FORWARD AND INVERSE SCATTERING ON MANIFOLDS WITH ASYMPTOTICALLY CYLINDRICAL ENDS

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Abstract. We study an inverse problem for a non-compact Riemannian manifold whose ends have the following properties : On each end, the Riemannian metric is assumed to be a short-range perturbation of the metric of the form $(dy)^2 + h(x, dx)$, $h(x, dx)$ being the metric of some compact manifold of codimension 1. Moreover one end is exactly cylindrical, i.e. the metric is equal to $(dy)^2 + h(x, dx)$. Given two such manifolds having the same scattering matrix on that exactly cylindrical end for all energies, we show that these two manifolds are isometric.

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1. Introduction

The aim of this paper is to study spectral properties and related inverse problems for a connected, non-compact Riemannian manifold Ω of dimension $n \geq 2$ with or without boundary. We assume that Ω is split into $N+1$ parts

$$\Omega = \mathcal{K} \cup \Omega_1 \cup \cdots \cup \Omega_N, \tag{1.1}$$

where $\mathcal{K}$ is an open, relatively compact set, and Ω_i, called an *end* of Ω, is diffeomorphic to $M_i \times (0, \infty)$, M_i being a compact manifold of dimension $n-1$. (See the figure 1.) More precisely, we assume that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, and we put $\mathcal{K} = \Omega \setminus \big(\cup_{i=1}^N \Omega_i \big)$. Denoting the local coordinates on M_i by x, we assume that M_i is equipped with a Riemannian metric $h_i(x, dx) = \sum_{p,q=1}^{n-1} h_{i,pq}(x) dx^p dx^q$. Letting y be the coordinate on $(0, \infty)$, we denote the local coordinates on Ω_i by $X = (x, y)$. We assume that the Riemannian metric G on Ω, which is denoted by $G_i = \sum_{p,q=1}^n g_{i,pq}(X) dX^p dX^q$ on Ω_i, has the following property

$$|\partial_X^\alpha (g_{i,pq}(X) - h_{i,pq}(x))| \leq C_\alpha (1+y)^{-1-\epsilon_0}, \quad \forall \alpha, \tag{1.2}$$

where $h_{i,pn}(x) = h_{i,np}(x) = 0$ if $1 \leq p \leq n-1$ and $h_{i,nn}(x) = 1$, and C_α is a constant. The metric $h_i(x, dx)$ on M_i is allowed to be different for different ends. We shall assume either Ω has no boundary or each M_i, consequently Ω itself, has a boundary. In the latter case, we impose Dirichlet or Neumann boundary condition on $\partial\Omega$. Let $H = -\Delta_G$, where Δ_G is the Laplace-Beltrami operator associated with the metric G. One can then define a scattering matrix $\widehat{S}(\lambda) = \big(\widehat{S}_{ij}(\lambda)\big)$, which is a bounded operator on $L^2(M_1) \oplus \cdots \oplus L^2(M_N)$, where $\lambda \in (E_0, \infty) \setminus \mathcal{E}(H)$ is the energy parameter, $E_0 = \inf \sigma_{ess}(H)$, and $\mathcal{E}(H)$ is the set of exceptional points to be defined in (3.34). Our goal is the following.

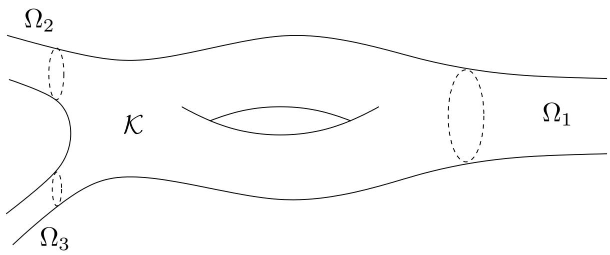

Figure 1. Manifold Ω has ends Ω_j, $j = 1, 2, \ldots, N$.

Theorem 1.1. *Suppose we are given two manifolds $\Omega^{(r)}$, $r = 1, 2$, of the form (1.1) having N_r ends, $\Omega_i^{(r)}$, $i = 1, \cdots, N_r$, equipped with the metric $G^{(r)}$ satisfying the assumption (1.2). Assume that $\Omega_1^{(1)} = \Omega_1^{(2)}$ and*

$$G_1^{(1)} = G_1^{(2)} = (dy)^2 + h_1(x, dx), \quad h_1(x, dx) = \sum_{j,k=1}^{n-1} h_{1,jk}(x) dx^j dx^k \tag{1.3}$$

on $\Omega_1^{(1)} = \Omega_1^{(2)}$, moreover $\widehat{S}_{11}^{(1)}(\lambda) = \widehat{S}_{11}^{(2)}(\lambda)$ for all $\lambda \in (E', \infty) \setminus (\mathcal{E}^{(1)} \cup \mathcal{E}^{(2)})$, where $\mathcal{E}^{(r)}$ is the set of exceptional points for $H^{(r)}$, and $E' = \max(E_0^{(1)}, E_0^{(2)})$. Then $\Omega^{(1)}$ and $\Omega^{(2)}$ are isometric as Riemannian manifolds with metrics $G^{(1)}$, $G^{(2)}$.

This means that if we observe waves coming in and going out of one end Ω_1, which is assumed to be non-perturbed, we can identify the whole manifold Ω. Note that in Theorem 1.1, neither the number of ends of each $\Omega^{(r)}$ nor the metric on the manifold $M_i^{(r)}$ are assumed to be known a-priori. The key idea of the proof is to introduce generalized eigenfunctions of the Laplace-Beltrami operator which are exponentially growing at infinity, and define the associated *non-physical* scattering amplitude. The crucial fact is that this non-physical scattering amplitude is the analytic continuation of the *physical* scattering amplitude. Then the physical scattering amplitude determines the non-physical scattering amplitude, which further determines the Neumann-Dirichlet map of the *interior* domain. By the boundary control method (see [3, 8, 9, 48, 52, 53]), one can determine the metric inside.

In this paper, we exclusively deal with the Neumann boundary condition. The other cases are treated similarly and in fact more easily. The forward problem of scattering is well-known for short-range perturbations (see e.g. [30, 31, 60, 61, 75, 33, 76, 44], see also [63]). The new issue we have to discuss in this paper is the difference of conormal derivatives on the boundary associated with unperturbed and perturbed metrics. Therefore, focusing on this point, we only explain the outline of the proof of the forward problem under the assumption (1.2) following the approach in [41], where spectral theory and inverse problems on hyperbolic spaces are developed in an elementary way.

In the Euclidean space, the first work on the multi-dimensional inverse problem was done by Faddeev in the case of potential scattering [27]. This was extended by Saito [69] for short-range potentials, and by Isozaki-Kitada [40] for long-range potentials. The determination of the obstacle from the scattering matrix of the wave equation was done by Schiffer and Lax-Phillips [56]. As for the metric perturbation problem in $\mathbb{R}^n$, we should stress that it is still unknown for the general short-range perturbations. However, although there seems to be no literature, it is known that, given the scattering matrices for all energies, one can compute the Dirichlet-Neumann map for a bounded domain for all energies, which enables us to recover the local perturbation of the metric by virtue of the boundary control method. In recent years, inverse scattering problems have been generalized for some non-compact Riemannian manifolds, see e.g. [32, 41, 43, 68].

In the cylindrical ends, the physical generalized eigenfunction of the Laplace-Beltrami operator admits the analytic continuation with respect to the energy parameter, and this analytically continued eigenfunction is exponentially growing as $y \to \infty$. This sort of non-physical exponentially growing generalized eigenfunction was first introduced by Faddeev to develop the multi-dimensional Gel'fand-Levitan theory ([26]). The exponentially growing solutions of Schrödinger equation was rediscovered in 1980's and were used to solve the inverse problem for the isotropic conductivity equation in dimensions $n > 2$ for C^2-smooth conductivities [72], even in a reconstructive way [64], and in dimension two for C^2-conductivities in [65] and finally for the L^∞-conductivities in [5], see review [28]. Later, also the anisotropic inverse conductivity problem has been solved by applying the exponentially growing solutions in dimension two [6, 71]. These solutions have also been crucial in the study of multidimensional inverse scattering problem in the Euclidean space [66, 35].

The interesting fact is that this apparently mysterious exponentially growing generalized eigenfunctions appear naturally in the cylindrical domain. Using these exponentially growing eigenfunctions, it is possible to obtain, from $\hat{S}_{11}(\lambda)$, the entry of the scattering matrix corresponding to Ω_1, the Gel'fand spectral data on a part of the boundary $\Gamma = M_1 \times \{1\}$ of the non-compact manifold $\Omega^1 = \Omega \setminus (M_1 \times (1, \infty))$. The Gel'fand boundary data for this case is the family of the Neumann-Dirichlet map, $\Lambda(z)$, $\Lambda(z) f = u\big|_\Gamma$, where u is the solution to the boundary value problem

$$\begin{cases} (-\Delta_G - z)u = 0 \quad \text{in} \quad \Omega^1, \\ \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega^1, \\ \partial_\nu u = \partial_\nu y = f \quad \text{on} \quad \Gamma. \end{cases}$$

To solve this problem, we use the boundary control (BC) method (see [8] for the pioneering work and [9], [48] for the detailed exposition). We note that typically the BC method deals with inverse problem on compact manifolds. The case of non-compact manifold considered here requires substantial modifications into the method, since the spectrum is no more discrete and it is also impossible to use eigenfunctions as coordinate functions. A short description of the BC-method for non-compact manifolds was given in [50]. Here we provide detailed constructions for the considered case of a manifold with asymptotically cylindrical ends.

The structure of this paper is as follows. Sections 2, 3, 4 are devoted to a detailed analysis of scattering on manifolds with asymptotically cylindrical ends. After some preliminary estimates for the case of a half-cylinder with a product metric in §2, we discuss the spectral properties of the Laplacin in Ω in §3. Using these properties, we develop the scattering theory for such manifolds in §4. The remaining part of the paper, Sections 5, 6 are devoted to the inverse scattering. In §4, we show that $\widehat{S}_{11}(\lambda)$ determines the Neumann-Dirichelt map $\Lambda(z)$. An important step, which at the moment requires the product structure of the metric on $M_1 \times (0, \infty)$, is the recovery, from physical scattering matrix $\widehat{S}_{11}(\lambda)$, the non-physical scattering amplitude. At last, §6 is devoted to the development of the BC-method for non-compact manifolds. For the convenience of the reader, interested predominantly in the inversion methods, we make this section independent of the previous ones.

Our manifold Ω is a mathematical model of compound waveguides, e.g. settings of optical and electric cables, oil, gas and water pipelines, etc, which are the most typical geometric constructions encountered in the every-day life. As for the inverse problem, many works have been devoted so far to the distribution of resonances for the waveguides ([15], [7], [21], [22], [4], [16]). Identification or reconstruction of the domain or the medium for grating, layers or waveguides are studied by [19], [39], [67], [25]. In particular, a similar inverse problem for waveguides was considered by Eskin-Ralston-Yamamoto [25] when Ω is a slab, $(0, B) \times \mathbb{R}$, with the variable sound speed $c(x, y)$, where $c(x, y) = c(x)$ for large $|y|$. Christiansen [17] proved that in the planar waveguide $\mathbb{R} \times (-\gamma, \gamma) \setminus \mathcal{O}$, one can determine the obstacle $\mathcal{O}$ from one or two entries of the scattering matrix for high energies, provided $\mathcal{O}$ is strictly convex, compact with analytic boundaries. The present paper deals with the forward and inverse scattering problems for waveguide in a full generality.

The notation in this paper is standard. For a self-adjoint operator A, $\sigma(A)$, $\sigma_p(A)$ and $\sigma_{ess}(A)$ mean its spectrum, point spectrum and essential spectrum, respectively. For two Banach spaces $\mathcal{H}_1, \mathcal{H}_2$, $\mathbf{B}(\mathcal{H}_1; \mathcal{H}_2)$ means the space of all bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For an operator A on a Hilbert space $\mathcal{H}$, $D(A)$ denotes its domain of definition. For a Riemannian manifold $\mathcal{M}$, $H^m(\mathcal{M})$ denotes the usual Sobolev space of order m on $\mathcal{M}$. For a domain D and a Hilbert pace $\mathcal{H}$, $L^2(D; \mathcal{H}; d\mu)$ means the space of $\mathcal{H}$-valued L^2-functions on D with respect to the measure $d\mu$. If $\mathcal{H} = \mathbb{C}$, we omit it. For a differentiable manifold M and $p \in M$, $T_p(M)$ denotes the tangent space of M at p. A simplified version of our results is given in [42].

2. A-priori estimates in half-cylinders

The forward problem of scattering has a long history, and has been brought into a satisfactory stage in the case of short-range perturbations. For example, an early statement of the limiting absorption principle, which is the first important step for the study of the continuous spectrum, can be found in [36]. For the case of waveguides, it was proved by [70]. Assuming, roughly speaking, that the ends are purely Euclidean cylinders outside a compact set, the limiting absorption principle, eigenfunction expansion theorem, completeness of wave operators, representation of S-matrices have been studied by Eidus [23], Goldstein [30, 31], Lyford [60, 61], Wilcox [75], Guillot-Wilcox [33], Edward [44].

Christiansen [16], and Christiansen-Zworski [18] studied the waveguide problem in the framework of b-metric due to Melrose [62, 63]. Assuming that the ends, whose manifolds at infinity do not have boundaries, are not necessarily Euclidean allowing exponentially decaying perturbations, they derived the trace formula and spectral asymptotics.

Our assumptions on the ends are similar to those of [15, 16, 18]. The difference is that we allow general short-range perturbations and also deal with boundary conditions for the manifolds at infinity. Although this is a folklore result, we feel it necessary to add the proof, since the main techniques have now been scattered in many papers. As the method of the proof of limiting absorption principle, we employ integration by parts due to Eidus. This is an elementary tool, however, gives no less deeper result than modern machineries. We show the completeness of wave operators by observing the behavior at infinity of solutions to the wave equation. This will give an intuition for the propagation of waves in the waveguide. We also deduce the eigenfunction expansion theorem from the behavior of the resolvent at infinity. This is an important intermediate step between the forward problem and the inverse problem.

As a preliminary, let us begin with proving some a-priori estimates for the operator $-\partial_y^2 - \Delta_h$ on $\Omega_0 = M \times \mathbb{R}_+$ with Neumann boundary condition, where $y \in R_+ = (0, \infty)$, M is a compact Riemannian manifold, and Δ_h is the Laplace-Beltrami operator associated with metric $h(x, dx)$ equipped on M.

2.1. **Besov type spaces on cylinder.** We define an abstract Besov type space, which was introduced by Hörmander [1] in the case of $\mathbb{R}^n$. Let M be the above mentioned compact manifold, and $(\, , \,)_M$, $\| \cdot \|_{L^2(M)}$ be inner product and norm of $L^2(M)$, respectively. We define intervals I_n by

$$I_n = \begin{cases} (2^{n-1}, 2^n], & n \geq 1, \\ (0, 1], & n = 0. \end{cases}$$

Let $\mathcal{B}$ be the Banach space of $L^2(M)$-valued functions on $(0, \infty)$ equipped with norm

$$\|f\|_{\mathcal{B}} = \sum_{n=0}^{\infty} 2^{n/2} \left(\int_{I_n} \|f(y)\|^2_{L^2(M)} dy \right)^{1/2}.$$

Its dual space is the set of $L^2(M)$-valued functions $u(y)$ satisfying

$$\|u\|_{\mathcal{B}^*} = \sup_{n \geq 0} 2^{-n/2} \left(\int_{I_n} \|v(y)\|^2_{L^2(M)} dy \right)^{1/2} < \infty.$$

It is easy to see that there exists a constant $C > 0$ such that

$$\begin{aligned} C^{-1} \sup_{n \geq 0} 2^{-n/2} \left(\int_{I_n} \|v(y)\|^2_{L^2(M)} dy \right)^{1/2} &\leq \left(\sup_{R>1} \frac{1}{R} \int_0^R \|u(y)\|^2_{L^2(M)} dy \right)^{1/2} \\ &\leq C \sup_{n \geq 0} 2^{-n/2} \left(\int_{I_n} \|v(y)\|^2_{L^2(M)} dy \right)^{1/2}. \end{aligned}$$

Therefore, we identify $\mathcal{B}^*$ with the space equipped with norm

$$\|u\|_{\mathcal{B}^*} = \left(\sup_{R>1} \frac{1}{R}\int_0^R \|u(y)\|^2_{L^2(M)}dy\right)^{1/2}.$$

We also use the following weighted L^2 space and weighted Sobolev space: For $s \in \mathbb{R}$,

$$L^{2,s} \ni f \Longleftrightarrow \|f\|_s^2 = \int_0^\infty (1+y)^{2s}\|f(y)\|^2_{L^2(M)}dy < \infty,$$

$$H^{m,s} \ni u \Longleftrightarrow \|u\|_{H^{m,s}} = \|(1+y)^s u\|_{H^m(M\times(0,\infty))} < \infty.$$

In the following, $\|\cdot\|$ means $\|\cdot\|_0$ and $(\cdot,\cdot)$ denotes the inner product of $L^2(M\times\mathbb{R}_+)$. It often denotes the coupling of two functions $f \in L^{2,s}$ and $g \in L^{2,-s}$ or $f \in \mathcal{B}$ and $g \in \mathcal{B}^*$. The following inclusion relations can be shown easily, and the proof is omitted.

Lemma 2.1. *For $s > 1/2$, we have*

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}.$$

We often make use of the following lemma, whose proof is also elementary and omitted.

Lemma 2.2. *Suppose $u \in \mathcal{B}^*$. Then*

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \|u(y)\|^2_{L^2(M)}dy = 0, \tag{2.1}$$

if and only if

$$\lim_{R\to\infty} \frac{1}{R}\int_0^\infty \rho\Big(\frac{y}{R}\Big)\|u(y)\|^2_{L^2(M)}dy = 0, \quad \forall \rho \in C_0^\infty((0,\infty)). \tag{2.2}$$

2.2. **A-priori estimates.** Let us consider the following equation in $\Omega_0 = M\times\mathbb{R}_+$:

$$\begin{cases} (-\partial_y^2 - \Delta_h - z)u = f \quad \text{in} \quad \Omega_0 \\ \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega_0, \end{cases} \tag{2.3}$$

z being a complex parameter, and ∂_ν conormal differentiation on the boundary. In the following, we often denote by $\|\partial_x^\alpha u\|$ the norm of derivatives of $|\alpha|$-th order of u without mentioning local coordinates.

Lemma 2.3. *Let $z \in \mathbb{C}$ be given. Then :*
(1) If $u, f \in L^{2,s}$ for some $s \in \mathbb{R}$, we have

$$\sum_{|\alpha|+l\le 2} \|\partial_x^\alpha \partial_y^l u\|_s \le C(\|u\|_s + \|f\|_s).$$

(2) If $u, f \in \mathcal{B}^*$, then we have

$$\|\partial_x u\|_{\mathcal{B}^*} + \|\partial_y u\|_{\mathcal{B}^*} \le C(\|u\|_{\mathcal{B}^*} + \|f\|_{\mathcal{B}^*}).$$

Proof. We shall prove (2). Pick $\chi(y) \in C_0^\infty(\mathbb{R})$ such that $\chi(y) = 1$ $(|y| < 1)$, $\chi(y) = 0$ $(|y| > 2)$ and put $\chi_R(y) = \chi(y/R)$. We take the inner product in $L^2(\Omega_0)$ of (2.3) and $\chi_R^2(y)u$. We then have

$$\|\chi_R\partial_y u\|^2 + \big(\chi_R\partial_y u, \frac{2}{R}\chi'(\frac{y}{R})u\big) + \|\chi_R\partial_x u\|^2 - z\|\chi_R u\|^2 = (f, \chi_R^2 u),$$

which implies

$$\|\chi_R\partial_y u\|^2 + \|\chi_R\partial_x u\|^2 \leq C\left(\frac{1}{R^2}\|\chi'(\frac{y}{R})u\|^2 + \|\chi_R u\|^2 + \|\chi_R f\|^2\right).$$

Then we have for $R > 1$

$$\begin{aligned} &\int_0^R \|\partial_y u\|^2_{L^2(M)}dy + \int_0^R \|\partial_x u\|^2_{L^2(M)}dy \\ \leq\ & C\left(\int_0^{2R} \|u\|^2_{L^2(M)}dy + \int_0^{2R} \|f\|^2_{L^2(M)}dy\right). \end{aligned}$$

Dividing by R and taking the supremum with respect to R, we obtain (2).

Let us prove (1). The 1st order derivatives are dealt with in the same way as above. We put $v = (1+y)^s u$. Then v satisfies $(-\partial_y^2 - \Delta_h - z)v = g$, where $g \in L^2(\Omega)$. By the a-priori estimates for elliptic operators, we have $v \in H^2(\Omega)$, which proves (1). □

Let $\lambda_1 < \lambda_2 \leq \cdots \to \infty$ be the eigenvalues of $-\Delta_h$, and P_n the associated eigenprojection. Then

$$\sqrt{z + \Delta_h} = \sum_{n=1}^\infty \sqrt{z - \lambda_n}\, P_n,$$

where for $\zeta = re^{i\theta}$, $(r > 0, 0 < \theta < 2\pi)$, we define $\sqrt{\zeta} = \sqrt{r}e^{i\theta/2}$.

Our next aim is to derive some a-priori estimates for solutions to the equation (2.3). We use the method of integration by parts due to Eidus ([23]). We put

$$P(z) = \sqrt{z + \Delta_h},$$

$$D_\pm(z) = \partial_y \mp iP(z).$$

Then the equation (2.3) is rewritten as

$$\partial_y D_\pm(z)u = \mp iP(z)D_\pm(z)u - f. \tag{2.2}$$

Lemma 2.4. *Let $\varphi(y) \in C^\infty(\mathbb{R})$ be such that $\varphi(y) \geq 0$. For a solution u of the equation (2.3), we put $w = D_+(z)u$. Then if* $\operatorname{Im} z \geq 0$ *we have for any $0 < a < b < \infty$*

$$\int_a^b \varphi'(y)\|w(y)\|^2_{L^2(M)}dy \leq 2\int_a^b \varphi(y)\big|(f, w)_{L^2(M)}\big|dy + \Big[\varphi\|w\|^2_{L^2(M)}\Big]_{y=a}^{y=b}.$$

Proof. Since w satisfies $\partial_y w = -iP(z)w - f$, we have

$$\int_a^b \varphi(y)(\partial_y w, w)_{L^2(M)}dy = -i\int_a^b \varphi(y)(P(z)w, w)_{L^2(M)}dy - \int_a^b \varphi(y)(f, w)_{L^2(M)}dy.$$

Taking the real part and integrating by parts, we have

$$\begin{aligned}&\Big[\varphi\|w\|^2_{L^2(M)}\Big]_a^b - \int_a^b \varphi'(y)\|w(y)\|^2_{L^2(M)}dy\\ = \quad & 2\int_a^b \varphi(y)(\operatorname{Im} P(z)w, w)_{L^2(M)}dy - 2\mathrm{Re}\int_a^b \varphi(y)(f, w)_{L^2(M)}dy.\end{aligned}$$

Taking notice of $\operatorname{Im} P(z) \geq 0$ for $\operatorname{Im} z \geq 0$, we get the lemma. □

Let $\mathbb{C}_+ = \{z \in \mathbb{C}\,; \operatorname{Im} z \geq 0\}$.

Lemma 2.5. *Let w be as in Lemma 2.4 and suppose that*

$$\lim_{R\to\infty} \frac{1}{R}\int_1^R \|w(y)\|^2_{L^2(M)}dy = 0. \tag{2.3}$$

Then there exists a constant $C > 0$ independent of $z \in \mathbb{C}_+$ such that

$$\|w(y)\|^2_{L^2(M)} \leq C\|f\|_{\mathcal{B}}\|w\|_{\mathcal{B}^*}, \quad \forall y \in \mathbb{R}.$$

Proof. Taking $\varphi(y) = 1$ in Lemma 2.4, we have

$$\begin{aligned}\|w(a)\|^2_{L^2(M)} &\leq \|w(b)\|^2_{L^2(M)} + 2\int_a^b |(f, w)_{L^2(M)}|dy\\ &\leq \|w(b)\|^2_{L^2(M)} + C\|f\|_{\mathcal{B}}\|w\|_{\mathcal{B}^*}.\end{aligned}$$

The assumption of the lemma implies $\liminf_{b\to\infty} \|w(b)\|_{L^2(M)} = 0$, which proves the lemma. □

Corollary 2.6. *Under the assumption of Lemma 2.5, there exists a constant $C > 0$ such that*

$$\|w\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad \forall z \in \mathbb{C}_+.$$

Proof. Lemma 2.5 implies that

$$\|w\|^2_{\mathcal{B}^*} = \sup_{R>1}\frac{1}{R}\int_0^R \|w(y)\|^2_{L^2(M)}dy \leq C\|f\|_{\mathcal{B}}\|w\|_{\mathcal{B}^*},$$

which proves this corollary. □

Theorem 2.7. *For a small $\delta > 0$, let*

$$J_\delta = \{z \in \mathbb{C}_+\ ;\ \mathrm{dist}\,\big(\operatorname{Re} z, \sigma(-\Delta_h)\big) > \delta\}.$$

Let u be a solution to (2.3) such that $w = D_+(z)u$ satisfies (2.3). Then there exists a constant $C > 0$ such that

$$\|u\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}$$

holds uniformly for $z \in J_\delta$.

Proof. Let $A(z) = \operatorname{Re} P(z) = (P(z) + P(z)^*)/2$. By the equation (2.2), we have

$$\partial_y(w, u)_{L^2(M)} = -i(P(z)w, u)_{L^2(M)} - (f, u)_{L^2(M)} + (w, \partial_y u)_{L^2(M)}.$$

In view of the formula

$$\begin{aligned}-i(P(z)w, u)_{L^2(M)} &= -2i(A(z)w, u)_{L^2(M)} + i(P(z)^*w, u)_{L^2(M)}\\ &= -2i(w, A(z)u)_{L^2(M)} + i(w, P(z)u)_{L^2(M)},\end{aligned}$$

we then have

$$\partial_y(w,u)_{L^2(M)} = -2i(w, A(z)u)_{L^2(M)} - (f,u)_{L^2(M)} + \|w\|^2_{L^2(M)}.$$

Using $w = \partial_y u - iP(z)u$, we compute

$$2i(w, A(z)u)_{L^2(M)} = 2i(\partial_y u, A(z)u)_{L^2(M)} + \|P(z)u\|^2_{L^2(M)} + (P(z)^2 u, u)_{L^2(M)}.$$

Summing up, we have arrived at

$$\begin{aligned}\partial_y(w,u)_{L^2(M)} = &-2i(\partial_y u, A(z)u)_{L^2(M)} - \|P(z)u\|^2_{L^2(M)} \\ &- ((z+\Delta_h)u, u)_{L^2(M)} - (f,u)_{L^2(M)} + \|w\|^2_{L^2(M)}.\end{aligned}$$

Taking the imaginary part and integrating in y, we have

$$\begin{aligned}\mathrm{Im}\Big[(w,u)_{L^2(M)}\Big]_{y=a}^{y=b} = &-2\mathrm{Re}\int_a^b (\partial_y u, A(z)u)_{L^2(M)} \\ &- \mathrm{Im}\, z\int_a^b \|u\|^2_{L^2(M)}dy - \mathrm{Im}\int_a^b (f,u)_{L^2(M)}dy.\end{aligned}$$

Since $A(z)$ is self-adjoint, we have by integration by parts

$$2\mathrm{Re}\int_a^b (\partial_y u, A(z)u)_{L^2(M)}dy = \Big[(A(z)u,u)_{L^2(M)}\Big]_{y=a}^{y=b}.$$

Using $\mathrm{Im}\, z \geq 0$, we obtain

$$\mathrm{Im}\Big[(w,u)_{L^2(M)}\Big]_{y=a}^{y=b} + \Big[(A(z)u,u)_{L^2(M)}\Big]_{y=a}^{y=b} \leq C\|f\|_{\mathcal{B}}\|u\|_{\mathcal{B}^*}, \tag{2.4}$$

where C is independent of $z \in \mathbb{C}_+$. We renumber the eigenvalues of $-\Delta_h$ in the increasing order $\mu_1 < \mu_2 < \cdots$ without counting multiplicities and put $\mu_0 = -\infty$, i.e. $\{\lambda_n\,; n = 1,2,\cdots\}$ and $\{\mu_n\,; n = 1,2\cdots\}$ are the same as subsets of $\mathbb{R}$. For a sufficiently small $\delta > 0$, we put

$$J_{n,\delta} = \{z \in \mathbb{C}_+\ ;\ \mu_{n-1} + \delta < \mathrm{Re}\, z < \mu_n - \delta\}.$$

Assume $z \in J_{n,\delta}$ and split u as $u = u_< + u_>$, where

$$u_< = \sum_{\lambda_j \leq \mu_{n-1}} P_j u, \quad u_> = \sum_{\lambda_j \geq \mu_n} P_j u,$$

Recall that P_j is the eigenprojection associated with λ_j. We also define $w_<$, $w_>$, $f_<$, $f_>$ similarly. Note that $w_< = D_+(z)u_<$. Let us remark that (2.3) and therefore (2.4) hold with w, u, f replaced by $w_<, u_<, f_<$ and $w_>, u_>, f_>$, respectively. For eigenvalues $\lambda_j \leq \mu_{n-1}$, we have $\mathrm{Re}\sqrt{z-\lambda_j} \geq \sqrt{\delta}$. Therefore

$$(A(z)u_<, u_<)_{L^2(M)} \geq \sqrt{\delta}\|u_<\|^2_{L^2(M)}. \tag{2.5}$$

Since $\partial_y u(0) = 0$, we have $w_<(0) = -iP(z)u_<(0)$. Therefore

$$\begin{aligned}-\mathrm{Im}\,(w_<(0), u_<(0))_{L^2(M)} &= \mathrm{Re}\,(P(z)u_<(0), u_<(0))_{L^2(M)} \\ &= (A(z)u_<(0), u_<(0))_{L^2(M)}.\end{aligned}$$

Letting $a = 0$, $b = t$ in (2.4), we then have

$$\mathrm{Im}\,(w_<(t), u_<(t))_{L^2(M)} + (A(z)u_<(t), u_<(t))_{L^2(M)} \leq C\|f\|_{\mathcal{B}}\|u\|_{\mathcal{B}^*}.$$

Using (2.5), we have

$$\|u_<(t)\|^2_{L^2(M)} \le C\big(\|w_<(t)\|^2_{L^2(M)} + \|f_<\|_{\mathcal{B}}\|u_<\|_{\mathcal{B}^*}\big).$$

Using Corollary 2.6, we then have for $R > 1$

$$\frac{1}{R}\int_0^R \|u_<(y)\|^2_{L^2(M)}dy \le C\left(\|f_<\|^2_{\mathcal{B}} + \|f_<\|_{\mathcal{B}}\|u_<\|_{\mathcal{B}^*}\right),$$

which implies

$$\|u_<\|_{\mathcal{B}^*} \le C\|f_<\|_{\mathcal{B}}. \tag{2.6}$$

On the other hand, if $\lambda_j \ge \mu_n$, we have $\mathrm{Re}\,(\lambda_j - z) \ge \delta$. Therefore

$$(-\partial_y^2 - \Delta_h - z)u_> = (-\partial_y^2 + B_z - i\mathrm{Im}\,z)u_> = f_>, \tag{2.7}$$

where B_z is a uniformly, with respect to z, strictly positive operator on $L^2(M)$. Hence, we have

$$\|u_>\|_{L^2} \le C\|f_>\|_{L^2}, \tag{2.8}$$

which by Lemma 2.1 implies

$$\|u_>\|_{\mathcal{B}^*} \le C\|f_>\|_{\mathcal{B}}. \tag{2.9}$$

The above two inequalities (2.6) and (2.9) prove the theorem. □

3. Manifolds with cylindrical ends

3.1. **Resolvent equation.** We return to the manifold $\Omega = \mathcal{K} \cup \Omega_1 \cup \cdots \cup \Omega_N$ introduced in §1. Fix a point $P_0 \in \mathcal{K}$ arbitrarily, and let $\mathrm{dist}(P, P_0)$ be the geodesic distance with respect to the metric G from P_0 to P. We put

$$\Omega_0(R) = \{P \in \Omega\,;\, \mathrm{dist}\,(P, P_0) < R\}, \quad \Omega_\infty(R) = \{P \in \Omega\,;\, \mathrm{dist}\,(P, P_0) \ge R\}.$$

For $R > 0$ large enough, take $\chi_0 \in C_0^\infty(\Omega)$ such that $\chi_0 = 1$ on $\Omega_0(R)$, $\chi_0 = 0$ on $\Omega_\infty(R+1)$. Define $\chi_j = 1 - \chi_0$ on Ω_j, $\chi_j = 0$ on $\Omega \setminus \Omega_j$. Then $\{\chi_j\}_{j=0}^N$ is a partition of unity on Ω.

Let Δ_G be the Laplace-Beltrami operator for the metric G on Ω endowed with Neumann boundary condition on $\partial\Omega$. The conormal differentiation with respect to G is denoted by ∂_ν. We put

$$H = -\Delta_G, \quad R(z) = (H - z)^{-1}.$$

As in §1, we identify Ω_j with $M_j \times (0, \infty)$, and let $h_j(x, dx)$ be the metric on M_j. We compare G with the unperturbed metric $G_j^{(0)} = (dy)^2 + h_j(x, dx)$ on Ω_j. Let $\Delta_{G_j^{(0)}}$ be the Laplace-Beltrami operator for $G_j^{(0)}$ with Neumann boundary condition on $\partial\Omega_j$. The associated conormal differentiation is denoted by $\partial_{\nu_j^{(0)}}$. We put

$$H_j^{(0)} = -\Delta_{G_j^{(0)}}, \quad R_j^{(0)}(z) = (H_j^{(0)} - z)^{-1}.$$

Our next concern is the difference between the boundary conditions for H and $H_j^{(0)}$. We put for large $R > 0$

$$\partial\Omega_j(R) = \partial\Omega \cap \Omega_j \cap \Omega_\infty(R).$$

Lemma 3.1. *There exists a real function $w(x,y) \in C^\infty(\Omega_j)$ such that*

$$\begin{cases} \partial_\nu w(x,y) = 0 \quad \text{on} \quad \partial\Omega_j(R) \\ w(x,y) = y + O(y^{-1-\epsilon_0}) \quad \text{as} \quad y \to \infty. \end{cases} \tag{3.1}$$

Proof. By the decay assumption (1.2), letting $w(x,y) = y + \widetilde{w}(x,y)$, we should have $\partial_\nu \widetilde{w} = -\partial_\nu y = O(y^{-1-\epsilon_0})$ on $\partial\Omega_j(R)$. Extending the vector field ν near the boundary and integrating along it, we get $\widetilde{w} = O(y^{-1-\epsilon_0})$. □

For $m \geq 0$ and $s \in \mathbb{R}$, we define the weighted Sobolev space on the boundary by

$$\psi \in H^{m,s}(\partial\Omega_j(R)) \Longleftrightarrow (1+y)^s\psi \in H^m(\partial\Omega_j(R)).$$

Lemma 3.2. *There exists an operator of extension $\widetilde{\mathcal{E}}_j$ such that for $m \geq 1/2$ and $\psi \in H^m(\partial\Omega_j(R))$*

$$\partial_\nu \widetilde{\mathcal{E}}_j \psi = \begin{cases} \psi \quad \text{on} \quad \partial\Omega_j(R), \\ 0 \quad \text{on} \quad \Omega \setminus (\Omega_j \cap \Omega_\infty(R-1/2)), \end{cases} \tag{3.2}$$

$$\operatorname{supp}(\widetilde{\mathcal{E}}_j\psi) \subset \Omega_j \cap \Omega_\infty(R-1). \tag{3.3}$$

For $m \geq 1/2$ and $s \geq 0$, it satisfies

$$\widetilde{\mathcal{E}}_j \in \mathbf{B}(H^{m,s}(\partial\Omega_j(R)); H^{m+3/2,s}(\Omega_j)). \tag{3.4}$$

Proof. Let $\mathcal{M}' = \Omega_j \cap \Omega_\infty(R-2)$. We smoothly modify the corner of $\mathcal{M}'$, i.e. $\{P \in \Omega_j \cap \partial\Omega\,;\, \mathrm{dist}(P,P_0) = R-2\}$, and let $\mathcal{M}$ be the resulting manifold. Let $\nu_{\mathcal{M}}$ be the unit outer normal to $\mathcal{M}$. By solving the elliptic boundary value problem

$$\begin{cases} (-\Delta_G + i)u = 0 \quad \text{in} \quad \mathcal{M}, \\ \partial_{\nu_{\mathcal{M}}} u = \psi \quad \text{on} \quad \partial\mathcal{M}, \end{cases} \tag{3.5}$$

we define $\widetilde{\mathcal{E}}_j\psi = \widetilde{\chi}_j u$, where $\widetilde{\chi}_j \in C^\infty(\Omega_j)$ is such that $\widetilde{\chi}_j = 1$ on $\Omega_j \cap \Omega_\infty(R-1/4)$, $\widetilde{\chi}_j = 0$ on $\Omega \setminus \big(\Omega_j \cap \Omega_\infty(R-1/2)\big)$. It then satisfies (3.2), (3.3). The property (3.4) for $s = 0$ follows from the standard estimate for the elliptic boundary value problem. Let $0 < s \leq 1 + \epsilon_0$ and take $\psi \in H^{m,s}(\partial\mathcal{M})$. For the solution u to the boundary value problem (3.5), we define $u_1 = (1+w(x,y))^s u$ and $\psi_1 = (1+w(x,y))^s\psi$, where $w(x,y)$ is constructed in Lemma 3.1. Then u_1 is a solution to the boundary value problem

$$\begin{cases} (-\Delta_G + L_1 + \kappa)u_1 = 0 \quad \text{in} \quad \mathcal{M}, \\ \partial_{\nu_{\mathcal{M}}} u_1 = \psi_2 \quad \text{on} \quad \partial\mathcal{M}, \end{cases}$$

where $\kappa > 0$ is sufficiently large, and L_1 is a 1st order differential operator with bounded coefficients, and $\psi_2 = \psi_1$ on $\partial\Omega_j(R)$. Since the mapping $\psi_2 \to u_1$ is bounded from $H^m(\partial\mathcal{M})$ to $H^{m+3/2}(\mathcal{M})$, we get (3.4) with $0 < s \leq 1+\epsilon_0$. Repeating this procedure, we can prove (3.4) for all $s > 0$. □

For $u \in H^2(\Omega_j)$ satisfying $\partial_{\nu_j^{(0)}} u = 0$ on $\partial\Omega_j(R)$, we have

$$\partial_\nu(\chi_j u) = w(x,y)^{-1-\epsilon_0} B_j u \quad \text{on} \quad \partial\Omega_j(R), \tag{3.6}$$

where

$$B_j = w(x,y)^{1+\epsilon_0}\Big(\chi_j\big(\partial_\nu - \partial_{\nu_j^{(0)}}\big) + \big(\partial_\nu \chi_j\big)\Big) \tag{3.7}$$

is a 1st order differential operator on $\partial\Omega_j(R)$ with bounded coefficients. We put

$$\mathcal{E}_j = w(x,y)^{-1-\epsilon_0}\widetilde{\mathcal{E}}_j. \tag{3.8}$$

Then by (3.1), (3.2) and (3.6), for $u \in H^2(\Omega)$ satisfying $\partial_{\nu_j^{(0)}} u = 0$ on $\partial\Omega_j(R)$ the following formula holds

$$\partial_\nu \mathcal{E}_j B_j u = \partial_\nu\left(\chi_j u\right) \quad \text{on} \quad \partial\Omega_j(R). \tag{3.9}$$

Moreover

$$y^{1+\epsilon_0}\mathcal{E}_j B_j \in \mathbf{B}(H^2(\Omega); H^2(\Omega)) \cap \mathbf{B}(H^{3/2}(\Omega); H^{3/2}(\Omega)). \tag{3.10}$$

Suppose u satisfies

$$\begin{cases} (-\Delta_{G_j^{(0)}} - z)u = f & \text{in} \quad \Omega_j, \\ \partial_{\nu_j^{(0)}} u = 0 & \text{on} \quad \Omega_j \cap \partial\Omega. \end{cases}$$

Then by (3.9), $v_j = \chi_j u - \mathcal{E}_j B_j u$ satisfies

$$\begin{cases} (-\Delta_G - z)v_j = \chi_j f + \mathcal{V}_j(z)u & \text{in} \quad \Omega, \\ \partial_\nu v_j = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.11}$$

where

$$\mathcal{V}_j(z) = [-\Delta_G, \chi_j] + \chi_j(\Delta_{G_j^{(0)}} - \Delta_G) + (\Delta_G + z)\mathcal{E}_j B_j. \tag{3.12}$$

Lemma 3.3. *Let $\widetilde{\chi}_j \in C^\infty(\Omega)$ be such that $\widetilde{\chi}_j = 1$ on $\Omega_j \cap \Omega_\infty(R-1)$ and $\widetilde{\chi}_j = 0$ outside $\Omega_j \cap \Omega_\infty(R-2)$. Then for $z \notin \mathbb{R}$, the following resolvent equations hold :*

$$R(z)\chi_j = \Big(\chi_j - \mathcal{E}_j B_j - R(z)\mathcal{V}_j(z)\Big)R_j^{(0)}(z)\widetilde{\chi}_j. \tag{3.13}$$

$$\chi_j R(z) = \widetilde{\chi}_j J_j^{-1} R_j^{(0)}(z) J_j \Big(\chi_j - (\mathcal{E}_j B_j)^* - \mathcal{V}_j(\overline{z})^* R(z)\Big), \tag{3.14}$$

where $J_j = \big(\det G/\det G_j^{(0)}\big)^{1/2}$, and the adjoint $*$ is taken with respect to the inner product of $L^2(\Omega)$ with volume element from the metric G. Moreover $R_j^{(0)}(z)J_j(\mathcal{E}_j B_j)^*$ and $R_j^{(0)}(z)J_j\mathcal{V}_j(\overline{z})^* R(z)$ are compact on $L^2(\Omega)$.

Proof. Let $u = R_j^{(0)}(z)\widetilde{\chi}_j f$ for $z \notin \mathbb{R}$. Then checking the boundary condition by (3.9), we have $v_j = \chi_j R_j^{(0)}(z)\widetilde{\chi}_j f - \mathcal{E}_j B_j R_j^{(0)}(z)\widetilde{\chi}_j f \in D(H)$, and by (3.11) $(H - z)v_j = \chi_j\widetilde{\chi}_j f + \mathcal{V}_j(z)u = \chi_j f + \mathcal{V}_j(z)u$, which implies (3.13).

By extending $f \in L^2(\Omega_j)$ to be 0 outside Ω_j, we regard $L^2(\Omega_j)$ as a closed subspace of $L^2(\Omega)$. The volume elements dV and $dV_j^{(0)}$ of G and $G_j^{(0)}$ satisfy $dV = J_j dV_j^{(0)}$. For $A \in \mathbf{B}(L^2(\Omega_j); L^2(\Omega_j))$, let A^* and $A^{*(j)}$ denote their adjoint operators with respect to the volume element dV and $dV_j^{(0)}$, respectively. Then it is easy to show that

$$A^* = J_j^{-1} A^{*(j)} J_j.$$

Taking $A = R_j^{(0)}(z)$, and noting that $R(z)^* = R(\overline{z})$ and $R_j^{(0)}(z)^{*(j)} = R_j^{(0)}(\overline{z})$, we prove (3.14). By (3.10) and (3.12), $\mathcal{E}_j B_j R_j^{(0)}(z)$ and $R(z)\mathcal{V}_j(z)J_j R_j^{(0)}(z)$ are compact on $L^2(\Omega)$, which implies the last assertion of the lemma. □

3.2. **Essential spectrum.**

Lemma 3.4. $\sigma_{ess}(H) = [0,\infty)$.

Proof. Lemma 3.3 implies $\chi_j R(z) - \widetilde{\chi}_j J_j^{-1} R_j^{(0)}(z) J_j \chi_j$ is compact. Therefore

$$R(z) = \sum_{j=1}^{N} \widetilde{\chi}_j J_j^{-1} R_j^{(0)}(z) J_j \chi_j + K(z), \tag{3.15}$$

where $K(z)$ is a compact operator and satisfies

$$\|K(z)\| \leq C|\mathrm{Im}\, z|^{-2}(1+|z|), \tag{3.16}$$

where $\|\cdot\|$ denotes the operator norm in $L^2(\Omega)$ and the constant C is independent of z. For $f(\lambda) \in C_0^\infty(\mathbb{R})$, there exists $F(z) \in C_0^\infty(\mathbb{C})$, called an almost analytic extension of f, such that $F(\lambda) = f(\lambda)$ for $\lambda \in \mathbb{R}$ and $|\overline{\partial_z} F(z)| \leq C_n |\mathrm{Im}\, z|^n$, $\forall n \geq 0$, and the following formula holds for any self-adjoint operator A :

$$f(A) = \frac{1}{2\pi i}\int_{\mathbb{C}} \overline{\partial_z} F(z)(z-A)^{-1} dz d\overline{z}. \tag{3.17}$$

(See e.g. [34] or [41].) We replace $(z-A)^{-1}$ by $-R(z)$ and plug (3.15). The inequality (3.16) implies $\|\overline{\partial_z} F(z) K(z)\| \leq C$, and the integral over $\mathbb{C}$ converges in the operator norm, hence it gives a compact operator. We then see that $\varphi(H) - \sum_{j=1}^N \widetilde{\chi}_j J_j^{-1}\varphi(H_j^{(0)}) J_j \chi_j$ is compact for any $\varphi(\lambda) \in C_0^\infty(\mathbb{R})$. Since $\sigma(H_j^{(0)}) = [0,\infty)$, we have $\varphi(H_j^{(0)}) = 0$ if $\varphi(\lambda) \in C_0^\infty((-\infty,0))$. Therefore $\varphi(H)$ is compact if $\varphi(\lambda) \in C_0^\infty((-\infty,0))$, which implies that $(-\infty,0)\cap\sigma_{ess}(H) = \emptyset$. For $\lambda \in (0,\infty) = \sigma(H_j^{(0)})$, one can construct $u_n \in D(H_j^{(0)})$ such that $\|u_n\| = 1$, $\|(H_j^{(0)} - \lambda)u_n\| \to 0$, and $\mathrm{supp}\, u_n \subset \{y > R_n\}$ with $R_n \to \infty$. Then letting $v_n = \chi_j u_n - \mathcal{E}_j B_j u_n$, we have $v_n \in D(H)$, $\|(H-\lambda)v_n\| \to 0$, $v_n \to 0$ weakly and $\|v_n\| > C$ uniformly in n with a constant $C > 0$. This implies $\lambda \in \sigma_{ess}(H)$. □

The set of thresholds for H is defined by

$$\mathcal{T}(H) = \bigcup_{j=1}^{N} \sigma_p(-\Delta_{h_j}), \tag{3.18}$$

where Δ_{h_j} is the Laplace-Beltrami operator on M_j. Replacing Ω_0 in §2 by Ω_j with $j = 1, \cdots, N$, we define the Besov type spaces $\mathcal{B}_j$, $\mathcal{B}_j^*$. We put

$$\|f\|_{\mathcal{B}} = \|\chi_0 f\|_{L^2(\Omega)} + \sum_{j=1}^{N} \|\chi_j f\|_{\mathcal{B}_j},$$

$$\|u\|_{\mathcal{B}^*} = \|\chi_0 f\|_{L^2(\Omega)} + \sum_{j=1}^{N} \|\chi_j f\|_{\mathcal{B}_j^*}.$$

The weighted L^2 space $L^{2,s}$ and the weighted Sobolev space $H^{m,s}$ are defined similarly.

3.3. **Radiation condition.** A solution $u \in \mathcal{B}^*$ of the reduced wave equation

$$\begin{cases} (H-\lambda)u = f \quad \text{in} \quad \Omega, \quad \lambda > 0, \\ \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

is said to satisfy the outgoing radiation condition if

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \|\chi_j\big(\partial_y - i\mathcal{P}_j(\lambda)\big)u\|^2_{L^2(M_j)}dy = 0, \quad 1 \leq \forall j \leq N, \tag{3.19}$$

where

$$\mathcal{P}_j(z) = \sqrt{z + \Delta_{h_j}}.$$

If $\partial_y - i\mathcal{P}_j(\lambda)$ is replaced by $\partial_y + i\mathcal{P}_j(\lambda)$, we say that u satisfies the incoming radiation condition. In the following, u is always assumed to satisfy the boundary condition $\partial_\nu u = 0$ on $\partial\Omega$.

Lemma 3.5. *Let $\lambda \in (0,\infty) \setminus \mathcal{T}(H)$. If $u \in \mathcal{B}^*$ satisfies $(H-\lambda)u = 0$ and the outgoing (or incoming) radiation condition, it also satisfies*

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \|\chi_j u\|^2_{L^2(M_j)}dy = 0, \qquad 1 \leq j \leq N.$$

Proof. We take $\rho(t) \in C_0^\infty((0,\infty))$ such that $\rho(t) \geq 0$, $\operatorname{supp}\rho(t) \subset (1,2)$ and $\int_0^\infty \rho(t)dt = 1$, and put

$$\varphi_R(y) = \chi\left(\frac{y}{R}\right), \quad \chi(t) = \int_t^\infty \rho(s)ds.$$

Then $\varphi_R(y) = 1$ for $y < R$ and $\varphi_R(y) = 0$ for $y > 2R$. We next construct $\psi_R \in C_0^\infty(\Omega)$ such that $\psi_R = 1$ on $\mathcal{K}$ and $\psi_R = \varphi_R$ on Ω_j for $1 \leq j \leq N$. Then we have

$$(i[H,\psi_R]u,u) = (i[H-\lambda,\psi_R]u,u) = 0.$$

By the construction of ψ_R, $[H,\psi_R] = 0$ on $\mathcal{K}$. By the assumption (1.2), on Ω_j the commutator has the form

$$i[H,\psi_R] = \frac{2i}{R}\rho(\frac{y}{R})\partial_y + L_{j,R}, \tag{3.20}$$

where $L_{j,R}$ is a 1st order differential operator whose coefficients have the form

$$\frac{1}{R}\widetilde{\chi}(\frac{y}{R})O(y^{-\epsilon_0})$$

and $\widetilde{\chi}(y)$ is either $\rho(y)$ or $\rho'(y)$. Let $v = (1+y)^{-\epsilon_0}u$. Then by Lemma 2.1 and Lemma 2.3 (1) (which also holds for Δ_{G_j}), $\partial_x v, \partial_y v \in L^{2,-\delta}$ for some $0 < \delta < 1/2$. Therefore

$$\frac{1}{R}\int_0^R \left(\|\partial_x v\|^2_{L^2(M_j)} + \|\partial_y v\|^2_{L^2(M_j)}\right)dy \leq \frac{C}{R^{1-2\delta}}\left(\|\partial_x v\|^2_{-\delta} + \|\partial_y v\|^2_{-\delta}\right),$$

which tends to 0 as $R \to \infty$. Therefore by Lemma 2.2

$$\lim_{R\to\infty}(L_{j,R}u,u)_{L^2(\Omega_j)} = 0.$$

Hence we have by using (3.20),

$$\lim_{R\to\infty}\sum_{j=1}^{N}\frac{1}{R}\int_0^\infty \rho\left(\frac{y}{R}\right)\left(\partial_y\chi_j u,\chi_j u\right)_{L^2(M_j)}dy = 0. \tag{3.21}$$

Assume that u satisfies the outgoing radiation condition. Using the inequality

$$\begin{aligned}&\left|\frac{1}{R}\int_0^\infty \rho(\frac{y}{R})\big((\partial_y - i\mathcal{P}_j(\lambda))\chi_j u,\chi_j u\big)_{L^2(M_j)}dy\right|\\ &\leq C\|u\|_{\mathcal{B}^*}\left(\frac{1}{R}\int_0^\infty \rho(\frac{y}{R})\,\|(\partial_y - i\mathcal{P}_j(\lambda))\chi_j u\|^2_{L^2(M_j)}\,dy\right)^{1/2}\end{aligned}$$

and (3.21), we then have

$$\lim_{R\to\infty}\sum_{j=1}^{N}\frac{1}{R}\int_0^\infty \rho\left(\frac{y}{R}\right)\left(\mathcal{P}_j(\lambda)\chi_j u,\chi_j u\right)_{L^2(M_j)}dy = 0. \tag{3.22}$$

As in the proof of Theorem 2.7, we split $\chi_j u$ into two parts,

$$\chi_j u_< = E_j((-\infty,\lambda))\chi_j u, \quad \chi_j u_> = E_j((\lambda,\infty))\chi_j u,$$

where $E_j(\cdot)$ is the spectral projection associated with $-\Delta_{h_j}$. Then by the short-range decay assumption of the metric,

$$(-\partial_y^2 - \Delta_{h_j} - \lambda)\chi_j u_> =: f_j \in L^2(\Omega_j).$$

Since $\lambda \notin \sigma(-\Delta_{M_j})$, arguing in the same way as in the proof of (2.7),

$$(-\partial_y^2 - \Delta_{h_j} - \lambda)\chi_j u_> = (-\partial_y^2 + B_j)\chi_j u_>,$$

where B_j is a self-adjoint operator on $L^2(M_j)$ such that $B_j \geq \delta(1-\Delta_{M_j})$, $\delta > 0$ being a constant. Therefore, $\mathcal{P}_j(\lambda)\chi_j u_> \in L^2(\Omega_j)$, hence

$$\lim_{R\to\infty}\frac{1}{R}\int_0^\infty \rho\left(\frac{y}{R}\right)\left(\mathcal{P}_j(\lambda)\chi_j u_>,\chi_j u_>\right)_{L^2(M_j)}dy = 0. \tag{3.23}$$

Since $\mathcal{P}_j(\lambda)\chi_j u_> = iC_j(\lambda)\chi_j u_>$, where $C_j(\lambda)$ is a strictly positive operator on $L^2(M_j)$, this also implies

$$\lim_{R\to\infty}\frac{1}{R}\int_0^\infty \rho\left(\frac{y}{R}\right)\|\chi_j u_>\|^2_{L^2(M_j)}dy = 0. \tag{3.24}$$

We show that

$$\lim_{R\to\infty}\sum_{j=1}^{N}\frac{1}{R}\int \rho\left(\frac{y}{R}\right)\left(\mathcal{P}_j(\lambda)\chi_j u_<,\chi_j u_<\right)_{L^2(M_j)}dy = 0. \tag{3.25}$$

In fact, in view of (3.22), splitting $u = u_< + u_>$ and using (3.23), to prove (3.25) we have only to show that

$$\lim_{R\to\infty}\frac{1}{R}\int \rho\left(\frac{y}{R}\right)\left(\mathcal{P}_j(\lambda)\chi_j u_>,\chi_j u_<\right)_{L^2(M_j)}dy = 0, \tag{3.26}$$

and the same assertion with $u_<$ and $u_>$ exchanged. Let us note that

$$\begin{aligned}|\big(\mathcal{P}_j(\lambda)\chi_j u_>,\chi_j u_<\big)_{L^2(M_j)}| &= |\big(\chi_j u_>,\chi_j\mathcal{P}_j(\lambda)^* u_<\big)_{L^2(M_j)}|\\ &\leq C\|\chi_j u_>\|\|\chi_j u_<\|.\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{R}\int \rho\Big(\frac{y}{R}\Big)|(\mathcal{P}_j(\lambda)\chi_j u_>, \chi_j u_<)_{L^2(M_j)}|dy \\
&\leq \left(\frac{1}{R}\int \rho\Big(\frac{y}{R}\Big)\|\chi_j u_>\|^2_{L^2(M_j)}dy\right)^{1/2}\left(\frac{C}{R}\int \rho\Big(\frac{y}{R}\Big)\|\chi_j u_<\|^2_{L^2(M_j)}dy\right)^{1/2} \\
&\leq C\left(\frac{1}{R}\int \rho\Big(\frac{y}{R}\Big)\|\chi_j u_>\|^2_{L^2(M_j)}dy\right)^{1/2},
\end{aligned}$$

since $\chi_j u_< \in \mathcal{B}^*$. By (3.24), this converges to 0. Similarly, we can prove (3.26) with $u_<$ and $u_>$ exchanged.

On the other hand, $(\mathcal{P}_j(\lambda)\chi_j u_<, \chi_j u_<) \geq C\|\chi_j u_<\|^2_{L^2(\Omega_j)}$ for a constant $C > 0$, which depends on λ. Therefore by (3.25)

$$\lim_{R\to\infty}\frac{1}{R}\int_0^\infty \rho\left(\frac{y}{R}\right)\|\chi_j u_<\|^2_{L^2(M_j)}dy = 0. \tag{3.27}$$

By (3.24) and (3.27), we complete the proof of the lemma. □

Lemma 3.6. *Suppose $u \in \mathcal{B}^*$ satisfies $(H-\lambda)u = f$ for $\lambda \in (0,\infty)\setminus\mathcal{T}(H)$ and $\partial_\nu u = 0$ on $\partial\Omega$. Assume also for some $1 \leq j \leq N$, $f \in L^{2,s}(\Omega_j)$ for any $s > 0$, and*

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R \|\chi_j u\|^2_{L^2(M_j)}dy = 0. \tag{3.28}$$

Then $u \in L^{2,s}(\Omega_j)$ for any $s > 0$. Moreover for any $s > 0$ and any compact interval $I \subset (0,\infty)\setminus\mathcal{T}(H)$, there exists a constant $C_{s,I} > 0$ such that

$$\|\chi_j u\|_{L^{2,s}(\Omega_j)} \leq C_{s,I}(\|u\|_{\mathcal{B}^*(\Omega_j)} + \|f\|_{L^{2,s+1}(\Omega_j)}), \quad \forall\lambda \in I. \tag{3.29}$$

Proof. We construct counterparts of $\mathcal{E}_j$ and B_j when the roles of G and $G_j^{(0)}$ are interchanged. Namely, there exists an operator of extension $\widetilde{\mathcal{E}_j}^{(0)}$ such that for $m \geq 1/2$ and $\psi \in H^m(\partial\Omega_j(R))$

$$\partial_{\nu_j^{(0)}}\widetilde{\mathcal{E}_j}^{(0)}\psi = \begin{cases}\psi & \text{on} \quad \partial\Omega_j(R), \\ 0 & \text{on} \quad \Omega\setminus(\Omega_j\cap\Omega_\infty(R-1/2)),\end{cases}$$

$$\mathrm{supp}\,(\widetilde{\mathcal{E}_j}^{(0)}\psi) \subset \Omega_j\cap\Omega_\infty(R-1),$$

$$\widetilde{\mathcal{E}_j}^{(0)} \in \mathbf{B}(H^{m,s}(\partial\Omega_j(R)); H^{m+3/2,s}(\Omega_j)), \quad m \geq 1/2, \quad s \geq 0.$$

If $\partial_\nu u = 0$ on $\partial\Omega$, we have

$$\partial_{\nu_j^{(0)}}(\chi_j u) = y^{-1-\epsilon_0}B_j^{(0)}u \quad \text{on} \quad \partial\Omega_j(R),$$

where

$$B_j^{(0)} = y^{1+\epsilon_0}\chi_j(\partial_{\nu_j^{(0)}} - \partial_\nu).$$

We put

$$\mathcal{E}_j^{(0)} = y^{-1-\epsilon_0}\widetilde{\mathcal{E}}_j^{(0)}.$$

Then

$$\partial_{\nu_j^{(0)}}\mathcal{E}_j^{(0)}B_j^{(0)}u = \partial_{\nu_j^{(0)}}(\chi_j u).$$

Suppose $u \in \mathcal{B}^*$ satisfies $(H-\lambda)u = f$, $\lambda \in (0,\infty)\setminus\mathcal{T}(H)$, and $\partial_\nu u = 0$ on $\partial\Omega$. We put $v_j = \chi_j u - \mathcal{E}_j^{(0)} B_j^{(0)} u$. Then v_j satisfies

$$\begin{cases} (-\partial_y^2 - \Delta_{h_j} - \lambda)v_j = \chi_j f + L_j u + (\partial_y^2 + \Delta_{h_j} + \lambda)\mathcal{E}_j^{(0)} B_j^{(0)} u & \text{in} \quad \Omega_j, \\ \partial_{\nu_j^{(0)}} v_j = 0 \quad \text{on} \quad \partial\Omega_j. \end{cases} \tag{3.30}$$

Here L_j is a 2nd order differential operator with coefficients decaying like $O(y^{-1-\epsilon_0})$. Note that $f_j := \chi_j f + L_j u + (\partial_y^2 + \Delta_{h_j} + \lambda)\mathcal{E}_j^{(0)} B_j^{(0)} u \in L^{2,1+\epsilon_0}(\Omega_j)$.

Let $v_{j,n} = (v_j(\cdot,y),\psi_n(\cdot))_{L^2(M_j)}$, where $\psi_n(x)$ is the normalized eigenvector associated with the eigenvalue λ_n of $-\Delta_{h_j}$. Then we have

$$(-\partial_y^2 - \mu_n)v_{j,n} = g_{j,n}, \quad \mu_n = \lambda - \lambda_n, \tag{3.31}$$

where $g_{j,n} \in L^{2,(1+\epsilon_0)/2}((-\infty,\infty))$, and $v_{j,n}(y) = g_{j,n}(y) = 0$ for $y < 0$. Let $r_0(z) = (-\partial_y^2 - z)^{-1}$ in $L^2(\mathbb{R})$, i.e.

$$\left(r_0(z)g\right)(y) = \frac{i}{2\sqrt{z}}\int_{-\infty}^{\infty} e^{i\sqrt{z}|y-y'|}g(y')dy',$$

where $\operatorname{Im}\sqrt{z} \geq 0$. Then as can be checked easily for any $s > 0$ and $\delta > 0$, there exists a constant $C_{s,\delta} > 0$ such that

$$\|(1+|y|)^s r_0(-a)(1+|y|)^{-s}\|_{\mathbf{B}(L^2(\mathbb{R});L^2(\mathbb{R}))} \leq C_{s,\delta}, \quad \forall a > \delta.$$

Therefore by (3.31), one can show that

$$E_j((\lambda,\infty))v_j \in L^{2,(1+\epsilon_0)/2}(\Omega_j), \tag{3.32}$$

where $E_j(\cdot)$ is the spectral projection associated with $-\Delta_{h_j}$.

For $\lambda_n < \lambda$, we study $v_{j,n}$ separately. By (3.28),

$$\lim_{R\to\infty}\frac{1}{R}\int_0^R |v_{j,n}(y)|^2 dy = 0. \tag{3.33}$$

In view of (3.33), we see that $v_{j,n}$ satisfies both of the outgoing and incoming radiation conditions. Adopting the outgoing radiation condition, we see that $v_{j,n}$ is written as $v_{j,n} = r_0(\mu_n + i0)g_{j,n}$, i.e.

$$v_{j,n}(y) = \frac{i}{2\sqrt{\mu_n}}\Big(\int_0^y e^{i\sqrt{\mu_n}(y-y')}g_{j,n}(y')dy' + \int_y^\infty e^{i\sqrt{\mu_n}(y'-y)}g_{j,n}(y')dy'\Big).$$

Note that $g_{j,n} \in L^1((0,\infty))$, since $g_{j,n} \in L^{2,(1+\epsilon_0)/2}((0,\infty))$. Therefore

$$\lim_{y\to\infty} v_{j,n}(y) = \frac{i}{2\sqrt{\mu_n}}\int_0^\infty e^{i\sqrt{\mu_n}(y-y')}g_{j,n}(y')dy'.$$

The condition (3.33) implies that this limit is equal to 0. which implies

$$v_{j,n}(y) = \frac{i}{2\sqrt{\mu_n}}\Big(-\int_y^\infty e^{i\sqrt{\mu_n}(y-y')}g_{j,n}(y')dy' + \int_y^\infty e^{i\sqrt{\mu_n}(y'-y)}g_{j,n}(y')dy'\Big).$$

Using the following Lemma 3.7 (Hardy's inequality), we have $(1+y)^{(\epsilon_0-1)/2}v_{j,n} \in L^2((0,\infty))$. Using (3.32), we then have $v_j \in L^{2,(-1+\epsilon_0)/2}((0,\infty))$. By Lemma 3.2 and the formula $\chi_j u = v_j + \mathcal{E}_j^{(0)} B_j^{(0)} u$, we have $u \in L^{2,(-1+\epsilon_0)/2}(\Omega_j)$. Thus we have seen that u gains the decay of order ϵ_0 in Ω_j. Then in (3.31), $g_{j,n} \in$

$L^{2,(1+2\epsilon_0)/2}((0,\infty)$. Hence $v_j \in L^{2,(-1+3\epsilon/2)}(\Omega_j)$. Repeating this procedure, we obtain $\chi_j u \in L^{2,m\epsilon_0}(\Omega_j)$, $\forall m > 0$. The estimate (3.29) can be proven by re-examining the above arguments. $\square$

Lemma 3.7. *Let $f(y) \in L^1((0,\infty))$ and put*

$$u(y) = \int_y^\infty f(t)dt.$$

The for $s > 1/2$

$$\int_0^\infty y^{2(s-1)}|u(y)|^2 dy \leq \frac{4}{(2s-1)^2} \int_0^\infty y^{2s}|f(y)|^2 dy.$$

For the proof, see [41] Chap. 3, Lemma 3.3. $\square$

Lemma 3.8. *Let $\sigma_{rad}(H)$ be the set of $\lambda \notin \mathcal{T}(H)$ for which there exists a non-trivial solution $u \in \mathcal{B}^*$ of the equation $(H-\lambda)u = 0$ satisfying the radiation condition. Then $\sigma_{rad}(H) = \sigma_p(H) \setminus \mathcal{T}(H)$. Moreover it is a discrete subset of $\mathbb{R} \setminus \mathcal{T}(H)$ with possible accumulation points in $\mathcal{T}(H)$ and ∞.*

Proof. The first part of the lemma is proved by Lemmas 3.5 and 3.6. Let I be a compact interval in $\mathbb{R} \setminus \mathcal{T}(H)$, and suppose there exists an infinite number of eigenvalues (counting multiplicities) in I. Let $u_n, n = 1, 2, \cdots$ be the associated orthonormal eigenvectors.

Take any $R > 0$ and let $\chi_0 = \chi_0^R$ be the function introduced in the beginning of this section. We decompose

$$u_n = \chi_0^R u_n + \sum_{j=0}^N (1-\chi_0^R)\chi_j u_n.$$

Then by (3.29), for any $\epsilon > 0$ there exists $R > 0$ such that $\|(1-\chi_0^R)u_n\|_{L^2(\Omega)} < \epsilon$ uniformly in n. By the compactness of the imbedding of $H^1_{loc}(\Omega)$ to $L^2_{loc}(\Omega)$, $\{\chi_R u_n\}_n$ is compact in $L^2(\Omega)$. Therefore $\{u_n\}_n$ contains a convergent subsequence, which is a contradiction. $\square$

It is known that the eigenvalues embedded in $\sigma_{ess}(H)$ can accumulate at $\tau(H)$ only from below, see [45].

The set of exceptional points $\mathcal{E}(H)$ is now defined by

$$\mathcal{E}(H) = \mathcal{T}(H) \bigcup \sigma_p(H). \tag{3.34}$$

Weyl's formula for the asymptotic distribution of eigenvalues on compact manifolds and Lemma 3.8 imply that $\mathcal{T}(H)$ is discrete and $\mathcal{E}(H)$ has only finite number of accumulation points on any compact interval in $\mathbb{R}$.

3.4. **Limiting absorption principle.** For a self-adjoint operator H defined in a Hilbert space $\mathcal{H}$, the limit

$$\lim_{\epsilon\to 0}(H-\lambda\mp i\epsilon)^{-1} =: (H-\lambda\mp i0)^{-1}, \quad \lambda\in\sigma(H),$$

does not exist as a bounded operator on $\mathcal{H}$. However if λ is in the continuous spectrum of H, it is sometimes possible to guarantee the existence of the above limit in $\mathbf{B}(\mathcal{X};\mathcal{X}^*)$, where $\mathcal{X},\mathcal{X}^*$ are Banach spaces such that $\mathcal{X}\subset\mathcal{H}=H^*\subset\mathcal{X}^*$, and $\mathcal{H}$ is identified with its dual space via Riesz' theorem. This fact, called the limiting absorption principle, is central in studying the absolutely continuous spectrum, and many works are devoted to it. We employ in this paper the classical method of integration by parts pioneered by Eidus [23]. The crucial step is to establish a-priori estimates as in §2 of this paper, and to show the uniqueness of solutions to the reduced wave equation satisfying the radiation condition. After this hard analysis part, the remaining arguments are almost routine.

We take any compact interval $I\subset(0,\infty)\setminus\mathcal{E}(H)$ and let

$$J=\{z\in\mathbb{C}\,;\,\mathrm{Re}\,z\in I,\ \mathrm{Im}\,z\neq 0\}.$$

We first note that Lemma 2.3 also holds for the solution to the equation

$$\begin{cases}(H-z)u=f \quad \text{in} \quad \Omega,\\ \partial_\nu u=0 \quad \text{on} \quad \partial\Omega,\end{cases}$$

by the standard elliptic regularity estimates. We put $u=R(z)f$ and $v_j=\chi_j u-\mathcal{E}_j^{(0)}B_j^{(0)}u$ as in the proof of Lemma 3.6, Then v_j satisfies (3.30) with λ replaced by z. We can then apply Theorem 2.7 to see that

$$\|\chi_j u\|_{\mathcal{B}^*}\le C_s\Big(\|f\|_{\mathcal{B}}+\|u\|_{-s}\Big), \tag{3.35}$$

for any $s>1/2$, where C is independent of $z\in J$. Once (3.35) is proved, we can repeat the arguments in Chap 2, §2 of [41] or those of Ikebe-Saito [37] without any essential change. Note that here and in the sequel, we use (,) to denote the inner product

$$(u,v)=\int_\Omega u\overline{v}dV$$

of $L^2(\Omega)$ as well as the coupling between $\mathcal{B}$ and $\mathcal{B}^*$, or $L^{2,s}$ and $L^{2,-s}$.

Lemma 3.9. *Take $s>1/2$ sufficiently close to $1/2$.*
(1) There exists a constant $C>0$ such that

$$\sup_{z\in J}\|R(z)f\|_{-s}\le C\|f\|_{\mathcal{B}}.$$

(2) For any $\lambda\in I$ and $f\in\mathcal{B}$, the strong limit $\lim_{\epsilon\to 0}R(\lambda\pm i\epsilon)f$ exists in $L^{2,-s}$.
(3) $I\ni\lambda\to R(\lambda\pm i0)f\in L^{2,-s}$ is continuous.

Sketch of the proof. Suppose the uniform bound (1) is not true. Then there exist a sequence $z_n\in J$ and $f_n\in\mathcal{B}$ such that $u_n=R(z_n)f_n$ satisfies $\|u_n\|_{-s}=1$ and $\|f_n\|_{\mathcal{B}}\to 0$. Without loss of generality, we can assume that $z_n\to\lambda\in I$. Using

(3.35) with $0 < s' < s$ and the compactness of the embedding of H^2_{loc} into L^2_{loc}, one can assume that u_n converges to some $u \in \mathcal{B}^*$, and u satisfies the equation $(H-\lambda)u = 0$ and the radiation condition (see Corollary 2.6). Therefore $u = 0$ by Lemma 3.8. However this contradicts $\|u_n\|_{-s} = 1$. The assertions (2) and (3) are proved in a similar manner. □

Using this lemma one can prove the following theorem.

Theorem 3.10. *(1) For any $\lambda \in I$, $\lim_{\epsilon\to 0} R(\lambda \pm i\epsilon)f$ exists in the weak-$*$ sense:*

$$\exists \lim_{\epsilon\to 0}(R(\lambda \pm i\epsilon)f, g) =: (R(\lambda \pm i0)f, g), \quad \forall f, g \in \mathcal{B}.$$

(2) There exists a constant $C > 0$ such that

$$\|R(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad \lambda \in I.$$

Moreover $R(\lambda \pm i0)f$ satisfies the outgoing radiation condition for $\lambda + i0$ and incoming radiation condition for $\lambda - i0$.
(3) For any $f, g \in \mathcal{B}$, $I \ni \lambda \to (R(\lambda \pm i0)f, g)$ is continuous.
(4) Let $E(\cdot)$ be the spectral decomposition of H. Then $E([0,\infty) \setminus \mathcal{E}(H))L^2(\Omega) = \mathcal{H}_{ac}(H)$, and we have the following orthogonal decomposition

$$L^2(\Omega) = \mathcal{H}_{ac}(H) \oplus \mathcal{H}_p(H).$$

Sketch of the proof. Since $L^{2,-s}$ $(s > 1/2)$ is dense in $\mathcal{B}^*$, (1) follow from Lemma 3.9 (2) and (3.35). The assertion (2) follows from Lemma 3.9 (1) and (3.35). The remaining assertions are proved in the same way as in Chap 2, §2 of [41] or Ikebe-Saito [37]. □

Let us recall that for a self-adjoint operator $H = \int_{-\infty}^{\infty} \lambda dE(\lambda)$, the absolutely continuous subspace for H, $\mathcal{H}_{ac}(H)$, is the set of u such that $(E(\lambda)u, u)$ is absolutely continuous with respect to $d\lambda$, and the point spectral subspace, $\mathcal{H}_p(H)$, is the closure of the linear hull of eigenvectors of H.

4. Forward problem

4.1. **Unperturbed spectral representations.** Let $\{\chi_j\}_{j=0}^N$ be the partition of unity defined in §3. Recall the spaces $\mathcal{B}$ and $\mathcal{B}^*$ introduced in §2. For two functions f, g on Ω, $f \simeq g$ means that

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \|\chi_j(y)\left(f(\cdot, y) - g(\cdot, y)\right)\|^2_{L^2(M_j)} dy = 0, \quad 1 \leq \forall j \leq N.$$

We also use the same notation $f \simeq g$ for f, g defined on Ω_j.

Green's function of $-d^2/dy^2 - \zeta$ on $(0,\infty)$ with Neumann boundary condition at $y = 0$ is

$$G(y, y'; \zeta) = \frac{i}{\sqrt{\zeta}}\begin{cases} \cos(\sqrt{\zeta}y)e^{i\sqrt{\zeta}y'}, & 0 < y < y', \\ e^{i\sqrt{\zeta}y}\cos(\sqrt{\zeta}y'), & 0 < y' < y. \end{cases}$$

Let $\lambda_{j,1} < \lambda_{j,2} \leq \cdots$ be the eigenvalues of $-\Delta_{h_j}$ with normalized eigenvectors $\varphi_{j,n}(x)$, $n = 1, 2, \cdots$. Without loss of generality, we assume that $\varphi_{j,n}(x)$'s are real-valued. Let $H_j^{(0)} = -\partial_y^2 - \Delta_{h_j}$ with Neumann boundary condition. Then $R_j^{(0)}(z) = (H_j^{(0)} - z)^{-1}$ is written as

$$\left(R_j^{(0)}(z)f\right)(x,y) = \sum_{n=1}^{\infty}\int_0^{\infty} G(y,y';z-\lambda_{j,n})\left(P_{j,n}f\right)(x,y')dy', \tag{4.1}$$

$$\left(P_{j,n}f\right)(x,y) = \langle f(\cdot,y),\varphi_{j,n}\rangle\varphi_{j,n}(x),$$

$\langle\ ,\ \rangle$ being the inner product of $L^2(M_j\,;\sqrt{\det(h_j)}\,dx)$. Note that $\det(h_{ij}) = \det G_j^{(0)}$. For $f(x,y) \in L^2\big(M_j \times (0,\infty); \big(\det G_j^{(0)}\big)^{1/2}dxdy\big)$, we define its cosine transform by

$$\mathcal{F}_{cos}(\lambda)f(x) = \pi^{-1/2}\lambda^{-1/4}\int_0^{\infty}\cos\left(y\sqrt{\lambda}\right)f(x,y)dy.$$

Lemma 4.1. *For $f \in \mathcal{B}$, and $\lambda \in (0,\infty)\setminus\sigma_p(-\Delta_{h_j})$, we have*

$$R_j^{(0)}(\lambda \pm i0)f \simeq \pm i\sqrt{\pi}\sum_{\lambda_{j,n}<\lambda}(\lambda-\lambda_{j,n})^{-1/4}e^{\pm iy\sqrt{\lambda-\lambda_{j,n}}}\mathcal{F}_{cos}(\lambda-\lambda_{j,n})P_{j,n}f(x).$$

Proof. We first show that the right-hand side of (4.1) is a bounded operator from $\mathcal{B}$ to $\mathcal{B}^*$. The sum over the terms in which $\lambda_{j,n} > \lambda$ is rewritten as

$$\begin{aligned} &A_j(\lambda)f \\ &:= \sum_{\lambda_{j,n}>\lambda}\frac{1}{2k_n}\int_0^{\infty}\left(e^{-k_n\,|y-y'|} + e^{-k_n\,(y+y')}\right)f_{j,n}(x,y')dy' \end{aligned}$$

where $f_{j,n} = P_{j,n}f$ and $k_n = \sqrt{\lambda_{j,n}-\lambda}$. Then we have

$$\begin{aligned} &\|A_j(\lambda)f(\cdot,y)\|^2_{L^2(M)} \\ &= \sum_{\lambda_{j,n}>\lambda}\frac{1}{4k_n^2}\left|\int_0^{\infty}\left(e^{-k_n\,|y-y'|} + e^{-k_n\,(y+y')}\right)\langle f(\cdot,y'),\varphi_{j,n}\rangle_{L^2(M_j)}dy'\right|^2 \\ &\le C_\lambda\sum_{\lambda_{j,n}>\lambda}\int_0^{\infty}\left|\langle f(\cdot,y),\varphi_{j,n}\rangle\right|^2 dy \\ &\le C_\lambda\|f\|^2_{L^2(M_j\times(0,\infty))}. \end{aligned}$$

Hence $A_j(\lambda) \in \mathbf{B}(L^2; L^\infty(\mathbb{R}_+; L^2(M_j)) \subset \mathbf{B}(\mathcal{B};\mathcal{B}^*)$. To estimate the term in which $\lambda_{j,n} < \lambda$, we put

$$u_{j,n}(x) = \int_0^{\infty} G(y,y';\lambda \pm i0 - \lambda_{j,n})f_{j,n}(x,y')dy'.$$

Then we have

$$|u_{j,n}(x)| \le C_\lambda\int_0^{\infty}|f_{j,n}(x,y)|dy.$$

Since

$$\|u_{j,n}\|_{\mathcal{B}^*} \le C\|u_{j,n}\|_{L^\infty}, \quad \|f_{j,n}\|_{L^1} \le C\|f_{j,n}\|_{\mathcal{B}},$$

We have proven that $R_j^{(0)}(\lambda \pm i0) \in \mathbf{B}(\mathcal{B};\mathcal{B}^*)$.

Now the assertion of the lemma is easy to prove if there exists $n_0 > 0$ such that $f_{j,n} = 0$ for $n \ge n_0$, and $f_{j,n}$ is compactly supported for $n < n_0$. Since such an f is dense in $\mathcal{B}$, we have proven the lemma. $\square$

The generalized eigenfunction of $H_j^{(0)}$ is defined for $\lambda > \lambda_{j,n}$

$$\Psi_{j,n}^{(0)}(x,y;\lambda) = \pi^{-1/2}(\lambda-\lambda_{j,n})^{-1/4}\cos\left(y\sqrt{\lambda-\lambda_{j,n}}\right)\varphi_{j,n}(x). \tag{4.2}$$

This $\Psi_{j,n}^{(0)}(x,y;\lambda)$ is often denoted by $\Psi_{j,n}^{(0)}(\lambda)$ in the sequel. It satisfies

$$\begin{cases} (-\Delta_{G_j^{(0)}}-\lambda)\Psi_{j,n}^{(0)}(\lambda) = 0 \quad \text{in} \quad \Omega_j, \\ \partial_{\nu_j^{(0)}}\Psi_{j,n}^{(0)}(\lambda) = 0 \quad \text{on} \quad \partial\Omega_j. \end{cases} \tag{4.3}$$

The Fourier transformation associated with $H_j^{(0)}$ is defined by

$$\mathcal{F}_j^{(0)}(\lambda)f = \sum_{n=1}^{\infty}\chi_{\lambda_{j,n}}(\lambda)\mathcal{F}_{j,n}^{(0)}(\lambda)f, \tag{4.4}$$

where $\chi_{\lambda_{j,n}}$ is the characteristic function of the interval $(\lambda_{j,n},\infty)$, and

$$\begin{aligned} \left(\mathcal{F}_{j,n}^{(0)}(\lambda)f\right)(x) &= \left(\int_{\Omega_j}\overline{\Psi_{j,n}^{(0)}(\lambda)}f dV_j^{(0)}\right)\varphi_{j,n}(x) \\ &= \mathcal{F}_{cos}(\lambda-\lambda_{j,n})P_{j,n}f(x), \end{aligned} \tag{4.5}$$

where $dV_j^{(0)} = \left(\det G_j^{(0)}\right)^{1/2}dxdy$. Define a subspace of $L^2((0,\infty);L^2(M_j);d\lambda)$ by

$$\begin{aligned} \widehat{\mathcal{H}}_j &= \sum_{n=1}^{\infty}L^2\big((\lambda_{j,n},\infty);d\lambda\big)\otimes\varphi_{j,n}(x) \\ &= \left\{\sum_{n=1}^{\infty}f_n(\lambda)\varphi_{j,n}(x)\,;\,f_n\in L^2\big((\lambda_{j,n},\infty)\,;\,d\lambda\big)\right\}. \end{aligned} \tag{4.6}$$

Then $\mathcal{F}_j^{(0)}$ defined by $\big(\mathcal{F}_j^{(0)}f\big)(\lambda) = \mathcal{F}_j^{(0)}(\lambda)f$ for $f\in C_0^\infty(\Omega_j)$ is uniquely extended to a unitary operator

$$\mathcal{F}_j^{(0)} : L^2(\Omega_j)\to\widehat{\mathcal{H}}_j.$$

We put

$$\mathbf{h} = \bigoplus_{j=1}^{N}L^2(M_j), \tag{4.7}$$

where $L^2(M_j) = L^2(M_j;\sqrt{\det(h_j)}\,dx)$, and and also

$$\mathcal{F}^{(0)} = (\mathcal{F}_1^{(0)},\cdots,\mathcal{F}_N^{(0)}). \tag{4.8}$$

By the computation similar to the one to be given in the proof of Lemma 4.3 below, one can show that

$$\frac{1}{2\pi i}\left(\Big[R_j^{(0)}(\lambda+i0)-R_j^{(0)}(\lambda-i0)\Big]f,f\right) = \|\mathcal{F}_j^{(0)}(\lambda)f\|_{L^2(M_j)}^2.$$

Therefore, $\mathcal{F}_j^{(0)}(\lambda)\in\mathbf{B}(\mathcal{B}\,;L^2(M_j))$, and $\mathcal{F}_j^{(0)}(\lambda)^*\in\mathbf{B}(L^2(M_j);\mathcal{B}^*)$.

Here we must pay attention to the following remarks. The first one is that in (4.4), $\mathcal{F}_j^{(0)}(\lambda)$ is a finite sum:

$$\mathcal{F}_j^{(0)}(\lambda) = \sum_{\lambda_{j,n}<\lambda}\mathcal{F}_{j,n}^{(0)}(\lambda). \tag{4.9}$$

The second remark is that the adjoint $*$ is taken in the following sense:

$$(\mathcal{F}_j^{(0)}(\lambda)f, h)_{L^2(M_j)} = (f, \mathcal{F}_j^{(0)}(\lambda)^* h)_{L^2(\Omega_j)} = \int_{\Omega_j} f\, \overline{\mathcal{F}_j^{(0)}(\lambda)^* h}\, dV_j^{(0)}, \tag{4.10}$$

$(h \in L^2(M_j))$. Therefore

$$\mathcal{F}_j^{(0)}(\lambda)^* = \sum_{\lambda_{j,n}<\lambda} \mathcal{F}_{j,n}^{(0)}(\lambda)^*, \tag{4.11}$$

and for $h \in L^2(M_j)$

$$\left(\mathcal{F}_{j,n}^{(0)}(\lambda)^* h\right)(x,y) = \Psi_{j,n}^{(0)}(x,y;\lambda)(h, \varphi_{j,n})_{L^2(M_j)}. \tag{4.12}$$

Since $\mathcal{F}_j^{(0)}(\lambda)^*$ satisfies $(H_j^{0)} - \lambda)\mathcal{F}_j^{(0)}(\lambda)^* = 0$, we have

$$\mathcal{F}_j^{(0)}(\lambda)^* \in \mathbf{B}(L^2(M_j); H^{2,-s}), \quad s > 1/2,$$

hence

$$\mathcal{F}_j^{(0)}(\lambda) \in \mathbf{B}(H^{-2,s}; L^2(M_j)), \quad s > 1/2.$$

4.2. **Perturbed spectral representations.** Using $\mathcal{E}_j$, B_j and $\mathcal{V}_j(z)$ in Subsection 3.1, for $\lambda > \lambda_{j,n}$ we define the generalized eigenfunction for H by

$$\Psi_{j,n,\pm}(\lambda) = (\chi_j - \mathcal{E}_j B_j)\, \Psi_{j,n}^{(0)}(\lambda) - R(\lambda \mp i0)\mathcal{V}_j(\lambda)\Psi_{j,n}^{(0)}(\lambda). \tag{4.13}$$

Here putting $s = (1+\epsilon_0)/2$, we regard $\mathcal{E}_j B_j$ and $\mathcal{V}_j(\lambda)$ in $\mathbf{B}(H^{2,-s}; L^{2,s})$. Note that $\Psi_{j,n,\pm}(\lambda) \in \mathcal{B}^*$. This definition easily implies

$$\begin{cases} (-\Delta_G - \lambda)\Psi_{j,n,\pm}(\lambda) = 0 \quad \text{in} \quad \Omega, \\ \partial_\nu \Psi_{j,n,\pm}(\lambda) = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.14}$$

The generalized Fourier transformation for H is defined by perturbing $\mathcal{F}_j^{(0)}$. We put for $\lambda > \lambda_{j,n}$

$$\mathcal{F}_{j,n,\pm}(\lambda) = \mathcal{F}_{j,n}^{(0)}(\lambda) J_j \big(\chi_j - (\mathcal{E}_j B_j)^* - \mathcal{V}_j(\lambda)^*\big) R(\lambda \pm i0), \tag{4.15}$$

where $J_j = \big(\det G / \det G_j^{(0)}\big)^{1/2}$. Note that $(\mathcal{E}_j B_j)^*, \mathcal{V}_j(\lambda)^* \in \mathbf{B}(L^{2,-s}; H^{-2,s})$, and $R(\lambda \pm i0) \in \mathbf{B}(L^{2,s}; H^{2,-s}) \cap \mathbf{B}(H^{-2,s}; L^{2,-s})$, hence (4.15) is well-defined.

Lemma 4.2. *For $f \in C_0^\infty(\Omega)$, we have*

$$\big(\mathcal{F}_{j,n,\pm}(\lambda) f\big)(x) = \left(\int_\Omega \overline{\Psi_{j,n,\pm}(\lambda)} f\, dV\right) \varphi_{j,n}(x), \tag{4.16}$$

where $dV = (\det(G))^{1/2}\, dxdy$.

Proof. We put $u = \big(\chi_j - (\mathcal{E}_j B_j)^* - \mathcal{V}_j(\lambda)^*\big) R(\lambda \pm i0) f$. Then by using (4.10)

$$\begin{aligned}(\mathcal{F}_{j,n,\pm}(\lambda) f, h)_{L^2(M_j)} &= (\mathcal{F}_{j,n}^{(0)}(\lambda) J_j u, h)_{L^2(M_j)} \\ &= \int_{\Omega_j} u\, \overline{\mathcal{F}_{j,n}^{(0)}(\lambda)^* h}\, J_j dV_j^{(0)}.\end{aligned}$$

We then use (4.12) to see that the right-hand side is equal to

$$\begin{aligned}
&\int_{\Omega_j} u\,\overline{\Psi^{(0)}_{j,n}(\lambda)}\,dV\,\overline{(h,\varphi_{j,n})}_{L^2(M_j)} \\
&= \Big(\big(\chi_j-(\mathcal{E}_jB_j)^*-\mathcal{V}_j(\lambda)^*\big)R(\lambda\pm i0)f,\Psi^{(0)}_{j,n}(\lambda)\Big)\,\overline{(h,\varphi_{j,n})}_{L^2(M_j)} \\
&= (f,\Psi_{j,n,\pm}(\lambda))\,(\varphi_{j,n},h)_{L^2(M_j)},
\end{aligned}$$

which proves the lemma. □

The adjoint operator $\mathcal{F}_{j,n,\pm}(\lambda)^*$ is defined by the following formula:

$$(\mathcal{F}_{j,n,\pm}(\lambda)f,h)_{L^2(M_j)} = (f,\mathcal{F}_{j,n,\pm}(\lambda)h^*)_{L^2(\Omega)},\quad h\in L^2(M_j). \tag{4.17}$$

Lemma 4.3. *The adjoint operator $\mathcal{F}_{j,n,\pm}(\lambda)^*$ has the following expression:*

$$\mathcal{F}_{j,n,\pm}(\lambda)^* = (\chi_j-\mathcal{E}_jB_j-R(\lambda\mp i0)\mathcal{V}_j(\lambda))\,\mathcal{F}^{(0)}_{j,n}(\lambda)^*, \tag{4.18}$$

where the adjoint $\mathcal{F}^{(0)}_{j,n}(\lambda)^*$ is taken in the sense of (4.10).

Proof. Let $u=\big(\chi_j-(\mathcal{E}_jB_j)^*-\mathcal{V}_j(\lambda)^*\big)R(\lambda\pm i0)f$. Then as is shown in the proof of Lemma 4.2,

$$\begin{aligned}
(\mathcal{F}_{j,n,\pm}(\lambda)f,h)_{L^2(M_j)} &= \int_{\Omega_j} u\,\overline{\mathcal{F}^{(0)}_{j,n}(\lambda)^*h}\,J_j dV^{(0)}_j \\
&= (u,\mathcal{F}^{(0)}_{j,n}(\lambda)^*h)_{L^2(\Omega)}.
\end{aligned}$$

Plugging the form of u, we see that the right-hand side is equal to

$$(f,(\chi_j-\mathcal{E}_jB_j-R(\lambda\mp i0)\mathcal{V}_j(\lambda))\mathcal{F}^{(0)}_{j,n}(\lambda)^*h)_{L^2(\Omega)},$$

which proves the lemma. □

We define

$$\mathcal{F}_{j,\pm}(\lambda) = \sum_{n=1}^{\infty}\chi_{\lambda_j,n}(\lambda)\mathcal{F}_{j,n,\pm}(\lambda) = \sum_{\lambda_{j,n}<\lambda}\mathcal{F}_{j,n,\pm}(\lambda), \tag{4.19}$$

$$\mathcal{F}_{\pm}(\lambda) = (\mathcal{F}_{1,\pm}(\lambda),\cdots,\mathcal{F}_{N,\pm}(\lambda)). \tag{4.20}$$

Lemma 4.4. *For any $\lambda\in(0,\infty)\setminus\mathcal{E}(H)$ and $f\in\mathcal{B}$, we have on Ω_j*

$$R(\lambda\pm i0)f \simeq \pm i\sqrt{\pi}\sum_{\lambda_{j,n}<\lambda}(\lambda-\lambda_{j,n})^{-1/4}e^{\pm iy\sqrt{\lambda-\lambda_{j,n}}}\mathcal{F}_{j,n,\pm}(\lambda)f. \tag{4.21}$$

Proof. This follows from (3.14), Lemma 4.1 and the definition (4.15). □

Lemma 4.5. *For any $\lambda\in(0,\infty)\setminus\mathcal{E}(H)$ and $f\in\mathcal{B}$, we have*

$$\frac{1}{2\pi i}\,((R(\lambda+i0)-R(\lambda-i0))f,f) = \|\mathcal{F}_{\pm}(\lambda)f\|^2_{\mathbf{h}}.$$

Proof. We prove the case for $\mathcal{F}_+(\lambda)$. We have only to prove the lemma when $f\in C_0^\infty(\Omega)$. We compute in a way similar to that in Lemma 3.5. Take $\rho(t)\in$

$C_0^\infty((0,\infty))$ such that $\int_0^\infty \rho(t)dt = 1$, and put $\chi(t) = \int_t^\infty \rho(s)ds$. Let $u = R(\lambda+i0)f$ and

$$\psi_R = \chi_0 + \sum_{j=1}^N \chi\big(\frac{y}{R}\big)\chi_j(y),$$

where $\{\chi_j\}_{j=0}^N$ is the partition of unity on Ω, and y in $\chi_j(y)$ is the local coordinate on Ω_j. We then have

$$([H-\lambda, \psi_R]u, u) = (\psi_R u, f) - (f, \psi_R u).$$

As $u \in \mathcal{B}^*$, by computing the commutator $[H, \psi_R]$, we then have

$$\lim_{R\to\infty} \sum_{j=1}^N \frac{2}{R}\Big(\rho\big(\frac{y}{R}\big)\chi_j(y)\partial_y u, u\Big) = (u,f) - (f,u).$$

Since $u = R(\lambda + i0)f$ satisfies the radiation condition (see Theorem 3.10 (2)), $(\partial_y - i\mathcal{P}_j(\lambda))\chi_j u \simeq 0$. Therefore

$$\lim_{R\to\infty} \sum_{j=1}^N \frac{2i}{R}\Big(\rho\big(\frac{y}{R}\big)\chi_j(y)\mathcal{P}_j(\lambda)u, u\Big) = (u,f) - (f,u).$$

Now we note that

$$\lim_{R\to\infty} \frac{1}{R}\Big(\rho\big(\frac{y}{R}\big)\chi_j(y)\mathcal{P}_j(\lambda)u, u\Big) = \lim_{R\to\infty}\frac{1}{R}\int_0^\infty \rho\big(\frac{y}{R}\big)\left(\mathcal{P}_j(\lambda)u, u\right)_{L^2(M_j)} dy.$$

Let $v_\pm$ be the term in the right-hand side of (4.21). Using Lemma 4.4, we first replace u of the right-hand side of $(\mathcal{P}_j(\lambda)u, u)_{L^2(M_j)}$ by $v_\pm$. We next move $\mathcal{P}_j(\lambda)$ to the right-hand side of the inner product, and replace u by $v_\pm$. Since $\mathcal{P}_{j,n}(\lambda)\varphi_{j,n}^{(0)} = \sqrt{\lambda - \lambda_{j,n}}\,\varphi_{j,n}^{(0)}$, we have $\mathcal{P}_j(\lambda)\mathcal{F}_{j,n,+}(\lambda) = \sqrt{\lambda-\lambda_{j,n}}\,\mathcal{F}_{j,n,+}(\lambda)$. The lemma then follows from a direct computation. $\square$

The formula in Lemma 4.5, when integrated with respect to λ over $(0,\infty)$, is a counterpart of the Parseval formula in the Fourier transformation, and a crucial step for the spectral representation. Using $\widehat{\mathcal{H}}_j$ in (4.6), we put

$$\widehat{\mathcal{H}} = \bigoplus_{j=1}^N \widehat{\mathcal{H}}_j. \tag{4.22}$$

The following theorem can be proved in the same way as in [38] or Chap. 3 of [41].

Theorem 4.6. *(1) For $\lambda \notin \mathcal{T}(H)$, $\mathcal{F}_\pm(\lambda) \in \mathbf{B}(\mathcal{B};\mathbf{h})$.*
(2) The operator $(\mathcal{F}_\pm f)(\lambda) = \mathcal{F}_\pm(\lambda)f$ defined for $f \in \mathcal{B}$ is uniquely extended to a partial isometry with initial set $\mathcal{H}_{ac}(H)$ and final set $\widehat{\mathcal{H}}$.
(3) $(\mathcal{F}_\pm Hf)(\lambda) = \lambda\,(\mathcal{F}_\pm f)(\lambda), \quad \forall\lambda \in (0,\infty)\setminus\mathcal{E}(H), \quad \forall f \in D(H)$.
(4) $\mathcal{F}_\pm(\lambda)^* \in \mathbf{B}(\mathbf{h};\mathcal{B}^*)$ is an eigenoperator of H with eigenvalue λ in the sense that

$$(H-\lambda)\mathcal{F}_\pm(\lambda)^*\psi = 0, \quad \forall\psi \in \mathbf{h}.$$

(5) For any compact interval $I \subset (0,\infty)\setminus \mathcal{T}(H)$ and $g \in \widehat{\mathcal{H}}$, we have

$$\int_I \mathcal{F}_\pm(\lambda)^* g(\lambda)\, d\lambda \in L^2(\Omega).$$

Let I_n be a finite union of compact intervals in $(0,\infty)\setminus\mathcal{E}(H)$ such that $I_n \subset I_{n+1}$, $\cup_{n=1}^\infty I_n = (0,\infty)\setminus\mathcal{E}(H)$. Then for any $f \in \mathcal{H}_{ac}(H)$, the inversion formula holds:

$$f = \mathrm{s}-\lim_{n\to\infty}\int_{I_n}\mathcal{F}_\pm(\lambda)^*(\mathcal{F}_\pm f)(\lambda)d\lambda.$$

4.3. **Time-dependent scattering theory.** Let $H_j^{(0)} = -\partial_y^2 - \Delta_{h_j}$ be the unperturbed Laplacian in the end Ω_j.

Theorem 4.7. *The wave operator $W_\pm : \bigoplus_{j=1}^N L^2(\Omega_j) \to L^2(\Omega)$ defined by*

$$W_\pm = \mathrm{s}-\lim_{t\to\pm\infty}\sum_{j=1}^N e^{it\sqrt{H}}\chi_j e^{-it\sqrt{H_j^{(0)}}}$$

exists and is complete, i.e. $\mathrm{Ran}\, W_\pm = \mathcal{H}_{ac}(H)$. *Moreover*

$$W_\pm = \big(\mathcal{F}_\pm\big)^*\mathcal{F}^{(0)}, \tag{4.23}$$

where $\mathcal{F}^{(0)}$ is the Fourier transformation defined by (4.8) for the system of Laplacians $(H_1^{(0)},\cdots,H_N^{(0)})$.

Sketch of the proof. We argue in the same way as in Chap. 2, Theorem 8.9 of [41]. Take $f \in \mathcal{H}_{ac}(H)$ such that $\big(\mathcal{F}_{j,n,+}f\big)(\lambda) \in C_0^\infty((\lambda_{j,n},\infty))$ and $\mathcal{F}_{j,n,+}f = 0$ except for a finite number of n. Then by Theorem 4.6 and Lemma 4.3

$$\begin{aligned} e^{-it\sqrt{H}}f &= \int_0^\infty e^{-it\sqrt{\lambda}}\mathcal{F}_+(\lambda)^*\big(\mathcal{F}_+f\big)(\lambda)d\lambda \\ &= \sum_{j,n}\int_0^\infty e^{-it\sqrt{\lambda}}\chi_j\left(\mathcal{F}_{j,n}^{(0)}(\lambda)\right)^*\big(\mathcal{F}_{j,n,+}f\big)(\lambda)d\lambda \\ &\quad -\sum_{j,m}\int_0^\infty e^{-it\sqrt{\lambda}}\mathcal{E}_jB_j\left(\mathcal{F}_{j,n}^{(0)}(\lambda)\right)^*\big(\mathcal{F}_{j,n,+}f\big)(\lambda)d\lambda \\ &\quad -\sum_{j,n}\int_0^\infty e^{-it\sqrt{\lambda}}R(\lambda-i0)\mathcal{V}_j(\lambda)\left(\mathcal{F}_{j,n}^{(0)}(\lambda)\right)^*\big(\mathcal{F}_{j,n,+}f\big)(\lambda)d\lambda. \end{aligned} \tag{4.24}$$

Because of the decay of $\mathcal{E}_j$, the 2nd term of the right-hand side tends to 0 in $L^2(\Omega)$. Letting $A = \sqrt{H}$, we have

$$(H - k^2 + i0)^{-1} = (A - k + i0)^{-1}(A+k)^{-1}.$$

We then put

$$g(k) = 2k(A+k)^{-1}\mathcal{V}_j(k^2)\left(\mathcal{F}_{j,n}^{(0)}(k^2)\right)^*\big(\mathcal{F}_{j,n,+}f\big)(k^2),$$

$$\widetilde{g}(t) = \int_0^\infty e^{-itk}g(k)dk.$$

We show that

$$\|\widetilde{g}(t)\| \le C(1+t)^{-1-\epsilon}, \quad t > 0. \tag{4.25}$$

In fact, take $h \in L^2(\Omega)$ and consider

$$
\begin{aligned}
(\widetilde{g}(t), h) &= \int_0^\infty 2ke^{-itk}\Big(\left(\mathcal{F}^{(0)}_{j,n}(k^2)\right)^* \left(\mathcal{F}_{j,n,+}f\right)(k^2), \mathcal{V}_j(k^2)(A+k)^{-1}h\Big)dk \\
&= \sum_n \int dV_j^{(0)} \int_0^\infty \left(e^{-i(tk+y\sqrt{k^2-\lambda_{j,n}})} + e^{-i(tk-y\sqrt{k^2-\lambda_{j,n}})}\right)\cdots dk.
\end{aligned}
$$

Here we have used the definition (4.5) of $\mathcal{F}^{(0)}_{j,n}$ and spitted $\cos(y\sqrt{k^2-\lambda_{j,n}})$ into $\frac{1}{2}\left(e^{-iy\sqrt{k^2-\lambda_{j,n}}} + e^{iy\sqrt{k^2-\lambda_{j,n}}}\right)$ to rewrite the inner product into the integral with respect to the measure $dV_j^{(0)} = \left(\det G_j^{(0)}\right)^{1/2}dxdy$. Since $\mathcal{V}_j(k^2)$ contains a factor $(1+y)^{-1-\epsilon}$, by the methods of stationary phase, one can prove

$$
|(\widetilde{g}(t), h)| \leq C(1+t)^{-1-\epsilon}\|h\|,
$$

which proves (4.25). We use the notation $f(t) \sim g(t)$ if $\|f(t) - g(t)\| \to 0$ as $t \to \infty$. In view of the following Lemma 4.8, we obtain as $t \to \infty$

$$
\begin{aligned}
e^{-it\sqrt{H}}f &\sim \sum_{j,n} \chi_j \int_0^\infty e^{-it\sqrt{\lambda}} \left(\mathcal{F}^{(0)}_{j,n}(\lambda)\right)^* \left(\mathcal{F}_{j,n,+}f\right)(\lambda)d\lambda \\
&= \sum_{j,n} \chi_j e^{-it\sqrt{H_j^{(0)}}} \left(\mathcal{F}^{(0)}_{j,n}\right)^* \mathcal{F}_{j,n,+}f,
\end{aligned}
$$

in $L^2(\Omega)$. This implies the existence of the limit

$$
\text{s} - \lim_{t\to\infty} \sum_{j=1}^N e^{it\sqrt{H_j^{(0)}}} \chi_j e^{-it\sqrt{H}} P_{ac}(H) = \left(\mathcal{F}^{(0)}\right)^* \mathcal{F}_+. \tag{4.26}
$$

Here, $P_{ac}(H)$ is the orthogonal projection onto $\mathcal{H}_{ac}(H)$. Since $\left(\mathcal{F}^{(0)}\right)^* \mathcal{F}_+$ is a partial isometry with initial set $\mathcal{H}_{ac}(H)$ and final set $L^2(\Omega)$, (4.26) also implies for $g = (g_1, \cdots, g_N) \in \bigoplus_{j=1}^N L^2(\Omega_j)$

$$
\|e^{it\sqrt{H}} \sum_{j=1}^N \chi_j e^{-it\sqrt{H_j^{(0)}}} g - \left(\mathcal{F}_+\right)^* \mathcal{F}^{(0)} g\| \to 0. \tag{4.27}
$$

Let us prove this fact. We put $U(t) = \sum_{j=1}^N e^{it\sqrt{H_j^{(0)}}} \chi_j e^{-it\sqrt{H}}$. Then (4.26) implies that $U(t) \to \left(\mathcal{F}^0\right)^* \mathcal{F}_+ =: U$ strongly, which implies

$$
U(t)^* \to U^* \quad \text{weakly.} \tag{4.28}
$$

We show that

$$
\|U(t)^* g\| \to \|g\| = \|U^* g\|, \quad g = (g_1, \cdots, g_N) \in \bigoplus_{j=1}^N L^2(\Omega_j). \tag{4.29}
$$

In fact, we have

$$
\|U(t)^* g\|^2 = \|\sum_{j=1}^N \chi_j e^{-it\sqrt{H_j^{(0)}}} g_j\|^2 = \sum_{j=1}^N \|\chi_j e^{-it\sqrt{H_j^{(0)}}} g_j\|^2.
$$

By the scattering property of $e^{-it\sqrt{H_j^{(0)}}}$, $\|(1-\chi_j)e^{-it\sqrt{H_j^{(0)}}}g_j\| \to 0$, which proves

$$\sum_{j=1}^{N} \|\chi_j e^{-it\sqrt{H_j^{(0)}}} g_j\|^2 \to \sum_{j=1}^{N} \|g_j\|^2 = \|g\|^2.$$

Now, (4.28) and (4.29) yield $\|U(t)^*g - U^*g\| \to 0$. This completes the proof of Theorem 4.7 for W_+. The assertion for W_- is proved similarly. □

Lemma 4.8. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. For $f(k) \in C_0((0,\infty);\mathcal{H})$, we put*

$$\widetilde{f}_\pm(t) = \int_0^\infty e^{\pm ikt} f(k)dk.$$

Then for any $\epsilon > 0$

$$\left\| \int_0^\infty (A - k \mp i\epsilon)^{-1} e^{\pm ikt} f(k)dk \right\| \leq \int_t^\infty \|\widetilde{f}_\pm(s)\| ds.$$

Proof. This is proved in [41], Chap. 2, Lemma 8.10. For the reader's convenience, we reproduce the proof. By virtue of the identity

$$(A - k \mp i\epsilon)^{-1} = \pm i \int_0^\infty e^{\mp is(A-k\mp i\epsilon)} ds,$$

we have

$$\int_0^\infty (A - k \mp i\epsilon)^{-1} e^{\pm ikt} f(k)dk = \pm i \int_0^\infty e^{\mp is(A\mp i\epsilon)} \widetilde{f}_\pm(s+t) ds,$$

which proves the lemma. □

4.4. **S-matrix.** The scattering operator is defined by $S = \big(W_+\big)^* W_-$. We consider its Fourier transform : $\widehat{S} = \mathcal{F}^{(0)} S \left(\mathcal{F}^{(0)}\right)^*$.

Lemma 4.9. *We have a direct integral representation:*

$$(\widehat{S}f)(\lambda) = \widehat{S}(\lambda)f(\lambda), \quad \forall f \in \widehat{\mathcal{H}}, \quad \forall \lambda > 0,$$

where $\widehat{S}(\lambda) = \big(\widehat{S}_{jk}(\lambda)\big)_{1\leq j,k\leq N}$ is a bounded operator on $\mathbf{h}$ called the S-matrix, and is written as follows

$$\widehat{S}_{jk}(\lambda) = \delta_{jk} - 2\pi i \mathcal{F}_{j,+}(\lambda)\mathcal{V}_k(\lambda) \left(\mathcal{F}_k^{(0)}(\lambda)\right)^*.$$

Proof. Lemma 4.5 implies

$$\frac{1}{2\pi i}\left(R(\lambda + i0) - R(\lambda - i0)\right) = \mathcal{F}_\pm(\lambda)^* \mathcal{F}_\pm(\lambda).$$

By Lemma 4.3, we then have

$$\mathcal{F}_{k,+}(\lambda)^* - \mathcal{F}_{k,-}(\lambda)^* = 2\pi i \mathcal{F}_+(\lambda)^* \mathcal{F}_+(\lambda)\mathcal{V}_k(\lambda)\mathcal{F}_k^{(0)}(\lambda)^*.$$

Then we have by Theorem 4.6 (2), for $f, g \in \widehat{\mathcal{H}}$

$$\big((\mathcal{F}_+ - \mathcal{F}_-)(\mathcal{F}_+)^* f, g\big) = -2\pi i \sum_{k=1}^{N} \int_0^\infty \Big(f(\lambda), \mathcal{F}_+(\lambda)\mathcal{V}_k(\lambda)\left(\mathcal{F}_k^{(0)}(\lambda)\right)^* g(\lambda)\Big)_{\mathbf{h}} d\lambda.$$

By (4.23), $\widehat{S} = \mathcal{F}_+\big(\mathcal{F}_-\big)^*$. Hence the lemma follows. □

Let $\mathbf{h}_j(\lambda)$ be the linear subspace of $L^2(M_j)$ spanned by $\varphi_{j,n}$ such that $\lambda_{j,n} < \lambda$ and put

$$\mathbf{h}(\lambda) = \bigoplus_{j=1}^{N} \mathbf{h}_j(\lambda).$$

Then $\widehat{S}(\lambda)$ is a partial isometry on $\mathbf{h}$ with initial and final set $\mathbf{h}(\lambda)$. The scattering amplitude is defined by

$$A_{jk}(\lambda) = \mathcal{F}_{j,+}(\lambda)\mathcal{V}_k(\lambda)\left(\mathcal{F}_k^{(0)}(\lambda)\right)^*.$$

Let $A_{jm,kn}(\lambda) : L^2(M_k) \to L^2(M_j)$ be given by

$$A_{jm,kn}(\lambda) = \mathcal{F}_{j,m,+}(\lambda)\mathcal{V}_k(\lambda)\left(\mathcal{F}_{k,n}^{(0)}(\lambda)\right)^*. \tag{4.30}$$

We then have

$$\widehat{S}_{jk}(\lambda) - \delta_{jk}I_j = -2\pi i \sum_{\lambda_{j,m}<\lambda,\, \lambda_{k,n}<\lambda} A_{jm,kn}(\lambda),$$

where I_j is the identity operator on $L^2(M_j)$. When j, k and the energy $\lambda > 0$ is fixed, $\big(A_{jm,kn}(\lambda)\big)$ is a finite matrix of size (d_j, d_k), where $d_j = \#\{m;\, \lambda_{j,m} < \lambda\}$. Let $\mathcal{A}_{jm,kn}(\lambda)$ be defined by

$$\mathcal{A}_{jm,kn}(\lambda) = \left(A_{jm,kn}(\lambda)\varphi_{k,n}, \varphi_{j,m}\right)_{L^2(M_j)}. \tag{4.31}$$

Then we have

$$A_{jm,kn}(\lambda)h = \mathcal{A}_{jm,kn}(\lambda)(h, \varphi_{k,n})_{L^2(M_k)}\varphi_{j,m}, \quad \forall h \in L^2(M_k). \tag{4.32}$$

The scattering amplitude is computed from the asymptotic expansion of the generalized eigenfunction in the following way.

Lemma 4.10.

$$P_{j,m}\left(\Psi_{k,n,-}(\lambda) - \chi_j\Psi_{k,n}^{(0)}(\lambda)\right) \simeq -\frac{i\sqrt{\pi}e^{iy\sqrt{\lambda-\lambda_{j,m}}}}{(\lambda-\lambda_{j,m})^{1/4}}\mathcal{A}_{jm,kn}(\lambda)\varphi_{j,m}.$$

Proof. This directly follows from (4.13) and Lemma 4.4. □

5. From scattering data to boundary data

5.1. **Non-physical scattering amplitude.** In this section, we observe waves coming in from and going out of the end Ω_1 assuming that

$$G_1 = (dy)^2 + h_1(x, dx) \quad \text{on} \quad \Omega_1. \tag{5.1}$$

This amounts to studying the scattering amplitude $A_{1m,1n}(\lambda)$ of (4.30), which is rewritten as

$$\begin{aligned} A_{1m,1n}(\lambda) = \mathcal{F}_{cos}(\lambda-\lambda_{1,m})P_{1,m}J_1\big(\chi_1 - \mathcal{V}_1(\lambda)^*R(\lambda+i0)\big) \\ \cdot\, \mathcal{V}_1(\lambda)\left(\mathcal{F}_{cos}(\lambda-\lambda_{1,n})\right)^* P_{1,n}. \end{aligned} \tag{5.2}$$

Note that $B_1 = 0$, because of the assumption (5.1). By the expression (3.12), $\mathcal{V}_1(\lambda)$ and $\mathcal{V}_1(\lambda)^*$ are independent of λ and compactly supported in the y-variable. Therefore, $A_{1m,1n}(\lambda)$ defined for $\lambda > \max\{\lambda_{1,m}, \lambda_{1,n}\}$ is analytically continued to the upper half plane $\mathbb{C}_+ = \{\mathrm{Im}\,\lambda > 0\}$. This analytic continuation can be extended to a continuous function on $\mathbb{C}_+ \cup (\mathbb{R} \setminus \mathcal{E}(H))$. We denote the obtained function for $\{\lambda < \max\{\lambda_{1,m}, \lambda_{1,n}\}\} \setminus \mathcal{E}(H)$ by $A^{(nph)}_{1m,1n}(\lambda)$ and call it the *non-physical* scattering amplitude. These functions can be represented by (5.2), where $\mathcal{F}_{cos}(\lambda - \lambda_{1,m})$ and $\mathcal{F}_{cos}(\lambda - \lambda_{1,n})$ are replaced by their analytic continuations. Let

$$\Phi^{(0)}_{1,n}(x, y; \lambda) = \pi^{-1/2} e^{-\pi i/4} (\lambda_{1,n} - \lambda)^{-1/4} \cosh\Big(y\sqrt{\lambda_{1,n} - \lambda}\Big)\varphi_{1,n}(x), \tag{5.3}$$

and put, similarly to (4.5)

$$\left(\mathcal{F}_{cosh}(\lambda_{1,n} - \lambda) P_{1,n} f\right)(x) = \left(\int_{\Omega_1} \overline{\Phi^{(0)}_{1,n}(\lambda)} f \, dV^{(0)}_1\right) \varphi_{1,n}(x).$$

In the following, we always assume that $\lambda \notin \mathcal{E}(H)$. The explicit form of $A^{(nph)}_{1m,1n}(\lambda)$ is given by the following lemma. Recall that the non-physical scattering amplitude $A^{(nph)}_{1m,1n}(\lambda)$ coincides with the physical scattering amplitude $A_{1m,1n}(\lambda)$ for $\lambda > \max\{\lambda_{1,m}, \lambda_{1,n}\}$.

Lemma 5.1. *(1) If* $\lambda_{1,m} < \lambda < \lambda_{1,n}$*,*

$$\begin{aligned} A^{(nph)}_{1m,1n}(\lambda) = {} & \mathcal{F}_{cos}(\lambda - \lambda_{1,m}) P_{1,n} J_1\big(\chi_1 - \mathcal{V}_1(\lambda)^* R(\lambda + i0)\big) \\ & \cdot \mathcal{V}_1(\lambda) \left(\mathcal{F}_{cosh}(\lambda_{1,n} - \lambda)\right)^* P_{1,n}. \end{aligned}$$

(2) If $\lambda_{1,n} < \lambda < \lambda_{1,m}$*,*

$$\begin{aligned} A^{(nph)}_{1m,1n}(\lambda) = {} & \mathcal{F}_{cosh}(\lambda_{1,m} - \lambda) P_{1,m} J_1\big(\chi_1 - \mathcal{V}_1(\lambda)^* R(\lambda + i0)\big) \\ & \cdot \mathcal{V}_1(\lambda) \left(\mathcal{F}_{cos}(\lambda - \lambda_{1,n})\right)^* P_{1,n}. \end{aligned}$$

(3) If $\lambda < \min\{\lambda_{1,m}, \lambda_{1,n}\}$*,*

$$\begin{aligned} A^{(nph)}_{1m,1n}(\lambda) = {} & \mathcal{F}_{cosh}(\lambda_{1,m} - \lambda) P_{1,m} J_1\big(\chi_1 - \mathcal{V}_1(\lambda)^* R(\lambda + i0)\big) \\ & \cdot \mathcal{V}_1(\lambda) \left(\mathcal{F}_{cosh}(\lambda_{1,n} - \lambda)\right)^* P_{1,n}. \end{aligned}$$

In accordance with (4.13), we define non-physical eigenfunction by

$$\Phi_{1,m,\pm}(\lambda) = \chi_1 \Phi^{(0)}_{1,m}(\lambda) - R(\lambda \mp i0)\mathcal{V}_1(\lambda)\Phi^{(0)}_{1,m}(\lambda). \tag{5.4}$$

Note that the *physical* eigenfunction $\Psi_{1,m,-}(\lambda)$ defined for $\lambda > \lambda_{1,m}$ is analytically continued through the upper half space $\mathbb{C}_+$ to the *nonphysical* eigenfunction $\Phi_{1,m,-}(\lambda)$ defined for $\lambda < \lambda_{1,m}$. The non-physical scattering amplitude is computed from the asymptotic behavior of non-physical eigenfunction in the following way.

We put

$$\mathcal{A}^{(nph)}_{1m,1n}(\lambda) = (A^{(nph)}_{1m,1n}(\lambda)\varphi_{1,n}, \varphi_{1,m})_{L^2(M_1)}.$$

Then we have for $h \in L^2(M_1)$

$$A^{(nph)}_{1m,1n}(\lambda) h = \mathcal{A}^{(nph)}_{1m,1n}(\lambda) \, (h, \varphi_{1,n})_{L^2(M_1)} \, \varphi_{1,m}.$$

Lemma 5.2. *(1) If* $\lambda_{1,m} < \lambda < \lambda_{1,n}$*, we have as* $y \to \infty$*,*

$$P_{1,m}\Big(\Phi_{1,n,-}(\lambda) - \Phi^{(0)}_{1,n}(\lambda)\Big) \simeq -\frac{i\sqrt{\pi}\, e^{iy\sqrt{\lambda - \lambda_{1,m}}}}{(\lambda - \lambda_{1,m})^{1/4}} \mathcal{A}^{(nph)}_{1m,1n}(\lambda)\varphi_{1,n}$$

(2) If $\lambda < \max\{\lambda_{1,m}, \lambda_{1,n}\}$, we have as $y \to \infty$,

$$P_{1,m}\Big(\Phi_{1,n,-}(\lambda) - \Phi^{(0)}_{1,n}(\lambda)\Big) \sim -\frac{e^{\pi i/4}\sqrt{\pi}e^{-y\sqrt{\lambda_{1,m}-\lambda}}}{(\lambda_{1,m}-\lambda)^{1/4}}\mathcal{A}^{(nph)}_{1m,1n}(\lambda)\varphi_{1,n},$$

with a super exponentially decreasing error, that is, with the error $r(y)$ satisfying $|r(y)| \leq C_N e^{-Ny}$ for any $N > 0$.

Proof. The assertion (1) is proved in the same way as in Lemma 4.8. By (4.1), letting $\zeta = \lambda - \lambda_{1,m}$, we have as $y \to \infty$

$$P_{1,m}R_1(\lambda + i0)f(x,y) \sim \frac{ie^{i\sqrt{\zeta}y}}{\sqrt{\zeta}}\int_0^\infty \cos\left(\sqrt{\zeta}y'\right) P_{1,m}f(x,y')dy'$$

with a super exponentially decaying error. This, together with (3.14) and Lemma 5.1, proves (2). □

5.2. **Splitting the manifold.** We take a compact hypersurface $\Gamma \subset \Omega_1$ having the following property.

(C-1) Γ splits Ω into a union: $\Omega = \Omega_{ext} \cup \Omega_{int}$ so that $\Omega_{ext} \cap \Omega_{int} = \Gamma$, Ω_{int} is a manifold with smooth boundary, and $\Omega_{ext} \subset \Omega_1$. (See figure 2.)

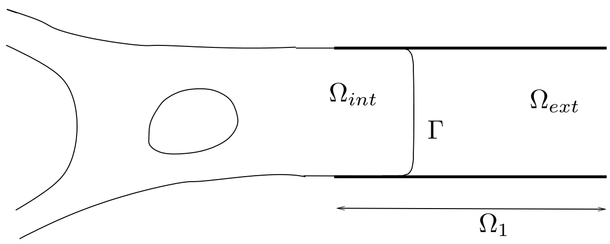

FIGURE 2. Surface Γ splits Ω to two parts, manifold Ω_{int} with a smooth boundary and its complement $\Omega_{ext} \subset \Omega_1$.

Let $\mathcal{O} \subset \Omega_{int}$ be an open, relatively compact set such that it has a smooth boundary not intersecting $\partial\Omega_{int}$ and that $\Omega_{int} \setminus \mathcal{O}$ is connected. Denote $\Omega_{\mathcal{O}} = \Omega_{int} \setminus \mathcal{O}$ and

$$\Gamma_{\mathcal{O}} = \begin{cases} \Gamma & \text{if} \quad \mathcal{O} = \emptyset, \\ \partial\mathcal{O} & \text{if} \quad \mathcal{O} \neq \emptyset. \end{cases}$$

We put for $f, g \in L^2(\Gamma_{\mathcal{O}})$

$$(f,g)_{\Gamma_{\mathcal{O}}} = \int_{\Gamma_{\mathcal{O}}} f(x)\overline{g(x)}dS_x,$$

dS_x being the measure induced from the metric G on $\Gamma_{\mathcal{O}}$. We put $H_{\mathcal{O}} = -\Delta_G$ in $\Omega_{\mathcal{O}}$ endowed with the Neumann boundary condition:

$$\partial_\nu v = 0 \quad \text{on} \quad \partial\Omega_{\mathcal{O}}, \tag{5.5}$$

ν being the unit normal to the boundary. If Ω has only one end, Ω_{int} is a bounded region. If Ω has more than one end, Ω_{int} is unbounded and the spectral theory developed for H applies also to $H_{\mathcal{O}}$. To see this, we have only to replace $\mathcal{K}$ by $\mathcal{K}\cup((\Omega_1\cap\Omega_{int})\setminus\mathcal{O})$, and to argue in the same way as in §3 and §4. Let $\mathcal{E}(H_{\mathcal{O}})$ be $\sigma_p(H_{\mathcal{O}})$ when $\Omega_{\mathcal{O}}$ is bounded, and the set of exceptional points for $H_{\mathcal{O}}$ when $\Omega_{\mathcal{O}}$ is unbounded.

Next we consider the case $\mathcal{O}=\emptyset$ so that $\Gamma_{\mathcal{O}}=\Gamma$.

Lemma 5.3. *Suppose $\lambda\notin\mathcal{E}(H)\cup\mathcal{E}(H_\emptyset)$, and let $\Psi_{1,n,-}(\lambda)$ and $\Phi_{1,n,-}(\lambda)$ be physical and non-physical eigenfunctions for H. Then the linear subspace spanned by $\partial_\nu\Psi_{1,n,-}(\lambda)\big|_\Gamma$, $\partial_\nu\Phi_{1,n,-}(\lambda)\big|_\Gamma$, $n=1,2,\dots$, is dense in $L^2(\Gamma)$.*

Proof. We show that, if $f\in L^2(\Gamma)$ satisfies

$$(f,\partial_\nu\Psi_{1,n,-}(\lambda))_\Gamma=(f,\partial_\nu\Phi_{1,n,-}(\lambda))_\Gamma=0,\quad\forall n\geq 1,\tag{5.6}$$

then $f=0$. We define an operator $\delta'_\Gamma\in\mathbf{B}((H^{1/2}(\Gamma))';H^{-2}(\Omega))$, where $(H^{1/2}(\Gamma))'$ is the dual space of $H^{1/2}(\Gamma)$, by

$$(\delta'_\Gamma f,w)=(f,\partial_\nu w)_\Gamma,\quad\forall w\in H^2(\Omega),$$

and put $u=R(\lambda-i0)\delta'_\Gamma f$ by duality. This means that , if $G_-(\lambda;X,X')$ is Green's function, i.e. the integral (Schwartz) kernel of $R(\lambda-i0)$,

$$u(X)=\left(R(\lambda-i0)\delta'_\Gamma f\right)(X)=\int_\Gamma\partial_{\nu'}G_-(\lambda;X,X')f(X')dS_{X'},$$

where $\partial_{\nu'}$ means the conormal differentiation with respect to the variable X'. Then $u\in\mathcal{B}^*$, and by (3.14), we have the following asymptotic expansion on Ω_1

$$u\simeq\sum_{\lambda_{1,n}<\lambda}C_n(\lambda)e^{-iy\sqrt{\lambda-\lambda_{1,n}}}\left(f,\partial_\nu\Psi_{1,n,-}(\lambda)\right)_\Gamma\varphi_{1,n}(x).$$

In particular, if $\lambda_{1,n}<\lambda$

$$(u,\varphi_{1,n})\simeq C_n(\lambda)e^{-iy\sqrt{\lambda-\lambda_{1,n}}}\left(f,\partial_\nu\Psi_{1,n,-}(\lambda)\right)_\Gamma,\tag{5.7}$$

$C_n(\lambda)$ being a constant. In a similar way, we have for $\lambda_{1,n}>\lambda$

$$(u,\varphi_{1,n})\sim C'_n(\lambda)e^{-y\sqrt{\lambda_{1,n}-\lambda}}\left(f,\partial_\nu\Phi_{1,n,-}(\lambda)\right)_\Gamma\tag{5.8}$$

modulo a super exponentially decaying term. Note that $u_n=\big(u,\varphi_{1,n}\big)$ satisfies the equation $(-\partial_y^2+\lambda_{1,n}-\lambda)u_n=0$ for $y>a$, a being a sufficiently large constant. In view of the assumption of (5.6) and (5.7), (5.8), we then have $\big(u,\varphi_{1,n}\big)=0$ for $y>a$, hence $u(x,y)=0$ for $y>a$. The unique continuation theorem then implies $u=0$ on Ω_{ext}. By the property of classical double layer potential, $\partial_\nu u$ is continuous across Γ, so that $\partial_\nu u|_\Gamma=0$.

Next we show that $u=0$ in Ω_{int}. In the region Ω_{int}, we have $(-\Delta_G-\lambda)u=0$. If Ω_{int} is bounded, then $u=0$ since λ is not a Neumann eigenvalue. If Ω_{int} is not bounded, u satisfies the incoming radiation condition, since so does u in Ω. Then $u=0$ in Ω_{int} by Lemma 3.4. As $u=R(\lambda-i0)\delta'_\Gamma f\in L^2_{loc}(\Omega)$, it follows from the above that $u=0$ in Ω. Applying $H-\lambda$, we have $\delta'_\Gamma f=0$ as a distribution, hence $f=0$ on Γ. $\square$

5.3. **Interior boundary value problem.** For $z \in \mathbb{C} \setminus \mathcal{E}(H_{\mathcal{O}})$, we consider the following boundary value problem

$$
\left\{
\begin{array}{l}
(H_{\mathcal{O}} - z)u = 0 \quad \text{in} \quad \Omega_{\mathcal{O}}, \\
\partial_\nu u = 0 \quad \text{on} \quad \partial\Omega_{\mathcal{O}} \setminus \Gamma_{\mathcal{O}}, \\
\partial_\nu u = f \in H_0^{1/2}(\Gamma_{\mathcal{O}}) \quad \text{on} \quad \Gamma_{\mathcal{O}}.
\end{array}
\right.
\tag{5.9}
$$

The incoming radiation condition is also imposed, if Ω_{int} is unbounded and $z \in \mathbb{R}$. The Neumann-Dirichlet map (N-D map) is then defined by

$$
\Lambda_{\mathcal{O}}(z)f = u\big|_{\Gamma_{\mathcal{O}}},
\tag{5.10}
$$

where u is the solution to (5.9). When $\mathcal{O} = \emptyset$, we use for the N-D map of the operator $H_\emptyset$ the notation $\Lambda_{\mathcal{O}}(z) = \Lambda(z)$.

Now we consider the operator theoretical meaning of the N-D map. Note that from now on $\mathcal{O}$ may be a non-empty set. We put $\mathcal{F} = (\mathcal{F}_c, \mathcal{F}_p)$, where $\mathcal{F}_c$ is the generalized Fourier transform for $H_{\mathcal{O}}$ (which is absent when Ω_{int} is bounded) and $\mathcal{F}_p$ is defined by

$$
\mathcal{F}_p : \mathcal{H}_p(H_{\mathcal{O}}) \ni u \mapsto ((u, \psi_1), (u, \psi_2), \cdots),
$$

and where $\mathcal{H}_p(H_{\mathcal{O}})$ is the point spectral subspace for $H_{\mathcal{O}}$ and ψ_i is the eigenfunction associated with the eigenvalue λ_i of $H_{\mathcal{O}}$. There are two kinds of generalized Fourier transformation, $\mathcal{F}_+$ and $\mathcal{F}_-$. Both choices will do as $\mathcal{F}_c$. Then $\mathcal{F}$ is a unitary

$$
\mathcal{F} : L^2(\Omega_{int}) \to \widehat{\mathcal{H}} \oplus \mathbb{C}^d,
\tag{5.11}
$$

where $d = \dim \mathcal{H}_p(H_{\mathcal{O}})$. If $d = \infty$, $\mathbb{C}^d$ is replaced by ℓ^2. Moreover, we have

$$
(H_{\mathcal{O}} - z)^{-1} = \int_0^\infty \frac{\mathcal{F}_c(\lambda)^* \mathcal{F}_c(\lambda)}{\lambda - z} d\lambda + \sum_{i=1}^d \frac{P_i}{\lambda_i - z},
\tag{5.12}
$$

where P_i are the eigenprojections associated with eigenvalues λ_i, numbered counting multiplicities by $i = 1, 2, \ldots, d$, and the right-hand side converges in the sense of strong limit in $L^2(\Omega_{\mathcal{O}})$.

Let $r_{\Gamma_{\mathcal{O}}} \in \mathbf{B}(H^1(\Omega_{\mathcal{O}}); H^{1/2}(\Gamma_{\mathcal{O}}))$ be the trace operator to $\Gamma_{\mathcal{O}}$,

$$
r_{\Gamma_{\mathcal{O}}} : H^1(\Omega_{\mathcal{O}}) \ni f \to f\big|_{\Gamma_{\mathcal{O}}} \in H^{1/2}(\Gamma_{\mathcal{O}}).
$$

We define $\delta_{\Gamma_{\mathcal{O}}} \in \mathbf{B}((H^{1/2}(\Gamma_{\mathcal{O}}))'; (H^1(\Omega_{\mathcal{O}}))')$ as the adjoint of $r_{\Gamma_{\mathcal{O}}}$:

$$
(\delta_{\Gamma_{\mathcal{O}}} f, w)_{L^2(\Omega_{int})} = (f, r_{\Gamma_{\mathcal{O}}} w)_{L^2(\Gamma_{\mathcal{O}})}, \quad f \in (H^{1/2}(\Gamma_{\mathcal{O}}))', \quad w \in H^1(\Omega_{\mathcal{O}}).
$$

With this in mind we write

$$
r_{\Gamma_{\mathcal{O}}} = \delta_{\Gamma_{\mathcal{O}}}^*.
$$

Lemma 5.4. *For $z \notin \mathcal{E}(H_{\mathcal{O}})$, the N-D map has the following representation*

$$
\begin{aligned}
\Lambda_{\mathcal{O}}(z) &= \delta_{\Gamma_{\mathcal{O}}}^* (H_{\mathcal{O}} - z)^{-1} \delta_{\Gamma_{\mathcal{O}}} \\
&= \int_0^\infty \frac{\delta_{\Gamma_{\mathcal{O}}}^* \mathcal{F}_c(\lambda)^* \mathcal{F}_c(\lambda) \delta_{\Gamma_{\mathcal{O}}}}{\lambda - z} d\lambda + \sum_{i=1}^d \frac{\delta_{\Gamma_{\mathcal{O}}}^* P_i \delta_{\Gamma_{\mathcal{O}}}}{\lambda_i - z}.
\end{aligned}
$$

Proof. For $f \in H_0^{1/2}(\Gamma_{\mathcal{O}})$, take $\widetilde{f} \in H^2(\Omega_{\mathcal{O}})$ such that $\partial_\nu \widetilde{f} = f$ on $\Gamma_{\mathcal{O}}$ and $\widetilde{f}$ has compact support in $\overline{\Omega_{int}}$. Then the solution u of (5.9) is written as $u = \widetilde{f} - (H_{\mathcal{O}} - z)^{-1}(-\Delta_G - z)\widetilde{f}$. Let $g = \mathcal{F}_c(\lambda)(\Delta_G + z)\widetilde{f}$. Then for any $h \in \widehat{\mathcal{H}}_{int}$, where $\widehat{\mathcal{H}}_{int}$ is defined by (4.22) with $j = 2, \dots, N$.

$$
\begin{aligned}
(\mathcal{F}_c(\lambda)(\Delta_G + z)\widetilde{f}, h) &= ((\Delta_G + z)\widetilde{f}, \mathcal{F}_c(\lambda)^* h) \\
&= (\partial_\nu \widetilde{f}, r_{\Gamma_{\mathcal{O}}}\mathcal{F}_c(\lambda)^* h)_{L^2(\Gamma_{\mathcal{O}})} + (\widetilde{f}, (\Delta_G + \overline{z})\mathcal{F}_c(\lambda)^* h) \\
&= (f, r_{\Gamma_{\mathcal{O}}}\mathcal{F}_c(\lambda)^* h)_{L^2(\Gamma_{\mathcal{O}})} + (\widetilde{f}, (-\lambda + \overline{z})\mathcal{F}_c(\lambda)^* h).
\end{aligned}
$$

This implies

$$
\mathcal{F}_c(\lambda)(\Delta_G + z)\widetilde{f} = \mathcal{F}_c(\lambda)\delta_{\Gamma_{\mathcal{O}}} f + (-\lambda + z)\mathcal{F}_c(\lambda)\widetilde{f},
$$

Hence

$$
\int_0^\infty \frac{\mathcal{F}_c(\lambda)^*\mathcal{F}_c(\lambda)(\Delta_G + z)\widetilde{f}}{\lambda - z} d\lambda = \int_0^\infty \frac{\mathcal{F}_c(\lambda)^*\mathcal{F}_c(\lambda)\delta_{\Gamma_{\mathcal{O}}} f}{\lambda - z} d\lambda - \mathcal{F}_c^*\mathcal{F}_c\widetilde{f}.
$$

Similarly,

$$
\sum_{i=1}^d \frac{P_i(\Delta_G + z)\widetilde{f}}{\lambda_i - z} = \sum_{i=1}^d \frac{P_i \delta_{\Gamma_{\mathcal{O}}} f}{\lambda_i - z} - \sum_{i=1}^d P_i \widetilde{f}.
$$

Since $\mathcal{F}_c^*\mathcal{F}_c\widetilde{f} + \sum_{i=1}^d P_i\widetilde{f} = \widetilde{f}$, by (5.12), these imply that

$$
u = (H_{\mathcal{O}} - z)^{-1}\delta_{\Gamma_{\mathcal{O}}} f,
$$

which proves the lemma. □

Let us call the set

$$
\Big\{\big(\lambda, \delta_{\Gamma_{\mathcal{O}}}^*\mathcal{F}_c(\lambda)^*\mathcal{F}_c(\lambda)\delta_{\Gamma_{\mathcal{O}}}\big)\,;\, \lambda \in (0,\infty) \setminus \mathcal{E}(H_{\mathcal{O}})\Big\} \cup \Big\{\Big(\lambda_i, \delta_{\Gamma_{\mathcal{O}}}^* P_i \delta_{\Gamma_{\mathcal{O}}}\Big)\Big\}_{i=1}^d, \tag{5.13}
$$

where $d = \dim \mathcal{H}_p(H_{\mathcal{O}})$, the boundary spectral projection (**BSP**) for $H_{\mathcal{O}}$ on $\Gamma_{\mathcal{O}}$. On the other hand, the set

$$
\Big\{\big(\lambda, \mathcal{F}_c(\lambda)\delta_{\Gamma_{\mathcal{O}}}\big)\,;\, \lambda \in (0,\infty) \setminus \mathcal{E}(H_{\mathcal{O}})\Big\} \cup \Big\{\Big(\lambda_i, \psi_i(x)\big|_{\Gamma_{\mathcal{O}}}\Big)\Big\}_{i=1}^d \tag{5.14}
$$

is called the boundary spectral data (**BSD**) on $\Gamma_{\mathcal{O}}$.

By using the formula (3.17), we have the following lemma.

Lemma 5.5. *For a bounded Borel function $\varphi(\lambda)$ with support in $\mathbb{R} \setminus \mathcal{T}(H_{\mathcal{O}})$, where $\mathcal{T}(H_{\mathcal{O}})$ is defined by (3.18) with $j = 2, \cdots, N$, we have*

$$
\delta_{\Gamma_{\mathcal{O}}}^*\varphi(H_{\mathcal{O}})\delta_{\Gamma_{\mathcal{O}}} = \int_0^\infty \varphi(\lambda)\delta_{\Gamma_{\mathcal{O}}}^*\mathcal{F}_c(\lambda)^*\mathcal{F}_c(\lambda)\delta_{\Gamma_{\mathcal{O}}} d\lambda + \sum_{i=1}^d \varphi(\lambda_i)\delta_{\Gamma_{\mathcal{O}}}^* P_i \delta_{\Gamma_{\mathcal{O}}}.
$$

Proof. By the formulae (3.17) and (5.12), this lemma holds for any $\varphi(\lambda) \in C_0^\infty(\mathbb{R} \setminus \mathcal{T}(H_{\mathcal{O}}))$. The general case the follows from the approximation. □

Usually BSD is referred as given data in the inverse boundary value problems. What is actually used in our reconstruction for the manifold is the BSP.

Lemma 5.6. *Let $\mathcal{O} \subset \Omega_{int}$. Then knowing the N-D map $\Lambda_{\mathcal{O}}(z)$ for all $z \notin \sigma(H_{\mathcal{O}})$ is equivalent to knowing the BSP for $H_{\mathcal{O}}$.*

Proof. By Lemma 5.4, one can compute the N-D map by using BSP. Taking $\varphi(\lambda)$ as the characteristic function of the interval $[a,t)$ and taking note of the remark after (3.34), we differentiate the formula in Lemma 5.5 with respect to t to recover $\delta_{\Gamma_{\mathcal{O}}}^* \mathcal{F}_c(t)^* \mathcal{F}_c(t) \delta_{\Gamma_{\mathcal{O}}}$ for $t \in \mathbb{R} \setminus \mathcal{E}(H_{\mathcal{O}})$. Since

$$\sum_{i=1}^{d} \frac{\delta_{\Gamma_{\mathcal{O}}}^* P_i \delta_{\Gamma_{\mathcal{O}}}}{\lambda_i - z} = \Lambda_{\mathcal{O}}(z) - \int_0^\infty \frac{\delta_{\Gamma_{\mathcal{O}}}^* \mathcal{F}_c(\lambda)^* \mathcal{F}_c(\lambda) \delta_{\Gamma_{\mathcal{O}}}}{\lambda - z} d\lambda,$$

one can obtain eigenvalues λ_i as the poles of the right-hand side. The residues in these poles provide us with $\delta_{\Gamma_{\mathcal{O}}}^* \sum_{\lambda_j=\lambda_i} P_j \delta_{\Gamma_{\mathcal{O}}}$. This determines the terms $\delta_{\Gamma_{\mathcal{O}}}^* P_j \delta_{\Gamma_{\mathcal{O}}}$ for indexes j such that $\lambda_j = \lambda_i$, up to an orthogonal transformation of the eigenspace associated to the eigenvalue λ_i, see [48, Lem. 4.9] or [49]. Thus we can determine the BSP for $H_{\mathcal{O}}$. □

We complete this section by the following result used later to prove Theorem 1.1. Let $\Omega^{(r)}$, $r = 1, 2$, be as in Theorem 1.1 We take Γ as above, which moreover has the following property: $G_1^{(1)} = G_1^{(2)}$ on $\Omega_{ext} = \Omega_{ext}^{(1)} = \Omega_{ext}^{(2)}$. We put the superscript $^{(r)}$ for all relevant operators and functions explained above. Let $\Lambda^{(r)}(\lambda)$, $r = 1, 2$ be the N-D map for $H_\emptyset^{(r)}$, that is, when $\mathcal{O} = \emptyset$. The basic idea of the following Lemma is due to Eidus [23].

Lemma 5.7. *Under the assumptions of Theorem 1.1, we have* $\Lambda^{(1)}(\lambda) = \Lambda^{(2)}(\lambda)$ *for* $\lambda \in (0,\infty) \setminus \cup_{r=1,2}(\mathcal{E}(H^{(r)}) \cup \mathcal{E}(H_\emptyset^{(r)}))$*, and BSP's for* $H_\emptyset^{(1)}$ *and* $H_\emptyset^{(2)}$ *coincide on* Γ.

Proof. Since $\widehat{S}_{11}^{(1)}(\lambda) = \widehat{S}_{11}^{(2)}(\lambda)$, the physical scattering amplitudes coincide, hence so do non-physical scattering amplitudes by analytic continuation. Let $u = \Psi_{1,n,-}^{(1)}(\lambda) - \Psi_{1,n,-}^{(2)}(\lambda)$ and $v = \Phi_{1,n,-}^{(1)}(\lambda) - \Phi_{1,n,-}^{(2)}(\lambda)$. Then since $H^{(1)} = H^{(2)} = -\partial_y^2 - \Delta_{h_1}$ on Ω_{ext}, u and v satisfy $(-\partial_y^2 - \Delta_{h_1} - \lambda)u = 0$ and $(-\partial_y^2 - \Delta_{h_1} - \lambda)v = 0$ in Ω_{ext}. Using Lemma 5.2 and arguing in the same way as in the proof of Lemma 5.3, we have $u = v = 0$ in Ω_{ext}. Therefore, $\Psi_{1,n,-}^{(r)}$ and $\Phi_{1,n,-}^{(r)}$ as well as their normal derivatives coincide for $r = 1, 2$ and for all $n \in \mathbf{Z}_+$. Since they satisfy the equation (5.9) for $H_\emptyset = H_\emptyset^{(r)}$, we have $\Lambda^{(1)}(\lambda) = \Lambda^{(2)}(\lambda)$ due to Lemma 5.3. The last statement now follows immediately from Lemma 5.6. □

6. Boundary control method for manifolds with asymptotically cylindrical ends

In this section we reconstruct the isometry type of the manifold (Ω_{int}, G) using given data.

Theorem 6.1. *Assume that we are given the set* Γ *as a differentiable manifold, the metric* G *on* Γ*, and the BSP for* $H_\emptyset$*. These data determine the manifold* (Ω_{int}, G) *up to an isometry.*

For proving this theorem, we use the boundary control (BC) method for inverse problems. The method goes back to [8] where it was used to recover the isotropic

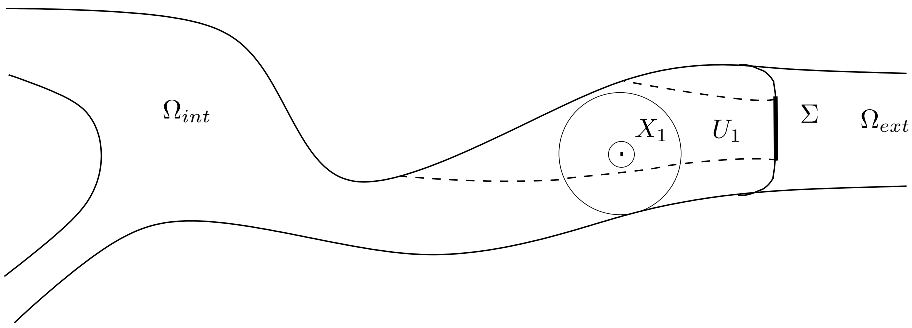

FIGURE 3. We will construct the manifold Ω_{int} by iterating local constructions. First, a neighborhood $U_1 \subset \Omega_{int}$ of $\Sigma \subset \Gamma$ is reconstructed. Next, a ball $\mathcal{O} = B(X_1, \rho) \subset U_1$ is removed from the manifold and data analogous to measurements on $\partial\mathcal{O}$ are constructed. After that, the metric is reconstructed in a larger ball $B(X_1, \tau)$, and the procedure is iterated to reconstruct the whole manifold Ω_{int}.

wave velocity in the acoustic equation in a domain in $\mathbb{R}^n$. In [11] it was developed to prove the analog of Theorem 6.1 for compact manifolds when BSD is given on $\partial\Omega$. The method was then extended to a large class of elliptic (and associated hyperbolic) operators on compact manifolds in e.g. [9, 46, 52, 49, 51], see also [48]. Later it was also extended to a number of inverse problems for systems on compact manifolds, e.g. [10, 55, 54]. The BC method combines Tataru's uniqueness results in the control theory for PDE's with Blagovestchenskii's identity that gives the inner product of the solutions of the wave equation in terms of the boundary data. This identity was originally used in the study of one-dimensional inverse problems, see [12, 13]. The reconstruction of non-compact manifolds is considered previously in the conference proceedings [50] and in [14] with different kind of data, using iterated time reversal for solutions of the wave equation. The reconstruction of (Ω_{int}, G) below is based on matching local reconstructions. Geometrically, this procedure is similar to the one described in [48], Sec. 4.4. However, the analytic technique used here is different. In [48] (see also [47]), the reconstruction is based on the combination of the use of Gaussian beams and the continuation of the eigenfunctions. In this section we develop a technique based on the continuation of Green's function and BSP which is suitable for the non-compact (as well as compact) manifolds.

The proof of Theorem 6.1 is divided into a series of lemmas. Our reconstruction of (Ω_{int}, G) is of recurrent nature. We will begin with the case when $\mathcal{O} = \emptyset$ so that we are given just the set $\Gamma_{\mathcal{O}} = \Gamma$ as a differentiable manifold, the metric on it, and the BSP for the operator $H_\emptyset$ on Γ. We apply the boundary control method to reconstruct the metric G on some neighborhood U_1 of Γ. Then, we will take a point $X_1 \in U_1 \setminus \Gamma$ and $\rho > 0$ such that $B(X_1, 2\rho) \subset U_1$, where $B(X_1, r)$ denotes the ball of radius r with center at X_1. We take $\mathcal{O} = B(X_1, \rho)$ and show that we can find the BSP for the operator $H_{\mathcal{O}}$ on $\Gamma_{\mathcal{O}} = \partial\mathcal{O}$. Then we apply the boundary control method starting from $\Gamma_{\mathcal{O}}$, which would allow us to recover (Ω_{int}, G) in a larger neighborhood $U_2 \supset U_1$ of Γ. Proceeding in this way, we will eventually recover the whole of (Ω_{int}, G). Therefore, our further considerations deal with arbitrary $\mathcal{O} \subset \Omega_{int}$ including the case $\mathcal{O} = \emptyset$.

6.1. **Blagovestchenskii's identity.** Let us first consider the initial boundary value problem

$$
\begin{cases} \partial_t^2 u = \Delta_G u, \quad \text{in} \quad \Omega_{\mathcal{O}} \times \mathbb{R}_+, \\ u\big|_{t=0} = \partial_t u\big|_{t=0} = 0, \quad \text{in} \quad \Omega_{\mathcal{O}}, \\ \partial_\nu u = f, \quad \text{in} \quad \partial\Omega_{\mathcal{O}} \times \mathbb{R}_+, \quad \operatorname{supp} f \subset \Gamma_{\mathcal{O}} \times \mathbb{R}_+. \end{cases} \tag{6.1}
$$

Lemma 6.2. *Assume that we are given the set $\Gamma_{\mathcal{O}}$ as a differentiable manifold, the metric G on $\Gamma_{\mathcal{O}}$ and the BSP for $H_{\mathcal{O}}$ on $\Gamma_{\mathcal{O}}$. Then for any given $f, h \in C_0^\infty(\Gamma_{\mathcal{O}} \times \mathbb{R}_+)$ and $t, s > 0$ these data uniquely determine*

$$
(u^f(t), u^h(s)) = \int_{\Omega_{\mathcal{O}}} u^f(X,t)\, \overline{u^h(X,s)}\, dV_X,
$$

where $u^f(t)$ and $u^h(t)$ are solutions of (6.1) with boundary data f and h, correspondingly.

Proof. Let

$$
S(t,\lambda) = \frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}}.
$$

Then the solution $u^f(t)$ is written as

$$
\begin{aligned} u^f(t) = & \int_0^t ds \int_0^\infty d\lambda S(t-s,\lambda)\mathcal{F}_c(\lambda)^*\mathcal{F}_c(\lambda)\delta_{\Gamma_{\mathcal{O}}} f(s) \\ & + \int_0^t ds \sum_{i=1}^d S(t-s,\lambda_i) P_i \delta_{\Gamma_{\mathcal{O}}} f(s). \end{aligned}
$$

Using the similar decomposition for $u^h(s)$ and the fact that $\mathcal{F}_c(\mu)\mathcal{F}_c(\lambda)^* = \delta(\mu-\lambda)$, we obtain the following formula:

$$
\begin{aligned} & (u^f(t), u^h(s)) \\ = & \int_0^t dt' \int_0^s ds' \int_0^\infty d\lambda\, \widetilde{S}(t-t', s-s', \lambda) \left(\delta_{\Gamma_{\mathcal{O}}}^* \mathcal{F}_c(\lambda)^* \mathcal{F}_c(\lambda) \delta_{\Gamma_{\mathcal{O}}} f(t'), h(s')\right)_{L^2(\Gamma_{\mathcal{O}})} \\ & + \int_0^t dt' \int_0^s ds' \sum_{i=1}^d \widetilde{S}(t-t', s-s', \lambda_i) \left(\delta_{\Gamma_{\mathcal{O}}}^* P_i \delta_{\Gamma_{\mathcal{O}}} f(t'), h(s')\right)_{L^2(\Gamma_{\mathcal{O}})}, \end{aligned} \tag{6.2}
$$

where $\widetilde{S}(t,s,\lambda) = S(t,\lambda)S(s,\lambda)$. Observe that the right-hand side depends only on BSP and the metric on $\Gamma_{\mathcal{O}}$. □

Above, the formula (6.2) is a generalization of Blagovestchenskii identity (see [48, Theorem 3.7]) for non-compact manifolds.

6.2. **Finite propagation property of waves.** Let us next introduce some notations. For $t > 0$ and $\Sigma \subset \Gamma_{\mathcal{O}}$ arbitrary, let

$$\Omega_{\mathcal{O}}(\Sigma, t) = \{X \in \Omega_{\mathcal{O}} \ ; \ d_{\mathcal{O}}(X, \Sigma) \leq t\}$$

be the domain of influence of Σ at time t. Here, $d_{\mathcal{O}}(X, Y)$ is the distance between X and Y in $\Omega_{\mathcal{O}}$. We use also the notation $\Omega_{\mathcal{O}}(Y, t) = \Omega_{\mathcal{O}}(\{Y\}, t)$. More generally, when $I = \{(\Sigma_j, t_j)\}_{j=1}^J$ is a finite collection of pairs (Σ_j, t_j), where $\Sigma_j \subset \Gamma_{\mathcal{O}}$ and $t_j > 0$, we denote

$$\Omega_{\mathcal{O}}(I) = \bigcup_{j=1}^J \Omega_{\mathcal{O}}(\Sigma_j, t_j) = \{X \in \Omega_{\mathcal{O}} \ ; \ d_{\mathcal{O}}(X, \Sigma_j) \leq t_j \text{ for some } j = 1, \ldots, J\}.$$

For any measurable set $B \subset \Omega_{\mathcal{O}}$, we denote $L^2(B) = \{v \in L^2(\Omega_{\mathcal{O}});\ v|_{\Omega_{\mathcal{O}} \setminus B} = 0\}$, identifying functions and their zero continuations.

Lemma 6.3. *Assume that we are given the set $\Gamma_{\mathcal{O}}$ as a differentiable manifold, the metric on $\Gamma_{\mathcal{O}}$ and the BSP for $H_{\mathcal{O}}$ on $\Gamma_{\mathcal{O}}$. Then, for any given $f \in C_0^\infty(\Gamma_{\mathcal{O}} \times \mathbb{R}_+)$, $T > 0$, and $I = \{(\Sigma_j, t_j)\}_{j=1}^J$, where $\Sigma_j \subset \Gamma_{\mathcal{O}}$ are open sets or single points, and $t_j < T$, we can determine*

$$a_{I,T}(f) = \int_{\Omega_{\mathcal{O}} \setminus \Omega_{\mathcal{O}}(I)} |u^f(T)|^2 \, dV. \tag{6.3}$$

Proof. When $\Sigma \subset \Gamma_{\mathcal{O}}$ is an open set and $h \in C_0^\infty(\Sigma \times \mathbb{R}_+)$, it follows from the finite velocity of wave propagation (see e.g. [57, Sec. 4.2], see also [41, Ch. 6]) that the wave $u^h(t) = u^h(\cdot, t)$ is supported in the domain $\Omega_{\mathcal{O}}(\Sigma, t)$ at time $t > 0$. It follows from Tataru's seminal unique continuation result, see [73, 74], that the set

$$\{u^h(t);\ h \in C_0^\infty(\Sigma \times \mathbb{R}_+)\} \tag{6.4}$$

is dense in $L^2(\Omega_{\mathcal{O}}(\Sigma, t))$, see e.g. [48, Theorem 3.10]. This clearly implies that, when $T > 0$ and $I = \{(\Sigma_j, t_j)\}_{j=1}^J$, where Σ_j are open and $t_j < T$, the set

$$\begin{aligned} X_I^T \ &:= \ \{u^h(T);\ h = h_1 + \cdots + h_J,\ h_j \in C_0^\infty(\Sigma_j \times [T - t_j, T])\} \\ &= \ \mathrm{span}_{j=1,\ldots,J} \{u^h(t_j);\ h \in C_0^\infty(\Sigma_j \times [0, t_j])\} \end{aligned}$$

is dense in $L^2(\Omega_{\mathcal{O}}(I))$.

Next, we consider the non-linear functional

$$a_{I,T}(f) = \inf\{\|u^{f-h}(T)\|^2_{L^2(\Omega_{\mathcal{O}})};\ h = h_1 + \cdots + h_J,\ h_j \in C_0^\infty(\Sigma_j \times [T - t_j, T])\}$$

where $f \in C_0^\infty(\Gamma_{\mathcal{O}} \times \mathbb{R}_+)$, $T > 0$, and $I = \{(\Sigma_j, t_j)\}_{j=1}^J$, $\Sigma_j \subset \Gamma_{\mathcal{O}}$ are open, and $t_j < T$. By the formula (6.2), the BSP and the metric on $\Gamma_{\mathcal{O}}$ determine the value $a_{I,T}(f)$ for any f. Moreover, as $u^{f-h}(T) = u^f(T) - u^h(T)$ and X_I^T is dense in $L^2(\Omega_{\mathcal{O}}(I))$, we see that

$$a_{I,T}(f) = \|(1 - \chi_{\Omega_{\mathcal{O}}(I)}) u^f(T)\|^2_{L^2(\Omega_{\mathcal{O}})}, \tag{6.5}$$

where $\chi_{\Omega_{\mathcal{O}}(I)}(x)$ is the characteristic function of the set $\Omega_{\mathcal{O}}(I)$ on $\Omega_{\mathcal{O}}$. This proves the lemma for the case when all Σ_j are open.

If for some j, the set Σ_j is just a point $X_j \in \Gamma_{\mathcal{O}}$, we define for those j's $\Sigma_j^{(k)} \subset \Gamma_{\mathcal{O}}$, $k = 1, 2, \cdots$ to be open neighborhoods of X_j such that $\overline{\Sigma_j^{(k+1)}} \subset \Sigma_j^{(k)}$ and $\bigcap_k \Sigma_j^{(k)} = \{X_j\}$. For those j's for which Σ_j is open, we define $\Sigma_j^{(k)} = \Sigma_j$. Denote the corresponding finite collection of $(\Sigma_j^{(k)}, t_j)$ by $I(k)$. Then

$$\Omega_{\mathcal{O}}(I(k+1)) \subset \Omega_{\mathcal{O}}(I(k)), \quad \Omega_{\mathcal{O}}(I) = \bigcap_{k=1}^{\infty} \Omega_{\mathcal{O}}(I(k)),$$

and for any $b \in L^2(\Omega_{\mathcal{O}})$,

$$\big(1 - \chi_{\Omega_{\mathcal{O}}(I(k))}\big)b \to \big(1 - \chi_{\Omega_{\mathcal{O}}(I)}\big)b, \quad \text{a.e.} \quad \text{as} \quad k \to \infty.$$

As $|\big(1 - \chi_{\Omega_{\mathcal{O}}(I(k))}\big)b(\cdot)| \leq |\big(1 - \chi_{\Omega_{\mathcal{O}}(I)}\big)b(\cdot)|$, a.e., using the monotone convergence theorem, we see that

$$a_{I(k),T}(f) \to \|\big(1 - \chi_{\Omega_{\mathcal{O}}(I)}\big)u^f(T)\|^2_{L^2(\Omega_{\mathcal{O}})} = a_{I,T}(f).$$

Thus, the BSP and the metric on $\Gamma_{\mathcal{O}}$ determine $a_{I,T}(f)$ for such $I's$. □

Definition 6.4. Let $I = \{(\Sigma_j, t_j)\}_{j=1}^J$, $I' = \{(\Sigma'_j, t'_j)\}_{j=1}^J$ and $T > 0$, where $\Sigma_j, \Sigma'_j \subset \Gamma_{\mathcal{O}}$ and $t_j, t'_j < T$. We say that the relation $I \geq I'$ is valid on manifold $\Omega_{\mathcal{O}}$ if

$$\Omega_{\mathcal{O}}(I') \setminus \Omega_{\mathcal{O}}(I) \text{ has measure zero.} \tag{6.6}$$

Lemma 6.5. *Let $I = \{(\Sigma_j, t_j)\}_{j=1}^J$, $I' = \{(\Sigma'_j, t'_j)\}_{j=1}^J$ and $T > 0$, where $\Sigma_j, \Sigma'_j \subset \Gamma_{\mathcal{O}}$ are open sets or single points and $t_j, t'_j < T$. Assume that we are given the set $\Gamma_{\mathcal{O}}$ as a differentiable manifold, the metric on $\Gamma_{\mathcal{O}}$, the BSP for $H_{\mathcal{O}}$ on $\Gamma_{\mathcal{O}}$, and the collections I and I'. Then we can determine whether the relation $I \geq I'$ is valid on manifold $\Omega_{\mathcal{O}}$ or not.*

Proof. The relation $I \geq I'$ is valid on manifold $\Omega_{\mathcal{O}}$ if and only if

$$a_{I,T}(f) \leq a_{I',T}(f) \quad \text{for all } f \in C_0^\infty(\Gamma_{\mathcal{O}} \times \mathbb{R}_+). \tag{6.7}$$

Indeed, the equivalence of (6.6) and (6.7) follows from (6.5) and the fact that, by Tataru's density result (6.4), the functions $u^f(T)$, $f \in C_0^\infty(\Gamma_{\mathcal{O}} \times \mathbb{R}_+)$, are dense in $L^2(\Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, T))$. As for given f, by Lemma 6.3, we can evaluate both sides of (6.7), using the BSP and the metric on $\Gamma_{\mathcal{O}}$, these data determine, for any pair (I, I'), if the relation $I \geq I'$ is valid or not. □

For any $X_0 \in \Omega_{int} \setminus \partial\Omega_{int}$, introduce the exponential map

$$\exp_{X_0} : (\xi, t) \mapsto \gamma_{(X_0,\xi)}(t),$$

where $\xi \in S_{X_0}(\Omega_{int}) = \{\eta \in T_{X_0}(\Omega_{int}); |\eta| = 1\}$ and $0 \leq t \leq s(X_0, \xi)$. Here $\gamma_{(X_0,\xi)}(t)$ is the geodesic on Ω, parametrized by the arclength, with $\gamma_{(X_0,\xi)}(0) = X_0$, $\dot\gamma_{(X_0,\xi)}(0) = \xi$, and $[0, s(X_0, \xi))$ is the maximal interval of t, when $\gamma_{(X_0,\xi)}(t)$ stays in Ω_{int}, that is, $s(X_0, \xi) = \sup\{t\,;\, \gamma_{(X_0,\xi)}([0, t)) \subset \Omega_{int} \setminus \partial\Omega_{int}\}$. Denote by

$$s(X_0) = \inf_{\xi \in S_{X_0}(\Omega)} s(X_0, \xi) \tag{6.8}$$

so that

$$B(X_0, s(X_0)) \subset \Omega_{int} \setminus \partial\Omega_{int}.$$

Define now

$$\tau(X_0, \xi) = \sup_{0<t<s(X_0)} \{t\,;\, d_\emptyset(\gamma_{(X_0,\xi)}(t), X_0) = t\}.$$

At last, define

$$\tau(X_0) = \inf_{\xi \in S_{X_0}(\Omega_{int})} \tau(X_0, \xi). \tag{6.9}$$

In geometric terms, the above definition of $\tau(X_0)$ means that in the ball $B(X_0, \tau(X_0)) \subset \Omega_{int} \setminus \partial\Omega_{int}$, it is possible to introduce the Riemannian normal coordinates

$$X \mapsto (\xi, t) : \xi \in S_{X_0}(\Omega_{int}), 0 \le t < \tau(X_0)$$

which satisfy $\gamma_{(X_0,\xi)}(t) = X$.

We also need the boundary exponential map

$$\exp_{\Gamma_\mathcal{O}} : \{(Z,t) \in \Gamma_\mathcal{O} \times \mathbb{R}_+\,;\, 0 \le t < s_\mathcal{O}(Z)\} \ni (Z,t) \to \gamma_{(Z,\nu)}(t) \in \Omega_\mathcal{O}.$$

Here ν is the interior unit normal (with respect to $\Omega_\mathcal{O}$) to $\Gamma_\mathcal{O}$ and

$$s_\mathcal{O}(Z) = \sup\{t > 0\,;\, \gamma_{(Z,\nu)}((0,t)) \subset \Omega_\mathcal{O} \setminus \partial\Omega_\mathcal{O}\}. \tag{6.10}$$

For any $Z \in \Gamma_\mathcal{O}$, let

$$\tau_\mathcal{O}(Z) = \sup_{0\le t\le s_\mathcal{O}(Z)} \{t\,;\, d_\mathcal{O}(\gamma_{(Z,\nu)}(t), \Gamma_\mathcal{O}) = t\}. \tag{6.11}$$

In the following, we impose the following condition (C-2) on Σ.

(C-2) For $\mathcal{O} = \emptyset$, Σ is an open subset of Γ such that $d_\emptyset(\Sigma, \partial\Gamma) > 0$, and for $\mathcal{O} \ne \emptyset$, $\Sigma = \partial\mathcal{O}$.

We define

$$\tau_\mathcal{O}(\Sigma) = \inf_{Z\in\Sigma} \tau_\mathcal{O}(Z). \tag{6.12}$$

In geometric terms, the above definition of $\tau_\mathcal{O}(\Sigma)$ means that, in the set

$$L(\Sigma, \tau_\mathcal{O}(\Sigma)) = \{\gamma_{(Z,\nu)}(t)\,;\, Z \in \Sigma, 0 \le t < \tau_\mathcal{O}(\Sigma)\} \subset \big(\Omega_\mathcal{O} \setminus \partial\Omega_\mathcal{O}\big) \cup \Sigma,$$

it is possible to introduce the boundary normal coordinates

$$X \mapsto (Z,t), Z \in \Sigma, 0 \le t < \tau_\mathcal{O}(\Sigma)$$

satisfying $X = \gamma_{(Z,\nu)}(t)$. Observe that when $\mathcal{O} = B(X,\rho)$, $X \in \Omega_{int} \setminus \partial\Omega_{int}$ and $\rho > 0$ is small enough, then

$$\tau_\mathcal{O}(\partial\mathcal{O}) = \tau(X) - \rho.$$

Lemma 6.6. *Assume that $\Sigma \subset \Gamma_\mathcal{O}$ satisfies condition (C-2). Let $Y \in \Sigma$, $Z \in \Gamma_\mathcal{O}$, $t < \tau_\mathcal{O}(\Sigma)$, and $X = \gamma_{(Y,\nu)}(t)$. Assume that we are given the set $\Gamma_\mathcal{O}$ as a differentiable manifold, the metric on $\Gamma_\mathcal{O}$ and the BSP for $H_\mathcal{O}$ on $\Gamma_\mathcal{O}$. Then we can determine the distance $d_\mathcal{O}(X,Z)$ on $\Omega_\mathcal{O}$.*

Proof. Note that as $t < \tau_{\mathcal{O}}(\Sigma)$, the set $\Omega_{\mathcal{O}}(Y,t) \setminus \Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, t-\varepsilon)$ contains a non-empty open set for all $\varepsilon > 0$. For $s, \varepsilon > 0$, let us denote (see Fig. 4)

$$I_\epsilon(t) = \big\{(Y,t), (\Gamma_{\mathcal{O}}, t-\varepsilon)\big\}, \quad I'_\varepsilon(t,s) = \big\{(Z,s),\, (\Gamma_{\mathcal{O}}, t-\varepsilon)\big\}.$$

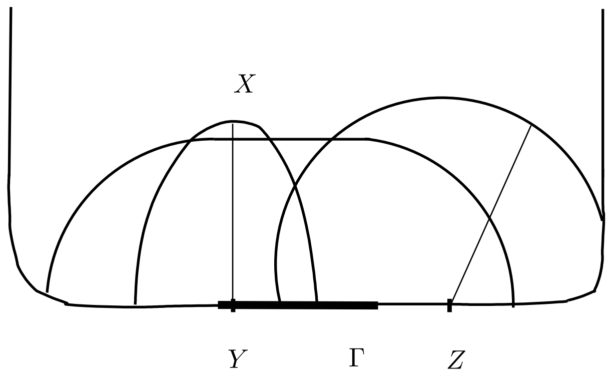

FIGURE 4. In the figure $\mathcal{O} = \emptyset$ and s is small enough so that the set $\Omega_{\mathcal{O}}(Y,t) \setminus \Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, t-\varepsilon)$ is not contained in $\Omega_{\mathcal{O}}(Z,s)$. This is the situation when $I'_\varepsilon(t,s) \not\geq I_\epsilon(t)$.

Let us next show that for any $r > 0$ there is $\varepsilon_0 > 0$ such that

$$\Omega_{\mathcal{O}}(Y,t) \setminus \Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, t-\varepsilon) \subset B(X,r), \quad \text{when} \quad \varepsilon < \varepsilon_0.$$

If this is not true, there are $r > 0$, a sequence $\varepsilon_j \to 0$, and $X_j \in \Omega_{\mathcal{O}}(Y,t) \setminus \Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, t-\varepsilon_j)$ such that $d_{\mathcal{O}}(X_j, X) \geq r$. As $\Omega_{\mathcal{O}}(Y,t)$ is compact, by considering a subsequence, we can assume that X_j converge to $\tilde{X} \in \Omega_{\mathcal{O}}(Y,t)$. Then

$$d_{\mathcal{O}}(\tilde{X}, Y) = \lim_{j\to\infty} d_{\mathcal{O}}(X_j, Y) \leq t,$$
$$d_{\mathcal{O}}(\tilde{X}, \Gamma_{\mathcal{O}}) = \lim_{j\to\infty} d_{\mathcal{O}}(X_j, \Gamma_{\mathcal{O}}) \geq t,$$

implying that Y is a closest point of $\Gamma_{\mathcal{O}}$ to $\tilde{X}$ and $d_{\mathcal{O}}(\tilde{X}, Y) = t$. Let us recall that the shortest curve from a point in $\Omega_{\mathcal{O}}$ to $\Gamma_{\mathcal{O}}$, which end point is an interior point of $\Gamma_{\mathcal{O}}$, is a normal geodesic to $\Gamma_{\mathcal{O}}$. Thus, we see that $\tilde{X} = \gamma_{(Y,\nu)}(t) = X$, which is in contradiction to $d(\tilde{X}, X) \geq r$. Thus, the existence of ε_0 for any r is proven.

The above implies that when $s > d(X,Z)$, the set $\Omega_{\mathcal{O}}(Y,t) \setminus \Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, t-\varepsilon)$ is contained in $\Omega_{\mathcal{O}}(Z,s)$ for all sufficiently small $\varepsilon > 0$ and therefore,

(6.13) there is $\varepsilon_1 > 0$ such that $I'_\varepsilon(t,s) \geq I_\varepsilon(t)$ for all $0 < \varepsilon < \varepsilon_1$.

On the other hand, for $s < d(X,Z)$, the set $\Omega_{\mathcal{O}}(Y,t) \setminus \Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, t-\varepsilon) \neq \emptyset$ do not intersect with $\Omega_{\mathcal{O}}(Z,s)$ at any $\varepsilon > 0$ small enough and thus (6.13) does not hold. Thus, by Lemma 6.5, we can find $d_{\mathcal{O}}(X,Z)$ for any $Z \in \Gamma_{\mathcal{O}}$ as the infimum of all $s > 0$ for which (6.13) hold. $\square$

For $\Sigma \subset \Gamma_{\mathcal{O}}$ satisfying (C-2) and $0 < T < \tau_{\mathcal{O}}(\overline{\Sigma})$, let $N_{\Sigma,T}$ and $M_{\Sigma,T}$ be the sets

$$
\begin{aligned}
&N_{\Sigma,T} = \{X \in \Omega_{\mathcal{O}};\ X = \gamma_{(Y,\nu)}(t),\ 0 \le t \le T,\ Y \in \overline{\Sigma}\},\\
&M_{\Sigma,T} = \{X \in \Omega_{\mathcal{O}};\ X = \gamma_{(Y,\nu)}(t),\ 0 < t < T,\ Y \in \Sigma\} \subset N_{\Sigma,T} = \overline{M_{\Sigma,T}}.
\end{aligned}
\tag{6.14}
$$

Note that $M_{\Sigma,T}$ is open in $\Omega_{\mathcal{O}}$.

6.3. **Boundary distance functions and reconstruction of topology.** Let us next consider the collection of the boundary distance functions associated with $\Gamma_{\mathcal{O}}$. For each $X \in \Omega_{\mathcal{O}}$, the corresponding *restricted boundary distance function*, $r_X \in C(\Gamma_{\mathcal{O}})$ (note that $\overline{\Gamma}_{\mathcal{O}}$ is compact) is given by

$$
r_X : \Gamma_{\mathcal{O}} \to \mathbb{R}_+, \quad r_X(Z) = d_{\mathcal{O}}(X, Z), \quad Z \in \Gamma_{\mathcal{O}}.
$$

The restricted boundary distance functions define *the boundary distance map* $R_{\mathcal{O}}$: $\Omega_{\mathcal{O}} \to C(\Gamma_{\mathcal{O}})$, $R_{\mathcal{O}}(X) = r_X$. The boundary distance representation of $N_{\Sigma,T} \subset \Omega_{\mathcal{O}}$ is the set

$$
R_{\mathcal{O}}(N_{\Sigma,T}) = \{r_X \in C(\Gamma_{\mathcal{O}});\ X \in N_{\Sigma,T}\},
$$

that is, the image of $N_{\Sigma,T}$ in $R_{\mathcal{O}}$. Clearly $R_{\mathcal{O}} : \Omega_{\mathcal{O}} \to C(\Gamma_{\mathcal{O}})$ is continuous.

Lemma 6.7. *Assume that we are given the set $\Gamma_{\mathcal{O}}$ as a differentiable manifold, the metric on $\Gamma_{\mathcal{O}}$, the BSP for $H_{\mathcal{O}}$ on $\Gamma_{\mathcal{O}}$, an open set $\Sigma \subset \Gamma_{\mathcal{O}}$ satisfying condition (C-2), and $0 < T < \tau_{\mathcal{O}}(\overline{\Sigma})$. Then we can determine the set*

$$
R_{\mathcal{O}}(N_{\Sigma,T}) = R_{\mathcal{O}}(\{\gamma_{(Y,\nu)}(t);\ Y \in \overline{\Sigma},\ 0 \le t \le T\}).
$$

Proof. By Lemma 6.6, for $Y \in \Sigma$, $t < T$ and $Z \in \Gamma_{\mathcal{O}}$, we can find $d_{\mathcal{O}}(X, Z)$ where $X = \gamma_{(Y,\nu)}(t)$ from BSP. This gives us the function $r_X(Z)$, $Z \in \Gamma_{\mathcal{O}}$, and for such X's. Thus, BSP and the metric on $\Gamma_{\mathcal{O}}$ determine the set $R_{\mathcal{O}}(M_{\Sigma,T})$. Using (6.14), we obtain $R_{\mathcal{O}}(N_{\Sigma,T})$ by closure of $R_{\mathcal{O}}(M_{\Sigma,T})$ in $C(\Gamma_{\mathcal{O}})$. □

Consider properties of $R_{\mathcal{O}}$. Assume that $r_X = r_Y$ for some $X, Y \in N_{\Sigma,T}$. Let $Z \in \Gamma_{\mathcal{O}}$ be the point where the function r_X attains its minimum. Then, it is the closest point of $\Gamma_{\mathcal{O}}$ to X. Thus, the shortest geodesic from X to Z is normal to $\Gamma_{\mathcal{O}}$, i.e. $X = \gamma_{(Z,\nu)}(t)$ with $t = r_X(Z)$. The same arguments show that Z is also the closest point of $\Gamma_{\mathcal{O}}$ to Y and $t = r_Y(Z)$, and hence $Y = \gamma_{(Z,\nu)}(t)$. Thus $X = Y$ and $R_{\mathcal{O}}$ is injective on $N_{\Sigma,T}$.

Thus, map $R_{\mathcal{O}} : N_{\Sigma,T} \to R_{\mathcal{O}}(N_{\Sigma,T})$ is a bijective continuous map defined on a compact set, implying that it is a homeomorphism. This implies that the map $R_{\mathcal{O}} : M_{\Sigma,T} \to R_{\mathcal{O}}(M_{\Sigma,T})$ is a homeomorphism. As BSP and the metric on $\Gamma_{\mathcal{O}}$ determine the manifold $R_{\mathcal{O}}(M_{\Sigma,T})$ with its topological structure inherited from $C(\Gamma_{\mathcal{O}})$, we see that these data determine the manifold $M_{\Sigma,T}$ as a topological space.

Lemma 6.8. *The set $R_{\mathcal{O}}(M_{\Sigma,T}) \subset C(\Gamma_{\mathcal{O}})$ can be endowed, in a constructive way, with a differentiable structure and a metric tensor $\tilde{G}$, so that $(R_{\mathcal{O}}(M_{\Sigma,T}), \tilde{G})$ becomes a manifold which is isometric to $(M_{\Sigma,T}, G)$ with $R_{\mathcal{O}}$ being an isometry.*

For compact manifolds, the result analogous to Lemma 6.8 is presented in detail in [48, Sect. 3.8]. Since the proof is based on local constructions, it works for non-compact manifolds without any change. However, for the convenience of the reader, we present this construction.

Proof. Let us define the evaluation functions, E_Z, $Z \in \Gamma_{\mathcal{O}}$,

$$E_Z : R_{\mathcal{O}}(M_{\Sigma,T}) \to \mathbb{R}, \quad E_Z(r_X) = r_X(Z) = d_{\mathcal{O}}(X,Z).$$

For $r(\cdot) \in R_{\mathcal{O}}(M_{\Sigma,T})$ corresponding to a point $X \in M_{\Sigma,T}$, i.e. $r(\cdot) = r_X(\cdot)$, we can choose points $Z_1, \dots, Z_n \in \Gamma_{\mathcal{O}}$ close to the nearest point of $\Gamma_{\mathcal{O}}$ to X so that $X \mapsto (d_{\mathcal{O}}(X, Z_j))_{j=1}^n$ forms a system of coordinates on $\Omega_{\mathcal{O}}$ near X, see [48, Lem. 2.14]. Similarly, the functions E_{Z_j}, $j = 1, \dots, n$, form a system of coordinates in $R_{\mathcal{O}}(M_{\Sigma,T})$ near r_X. These coordinates provide for $R_{\mathcal{O}}(M_{\Sigma,T})$ a differential structure which makes it diffeomorphic to manifold $M_{\Sigma,T}$.

Let us denote by $\tilde G$ the metric on $R_{\mathcal{O}}(M_{\Sigma,T})$ which makes it isometric to $(M_{\Sigma,T}, G)$, that is, $\tilde G = ((R_{\mathcal{O}})^{-1})^* G$. Let $r \in R_{\mathcal{O}}(M_{\Sigma,T})$ and $X \in M_{\Sigma,T}$ be such that $r = r_X$. Let Z_0 is a point where r obtains its minimum, that is, the closest point of $\Gamma_{\mathcal{O}}$ to X. When Z is close to Z_0, the differentials of functions E_Z are covectors of length 1 on $(R_{\mathcal{O}}(M_{\Sigma,T}), \tilde G)$, see [48, Lem. 2.15]. This is equivalent to the fact that the gradients of the distance functions $X \mapsto d_{\mathcal{O}}(X, Z)$ have length one. By this observation, it is possible to find infinitely many covectors dE_Z, $Z \in \Gamma_{\mathcal{O}}$ of length 1 at any point r of $R_{\mathcal{O}}(M_{\Sigma,T})$. Using such vectors, one can reconstruct the metric tensor $\tilde G$ at r. By the above considerations, BSP determines the manifold $(M_{\Sigma,T}, G)$ up to an isometry. □

6.4. **Continuation of the data.** Let us now consider the case when $\mathcal{O} = \emptyset$ and we are given the set Γ as a differentiable manifold, the metric G on Γ, and the BSP for $H_\emptyset$. Assume that there are two manifolds $\Omega_{int}^{(1)}$ and $\Omega_{int}^{(2)}$ such that Γ is isometric to subsets $\Gamma^{(j)} \subset \partial\Omega_{int}^{(j)}$ for $j = 1, 2$ and that the BSP for $H_\emptyset^{(j)}$, $j = 1, 2$, coincides with the given data. Let now $\Sigma \subset \Gamma$ satisfy condition (C-2) and

$$0 < T < \min(\tau_\emptyset^{(1)}(\overline{\Sigma}), \tau_\emptyset^{(2)}(\overline{\Sigma})).$$

Then the above constructions show that the manifolds

$$M_{\Sigma,T}^{(j)} = \{X \in \Omega_{int}^{(j)};\ X = \gamma_{(Y,\nu)}(t),\ 0 < t < T,\ Y \in \Sigma\}$$

with $j = 1$ and $j = 2$, are isometric. Thus, we can consider the set $M_{\Sigma,T}^{(1)}$, denoted by U_1 as a subset of both manifolds $\Omega_{int}^{(1)}$ and $\Omega_{int}^{(2)}$, and, by the previous considerations, we can construct a metric $\tilde G$ on it which makes $(U_1, \tilde G)$ isometric to $(M_{\Sigma,T}^{(j)}, G^{(j)})$, $j = 1, 2$.

We continue the construction by continuation of the data using Green's functions, cf. [58, 59]. To this end, let $z \in \mathbb{C} \backslash \mathbb{R}_+$ and consider the Schwartz kernel $G_{\mathcal{O}}(z; Y, Y')$

of the operator $(H_{\mathcal{O}} - z)^{-1}$. It satisfies the equation

$$(6.15) \qquad \begin{aligned} &(H_{\mathcal{O}} - z)G_{\mathcal{O}}(z;\cdot\,,Y') = \delta_{Y'}, \quad Y, Y' \in \Omega_{\mathcal{O}} = \Omega_{int} \setminus \mathcal{O}, \\ &\partial_\nu G_{\mathcal{O}}(z;\cdot\,,Y')|_{\partial\Omega_{\mathcal{O}}} = 0. \end{aligned}$$

We denote $G(z;Y,Y') = G_{\mathcal{O}}(z:Y,Y')$ when $\mathcal{O} = \emptyset$.

Lemma 6.9. *Let $U \subset \Omega_{int}$ be a connected neighborhood of an open set $\Sigma \subset \Gamma$, where Σ satisfies condition (C-2) with $\mathcal{O} = \emptyset$. Let $X_0 \in U \setminus \partial\Omega_{int}$ and $\rho > 0$ be such that $\mathcal{O} = B(X_0, \rho) \subset U \setminus \partial\Omega_{int}$. Assume that we are given the metric tensor G in U. Then BSP on Γ for the operator $H_\emptyset$ determines $G(z;Y,Y')$ for $Y, Y' \in U$ and $z \in \mathbb{C} \setminus \mathcal{E}(H_\emptyset)$. Moreover, these data determine BSP on $\Gamma_{\mathcal{O}}$ for the operator $H_{\mathcal{O}}$.*

Proof. By Lemma 5.6, BSP on Γ determines the N-D map $\Lambda(z)$ at $\Gamma\times\Gamma$. By Lemma 5.4, the Schwartz kernel of the N-D map $\Lambda(z)$ at $\Gamma \times \Gamma$ coincides with $G(z;Y,Y')$. Thus we know the function $G(z;Y,Y')$ for $Y, Y' \in \Sigma$. As the Neumann boundary values of $Y \mapsto G(z;Y,Y')$ on $\Gamma \setminus \{Y'\}$ vanish, using the Unique Continuation Principle for the elliptic equation (6.15) in the Y variable, we see that the values of $G(z;Y,Y')$ are uniquely determined for $Y' \in \Sigma$ and $Y \in U \setminus \{Y'\}$. Using the symmetry $\overline{G(z;Y,Y')} = G(\overline{z};Y',Y)$ and again the Unique Continuation Principle, now in the Y' variable, we can determine the values of $G(z;Y,Y')$ in $\{(Y,Y') \in U \times U;\ Y \neq Y'\}$. Considering $G(z;Y,Y')$ as a locally integrable function, we see that it is defined a.e. in $U \times U$.

For $Y' \in (\Omega_{\mathcal{O}} \cap U) \setminus \partial\Omega_{\mathcal{O}}$, denote by $G^{ext}_{\mathcal{O}}(z;Y,Y')$ a smooth extension of $G_{\mathcal{O}}(z;Y,Y')$ into $\mathcal{O}$. Then

$$(-\Delta_G - z)G^{ext}_{\mathcal{O}}(z;Y,Y') - \delta(Y,Y') = F(Y,Y') \in C^\infty(\Omega_{int}),$$

where $\operatorname{supp} F(\cdot,Y') \subset \overline{\mathcal{O}}$. Therefore,

$$G_{\mathcal{O}}(z;Y,Y') = G(z;Y,Y') + \int_{\mathcal{O}} G(z;Y,Y'')F(Y'',Y')dV_{Y''}.$$

In particular,

$$(6.16) \qquad \partial_{\nu(Y)}G(z;Y,Y') + \int_{\mathcal{O}} \partial_{\nu(Y)}G(z;Y,Y'')F(Y'',Y')dV_{Y''} = 0, \quad Y \in \partial\mathcal{O},$$

where $\nu(Y)$ is the unit normal to $\mathcal{O}$ at Y. On the other hand, if $F(\cdot\,,Y') \in C^\infty(U)$, $\operatorname{supp} F(\cdot\,,Y') \subset \overline{\mathcal{O}}$, satisfies (6.16), the function

$$(6.17) \qquad G(z;Y,Y') + \int_{\mathcal{O}} G(z;Y,Y'')F(Y'',Y')dV_{Y''}, \quad Y, Y' \in U \setminus \overline{\mathcal{O}},$$

is $G_{\mathcal{O}}(z;Y,Y')$. As we have in our disposal $G(z;Y,Y')$ for $Y, Y' \in U$, we can verify for a given F, condition (6.16).

Now, we return to $\Omega^{(1)}_{int}$, $\Omega^{(2)}_{int}$ with Γ and BSP on Γ being the same. We denote the associated functions appearing above by adding the superscript (j). Let (6.16) hold with $G(z;Y,Y')$, $F(Y'',Y')$ replaced by $G^{(1)}(z;Y,Y')$, $F^{(1)}(Y'',Y')$, respectively. Since $G^{(1)}(z;Y,Y') = G^{(2)}(z;Y,Y')$ on $U \times U$, (6.16) also holds with $G(z;Y,Y')$,

$F(Y'', Y')$ replaced by $G^{(2)}(z; Y, Y')$, $F^{(1)}(Y'', Y')$, respectively. Thus, for $Y, Y' \in U \setminus \overline{\mathcal{O}}$, we have

$$G^{(j)}_{\mathcal{O}}(z; Y, Y') = G^{(j)}(z; Y, Y') + \int_{\mathcal{O}} G^{(j)}(z; Y, Y'')F^{(1)}(Y'', Y')dV_{Y''}, \quad j = 1, 2,$$

so that

$$G^{(1)}_{\mathcal{O}}(z; Y, Y') = G^{(2)}_{\mathcal{O}}(z; Y, Y'), \quad z \in \mathbb{C} \setminus \mathbb{R}, \ Y, Y' \in U \setminus \overline{\mathcal{O}}.$$

In particular, this implies that $\Lambda^{(1)}_{\mathcal{O}}(z) = \Lambda^{(2)}_{\mathcal{O}}(z)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Then by Lemma 5.6, BSP's for $H^{(1)}_{\mathcal{O}}$ and $H^{(2)}_{\mathcal{O}}$ coincide. $\square$

Next we show that we can use these data to determine the critical distance which we use in the step-by-step construction of the manifold.

Lemma 6.10. *Let $X_0 \in \Omega_{int} \setminus \partial\Omega_{int}$ and $0 < \rho < \tau(X_0)/2$. Let $\mathcal{O} = B(X_0, \rho)$ and $\Gamma_{\mathcal{O}} = \partial\mathcal{O}$. Assume that we are given the set $\Gamma_{\mathcal{O}}$ as a differentiable manifold, the metric $G\big|_{\Gamma_{\mathcal{O}}}$ on $\Gamma_{\mathcal{O}}$, and the BSP for $H_{\mathcal{O}}$ on $\Gamma_{\mathcal{O}}$. Then these data determine $\tau_{\mathcal{O}}(\Gamma_{\mathcal{O}}) = \tau(X_0) - \rho$.*

Proof. Let us assume that $t_0 < \tau(X_0) - \rho$. Then, for any $Y \in \Gamma_{\mathcal{O}}$, the set $\Omega_{\mathcal{O}}(Y, t_0) \setminus \Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, t_0 - \varepsilon)$ contains an open neighborhood of $\gamma_{(Y,\nu)}(t_0 - \varepsilon/2)$ and, therefore, has positive measure. Hence, if $t < \tau(X_0) - \rho$, then the condition

$$\forall Y \in \Gamma_{\mathcal{O}} \ \forall \varepsilon > 0 : \ I_{\epsilon,t} := \{(\Gamma_{\mathcal{O}}, t - \varepsilon)\} \not\geq I'_{Y,t} := \{(Y, t)\} \tag{6.18}$$

is valid.

Let us next assume that condition (6.18) is valid and consider its consequences.

First, observe that by (6.8) and (6.9), we have either

(a) $s(X_0) = \tau(X_0)$ and there is $Y \in \Gamma_{\mathcal{O}}$ such that $X = \gamma_{(Y,\nu)}(\tau(X_0) - \rho) \in \partial\Omega_{int}$,

or

(b) $s(X_0) > \tau(X_0)$ and there are $Y \in \Gamma_{\mathcal{O}}$ and s such that $s(X_0) > s > \tau(X_0) - \rho$ and $d_{\mathcal{O}}(\gamma_{(Y,\nu)}(s), \Gamma_{\mathcal{O}}) < s$.

Let us consider these two cases separately.

(a) It follows from (6.8) and (6.9) that X is a closest point to X_0 on $\partial\Omega_{int}$. Therefore, the geodesic $\gamma_{(Y,\nu)}$ intersects $\partial\Omega_{int}$ normally at $X = \gamma_{(Y,\nu)}(s)$, $s = \tau(X_0) - \rho$.

Assume next that $t > 0$ is such that

$$\forall \varepsilon > 0 : \ I_{\epsilon,t} \not\geq I'_{Y,t}. \tag{6.19}$$

Then for any $\varepsilon > 0$ there is

$$X_\varepsilon \in \Omega_{\mathcal{O}}(Y, t) \setminus \Omega_{\mathcal{O}}(\Gamma_{\mathcal{O}}, t - \epsilon). \tag{6.20}$$

As $\Omega_{\mathcal{O}}(Y, t)$ is relatively compact, there are $\varepsilon_n \to 0$ and $X_n = X_{\varepsilon_n}$ such that $X_n \to X' \in \Omega_{int}$ as $n \to \infty$. Then

$$d_{\mathcal{O}}(X', Y) = t, \quad d_{\mathcal{O}}(X', \Gamma_{\mathcal{O}}) = t. \tag{6.21}$$

This shows that Y is the closest point of $\Gamma_{\mathcal{O}}$ to X' in $\Omega_{\mathcal{O}}$. Consider a shortest curve $\mu(s)$ from Y to X'. By [2], a shortest curve between two points on a manifold with boundary is a C^1-curve. Moreover, it is a geodesic on $\Omega_{\mathcal{O}} \setminus \partial\Omega_{\mathcal{O}}$. Since $\mu(s)$ is a shortest curve from X' to $\partial\Omega_{\mathcal{O}}$, it is normal to $\partial\Omega_{\mathcal{O}}$ at Y. Thus $\mu(s) = \gamma_{Y,\nu}(s)$, $s \leq \tau(X_0) - \rho$. However, $\gamma_{(Y,\nu)}(s)$ hits $\partial\Omega_{int}$ normally at $s = \tau(X_0) - \rho$. Therefore, by the short-cut arguments, we see that the curve $\gamma_{(Y,\nu)}([0, \tau(X_0) - \rho]) \subset \Omega_{\mathcal{O}}$ can not be extended to a longer curve which is a shortest curve between Y and its other end point. Thus $\mu \subset \gamma_{(Y,\nu)}([0, \tau(X_0) - \rho])$, implying that $t = d_{\mathcal{O}}(Y, X') \leq \tau(X_0) - \rho$. Hence in the case (a) the condition (6.18) implies that $t \leq \tau(X_0) - \rho$.

(b) In this case arguments are similar but slightly simpler. Again, assume that $t > 0$ is such that (6.19) is satisfied. Again, there are $\epsilon_n > 0$ and $X_n = X_{\varepsilon_n}$ satisfying (6.20), such that $X_n \to X'$ and $X' \in \Omega_{int}$ satisfies (6.21). Moreover, a shortest curve $\mu(s)$ from Y to X' coincides with the normal geodesic $\gamma_{(Y,\nu)}(s)$ for small values of s. Since the geodesic $\gamma_{(Y,\nu)}([0, s'])$ is a shortest curve between its end points for $s' \leq \tau(X_0) - \rho$ but not for $s(X_0) - \rho > s' > \tau(X_0) - \rho$, we see that $\mu \subset \gamma_{(Y,\nu)}([0, \tau(X_0) - \rho])$ and thus $t \leq \tau(X_0) - \rho$.

Therefore, in both cases (a) and (b), the condition (6.18) implies that $t \leq \tau(X_0) - \rho$. Combining these facts, we see that

$$\tau(X_0) - \rho = \sup\{t > 0\,;\ \text{condition (6.18) is satisfied for } t\}.$$

The lemma then follows from this and Lemma 6.5. $\square$

6.5. **Proof of Theorem 6.1.** We are now in a position to complete the proof of Theorem 6.1.

6.5.1. *Local reconstruction of Riemannian structure.* We start our considerations with $\mathcal{O} = \emptyset$. Let $\Sigma \subset \Gamma$ satisfies condition (C-2) and $T > 0$ be sufficiently small. In fact, we can consider any $0 < T < \tau_\emptyset(\overline{\Sigma})$. Using Lemma 6.7 we see that the set $R_\emptyset(M_{\Sigma,T}) \subset C(\Gamma)$ is uniquely determined. On this set we introduce the boundary normal coordinates,

$$r(\cdot) \mapsto (Z, t), \quad t = \min_{Z' \in \Sigma} r(Z'),$$

where Z is the unique point on Σ on which $r(\cdot)$ attains its minimum. Observe that these coordinates on $R_\emptyset(M_{\Sigma,T})$ coincide with the boundary normal coordinates of the point $X \in \Omega_{int}$ such that

$$r(\cdot) = r_X(\cdot).$$

Thus, $R_\emptyset(M_{\Sigma,T})$ with the above coordinates is diffeomorphic to $M_{\Sigma,T}$.

Next we use Lemma 6.8 to endow $R_\emptyset(M_{\Sigma,T})$ with Riemannian metric, $\tilde{G}$, so that $(R_\emptyset(M_{\Sigma,T}), \tilde{G})$ is isometric to the manifold $(M_{\Sigma,T}, G)$.

Remark. For the inverse scattering problem considered in the introduction, Section 6.5.1 is not necessary, because we know *a priori* the Riemannian structure of the open set $(\Omega_{int} \setminus \partial\Omega_{int}) \cap \Omega_1$. However, to make the results of §6 appropriate for general non-compact manifolds with asymptotically cylindrical ends, we have included this step.

6.5.2. *Iteration of local reconstruction.* To describe the procedure which we will iterate, let us assume that $U_1 \subset \Omega_{int}$ is a connected neighborhood $\Sigma \subset \Gamma$ which satisfies condition (C-2) with $\mathcal{O} = \emptyset$ and that we know the Riemannian manifold (U_1, G) up to an isometry. Since the set $(R_\emptyset(M_{\Sigma,T}, \tilde{G}))$ is already determined, we can take $U_1 = M_{\Sigma,T}$, where $T > 0$ is sufficiently small .

Choose $X_1 \in U_1$ and $\rho > 0$ such that $\mathcal{O} = B(X_1, \rho) \subset U_1$. By Lemma 6.9 we can determine $G(z; Y, Y')$ for all $Y, Y' \in U_1$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Moreover, it gives us BSP on $\partial\mathcal{O}$. Therefore by Lemma 6.10, these data determine $\tau_{\mathcal{O}}(\Gamma_{\mathcal{O}})$, hence $\tau(X_1) = \tau_{\mathcal{O}}(\Gamma_{\mathcal{O}}) + \rho$. Take any $X \in B(X_1, \tau) \setminus \mathcal{O}$, where $\tau = \tau(X_1)$, and let Y be the intersection of $\partial\mathcal{O}$ and the geodesic with end points X_1 and X. Taking any $Z \in \partial\mathcal{O}$ and applying Lemma 6.6, we can then find $d_{\mathcal{O}}(X, Z)$.

Using, similarly to the above, Lemmas 6.7 and 6.8, we can find the image of the embedding $R_{\mathcal{O}} : B(X_1, \tau) \setminus \mathcal{O} \to C(\partial\mathcal{O})$. We then recover, in the boundary normal coordinates associated with $\partial\mathcal{O}$, i.e. the Riemannian normal coordinates centered at X_1, the metric tensor G on $B(X_1, \tau) \setminus B(X_1, \rho)$, and, since G on $B(X_1, \rho)$ is known, on the whole $B(X_1, \tau)$. This construction makes it possible to introduce the structure of the differentiable manifold on $U_1 \bigsqcup B(X_1, \tau)$ which we considered, by now, as a disjoint union of two Riemannian manifolds. Next we glue these two components together. To this end we observe that, since $\mathcal{O} \subset U_1$, we have in our disposal Green's function $G(z; Y, Y')$ for $Y, Y' \in \mathcal{O}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. The set $\mathcal{O}$ can be considered also as the subset $B(X_1, \rho)$ of $B(X_1, \tau)$, and thus we know the function $G(z; Y, Y')$ for $Y, Y' \in B(X_1, \rho)$ e.g. in the Riemannian normal coordinates centered at X_1. Thus, using the Unique Continuation Principle, we can determine, in the Riemannian normal coordinates, the function $G(z; Y, Y')$ for all $Y \in B(X_1, \tau)$ and $Y' \in B(X_1, \rho)$.

Since $Y' \mapsto G(z; Y, Y')$ is a smooth function in $\Omega_{int} \setminus \{Y\}$ and $G(z; Y, Y') \to \infty$ as $Y' \to Y$, we see that for $Y_1, Y_2 \in \Omega_{int}$, we have $Y_1 = Y_2$ if and only if $G_z(Y_1, Y') = G_z(Y_2, Y')$ for all $Y' \in \Omega_{int}$, $z \in \mathbb{C} \setminus \mathbb{R}$. Using the Unique Continuation Principle, this is equivalent to $G(z; Y_1, Y') = G(z; Y_2, Y')$ for all $Y' \in B(X_1, \rho)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Next, let us define that the points $X_U \in U_1$ and $X_B \in B(X_1, \tau)$ are equivalent and denote $X_U \sim X_B$ if $G(z; X_U, Y') = G(z; X_B, Y')$ for all $Y' \in B(X_1, \rho)$, $z \in \mathbb{C} \setminus \mathbb{R}$. Then the manifold $U_2 = U_1 \cup B(X_1, \tau) \subset \Omega_{int}$ is diffeomorphic to manifold $(U_1 \bigsqcup B(X_1, \tau))/\sim$, which is obtained by glueing together the equivalent points on U_1 and $B(X_1, \tau)$. As we know the metric tensor on both U_1 and $B(X_1, \rho)$, we have reconstructed a Riemannian manifold $(U_2, G) \subset (\Omega_{int}, G)$ up to an isometry.

6.5.3. *Maximal reconstruction.* Let us iterate the above process, that is, we start from an open set $\Sigma \subset \Gamma$ satisfying condition (C-2) with $\mathcal{O} = \emptyset$, construct its neighborhood U_1, and iterate the construction by choosing at each step $j = 1, 2, \dots$

a point $X_j \in U_j$ and constructing a Riemannian manifold isometric to $U_{j+1} = U_j \cup B(X_j, \tau(X_j)) \subset \Omega_{int}$.

Consider the open sets in $\Omega_{int} \setminus \partial\Omega_{int}$ which can be reconstructed, with the metric, when we are given the set Γ with its metric and the BSP on Γ. As the collection of these sets is closed with respect to taking the union, consider maximal open set $U_{max} \subset \Omega_{int} \setminus \partial\Omega_{int}$ which can be reconstructed, with its metric, from the set Γ with its metric and the BSP on Γ. Let us show that $U_{max} = \Omega_{int} \setminus \partial\Omega_{int}$. Since $\Omega_{int} \setminus \partial\Omega_{int}$ is connected, it suffices to show that U_{max} is open and closed in Ω_{int}. By construction, U_{max} is open. Let now $X \notin \partial\Omega_{int}$ be a limit point of U_{max}, i.e., $X = \lim_{n\to\infty} X_n$, $X_n \in U_{max}$. Denote $a = d(X, \partial\Omega_{int})$ so that if $Y \in B(X, a/4)$, then $s(Y) \geq 3a/4$, see (6.8). Since the cut locus distance of the Riemannian normal coordinates is continuous with respect to the center, see e.g. [50, Sec. 2.1] or [29], there is $\delta > 0$ such that $\tau(Y) \geq \delta$ for all $Y \in B(X, a/4)$.

Let now $X_n \in U_{max}$ satisfy the inequality $d(X_n, X) < \sigma = \min(a/4, \delta/4)$. Let us assume that X_n has a neighborhood $B(X_n, \rho_n)$, with a sufficiently small $\rho_n < d(X_n, X)$, which can be reconstructed using $N(n)$ iteration steps, that is, $B(X_n, \rho_n) \subset U_{N(n)}$. Then $\tau(X_n) > 4\sigma$ so that $X \in B(X_n, \tau(X_n))$. By Lemma 6.9, we can find the BSP for the operator $H_{\mathcal{O}}$ with $\mathcal{O} = B(X_n, \tau(X_n))$ and, using one more iteration step, reconstruct the Riemannian structure on $U_{N(n)} \cup B(X_n, \tau(X_n))$ which includes the point X. Therefore, the point X is in U_{max}. This shows that U_{max} is relatively open and closed in $\Omega_{int} \setminus \partial\Omega_{int}$. Thus, $U_{max} = \Omega_{int} \setminus \partial\Omega_{in}$.

The above shows that using an enumerable number of iteration steps we can construct a Riemannian manifold isometric to $(\Omega_{int} \setminus \partial\Omega_{int}, G)$. Thus we have reconstructed the Riemannian manifold $(\Omega_{int} \setminus \partial\Omega_{int}, G)$ up to an isometry.

It remains to identify the differentiable and Riemannian structures near $\partial\Omega_{int}$. Observe that Ω_{int} is just the closure of $\Omega_{int} \setminus \partial\Omega_{int}$ with respect to the distance function generated by the metric G on $\Omega_{int} \setminus \partial\Omega_{int}$. Moreover, for any open relatively compact set $\Sigma \subset \partial\Omega_{int}$, there exists $\delta > 0$ such that $\tau_\emptyset(\overline{\Sigma}) \geq \delta > 0$.

Let $0 < t < \delta$ and consider the set

$$\Sigma_t = \{X \in \Omega_{int} \setminus \partial\Omega_{int}\,;\, d(X, \partial\Omega_{int}) = t,\ d(X, Z) = t,\ \text{for some } Z \in \Sigma\}.$$

This implies that for $X \in \Sigma_t$ the closest point $Z \in \Omega_{int}$ is in Σ and $X = \gamma_{Z,\nu}(t)$. Therefore, Σ_t is a smooth $(n-1)$-dimensional open submanifold in Ω_{int} of points having the form $X = \gamma_{(Z,\nu)}(t)$, $Z \in \Sigma$. This makes it possible to introduce the boundary normal coordinates in $M_{\Sigma,\delta}$ which provides the differentiable structure near Σ. Writing the metric tensor G in these coordinates and extending this tensor continuously on Σ, we find the metric tensor in Ω_{int} in the boundary normal coordinates associated to Σ. □

6.6. **Proof of Theorem 1.1.** Having Theorem 6.1 in our disposal, Theorem 1.1 follows immediately from Lemma 5.7. □

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