u}'}$. At $g=1$ it can be expressed as

$$I_{u,u'} = \frac{e^{\frac{\pi i}{4}}\theta_3(u+X)\theta_4(u'+X) + e^{-\frac{\pi i}{4}}\theta_4(u+X)\theta_3(u'+X)}{\theta_3(X)\theta_4(X)}. \tag{6.7}$$

By using the following theta function identity

$$\begin{aligned}&\theta_1(z-w)\theta_2(0)\theta_3(X)\theta_4(z+w+X)\\ &\quad = \theta_1(z)\theta_2(w)\theta_4(z+X)\theta_3(w+X) - \theta_2(z)\theta_1(w)\theta_3(z+X)\theta_4(w+X)\end{aligned} \tag{6.8}$$

and (6.6), one obtains

$$I_{u,u'} = e^{-\frac{\pi i}{4}}\frac{\theta_1(u-u')\theta_2(0)}{\theta_1(u)\theta_2(u')}\frac{\theta_4(u+u'+X)}{\theta_4(X)}. \tag{6.9}$$

One finds that w-dependence now appears only through θ_4's. This is the form obtained in [22], and gives an AdS_3 analog of the form found in [9] for dS_3. We note that our construction however uses only the ordinary Riemann theta function instead of the Prym theta function. The above result also shows that the factorized form (5.10) in the degenerate limit generally holds for $g=1$. The factorization for $g=2$ found there may be regarded as a consequence of a generalization of the discussion here to the case of arbitrary genus.

7. Conclusions

We have constructed space-like constant mean curvature surfaces of the general finite-gap type in AdS_3. In a special case with vanishing mean curvature, our results provide a fully explicit form of the general finite-gap open string solutions in AdS_3 with Euclidean world-sheet. The points in the construction are concerning the reality condition and the Virasoro constraints without extra directions. These are not achieved by a simple modification of the results [9–12], but rather need analysis analogous to that in [8].

We have also analyzed properties of the solutions. Generically, the solutions oscillate with sign changes. In a degenerate (soliton) limit, the solutions describe the surfaces with null boundaries coming from the world-sheet boundary. For the genus of the spectral curve $g \geq 2$, the surfaces in addition have other type of boundaries mapped from internal part of the world-sheet. The case of $g=1$ is special in that it has solutions with only null boundaries.

The analysis here is an extension of the search of the null boundary solutions among finite-gap solutions carried out for $g = 1$ in [22]. The solutions in this paper generically have different analytic structures from those for the null boundary solutions of Alday–Maldacena [3], and thus belong to a different class. We have also discussed the relation to the construction for dS_3 [9] based on the Prym theta function.

The construction in this paper can be extended to the string solutions in AdS_3 with Minkowski world-sheet, by reanalyzing the antiholomorphic reduction condition (3.4) and the resulting reality conditions accordingly. The large class of classical solutions from both Euclidean and Minkowski world-sheets may shed light on the study of strings in AdS_3. In particular, finding non-oscillating solutions with appropriate boundaries would be useful for exploring the scattering amplitudes and the Wilson loops in gauge theories.

Acknowledgments

We would like to thank A. Kato and K. Mohri for useful discussions. The work of K.S. and Y.S. is supported in part by Grant-in-Aid for Scientific Research from the Japan Ministry of Education, Culture, Sports, Science and Technology.

References

[1] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Nucl. Phys. B **636** (2002) 99 [arXiv:hep-th/0204051].

[2] L. F. Alday and J. M. Maldacena, JHEP **0706** (2007) 064 [arXiv:0705.0303 [hep-th]].

[3] L. F. Alday and J. Maldacena, JHEP **0911** (2009) 082 [arXiv:0904.0663 [hep-th]].

[4] L. F. Alday, D. Gaiotto and J. Maldacena, arXiv:0911.4708 [hep-th].

[5] V. A. Kazakov and K. Zarembo, JHEP **0410** (2004) 060 [arXiv:hep-th/0410105].

[6] N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo, Commun. Math. Phys. **263** (2006) 659 [arXiv:hep-th/0502226].

[7] N. Gromov and P. Vieira, JHEP **0902** (2009) 040 [arXiv:0807.0437 [hep-th]].

[8] A. I. Bobenko, Math. Ann. **290** (1991) 209.

[9] I. M. Krichever, Funct. Anal. Appl. **28** (1994) 21.

[10] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, JHEP **0405** (2004) 024 [arXiv:hep-th/0402207].

[11] N. Dorey and B. Vicedo, JHEP **0607** (2006) 014 [arXiv:hep-th/0601194].

[12] N. Dorey and B. Vicedo, JHEP **0703** (2007) 045 [arXiv:hep-th/0606287].

[13] S. Ryang, Phys. Lett. B **659** (2008) 894 [arXiv:0710.1673 [hep-th]].

[14] A. Jevicki, K. Jin, C. Kalousios and A. Volovich, JHEP **0803** (2008) 032 [arXiv:0712.1193 [hep-th]].

[15] D. Astefanesei, S. Dobashi, K. Ito and H. Nastase, JHEP **0712** (2007) 077 [arXiv:0710.1684 [hep-th]].

[16] S. Dobashi, K. Ito and K. Iwasaki, JHEP **0807** (2008) 088 [arXiv:0805.3594 [hep-th]].

[17] A. Mironov, A. Morozov and T. N. Tomaras, JHEP **0711** (2007) 021 [arXiv:0708.1625 [hep-th]]; Phys. Lett. B **659** (2008) 723 [arXiv:0711.0192 [hep-th]].

[18] H. Itoyama, A. Mironov and A. Morozov, Nucl. Phys. B **808** (2009) 365 [arXiv:0712.0159 [hep-th]].

[19] H. Itoyama and A. Morozov, Prog. Theor. Phys. **120** (2008) 231 [arXiv:0712.2316 [hep-th]].

[20] C. M. Sommerfield and C. B. Thorn, Phys. Rev. D **78** (2008) 046005 [arXiv:0805.0388 [hep-th]].

[21] H. Dorn, G. Jorjadze and S. Wuttke, JHEP **0905** (2009) 048 [arXiv:0903.0977 [hep-th]].

[22] K. Sakai and Y. Satoh, JHEP **0910** (2009) 001 [arXiv:0907.5259 [hep-th]].

[23] H. Dorn, JHEP **1002** (2010) 013 [arXiv:0910.0934 [hep-th]].

[24] A. Jevicki and K. Jin, arXiv:0911.1107 [hep-th].

[25] B. A. Burrington and P. Gao, arXiv:0911.4551 [hep-th].

[26] H. Dorn, N. Drukker, G. Jorjadze and C. Kalousios, arXiv:0912.3829 [hep-th].

[27] E. D. Belokolos, A. I. Bobenko, V. Z. Enolski, A. R. Its and V. B. Matveev, “Algebro-geometric approach in the theory of integrable equations”, Springer Series in Nonlinear Dynamics, Springer, Berlin (1994).

[28] A. Jevicki and K. Jin, JHEP **0906** (2009) 064 [arXiv:0903.3389 [hep-th]].

[29] N. Beisert, V. A. Kazakov and K. Sakai, Commun. Math. Phys. **263** (2006) 611 [arXiv:hep-th/0410253].