Weakly hyperbolic systems with Hölder continuous coefficients

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ABSTRACT – We study the Cauchy Problem for hyperbolic systems with multiple characteristics and nonsmooth coefficients depending on time. We prove in particular that, if the leading coefficients are α-Hölder continuous, and the system has size $m \leq 3$, then the Cauchy Problem is well posed in each Gevrey class of exponent $s < 1+\alpha/m$.

§1. Introduction

We consider the Cauchy problem, on $[0,T] \times \mathbf{R}_x$, for the system

$$\begin{cases} \partial_t U \;=\; A(t)\partial_x U \;+\; B(t)U \\ U(0,x) \;=\; U_0(x), \end{cases} \tag{1}$$

where $U \in \mathbf{C}^m$, $A(t)$ is a $m \times m$ matrix with *real eigenvalues* $\{\lambda_1(t), \cdots, \lambda_m(t)\}$. We say that (1) is well posed in a class $\mathcal{X}$ of functions on $\mathbf{R}_x$, when, for all $U_0 \in [\mathcal{X}]^m$, it admits a unique solution $U \in C^1([0,T],[\mathcal{X}]^m)$.

If the entries of $A(t)$ are sufficiently smooth functions of t (e.g., of class C^2), we know by Bronshtein and Kajitani ([1], [9], see also [5]) that (1) is well posed in the Gevrey class $\gamma^s = \gamma^s(\mathbf{R}_x)$ provided

$$1 < s \;<\; 1 + \frac{1}{m-1} \;.$$

When the leading coefficients are only Hölder continuous, i.e., $A(t) \in C^{0,\alpha}$ for some $\alpha \leq 1$, we expect a similar conclusion with $1 < s < \bar{s}$, for some smaller bound $\bar{s} = \bar{s}(m,\alpha)$. The first result in this direction, due to Colombini, Jannelli and Spagnolo [4], was concerned with the scalar equation

$$\partial_t^2 u \;=\; a(t)\partial_x^2 u \;+\; b(t)\partial_x u\,, \qquad a(t) \;\geq\; 0\,, \quad a(t) \in C^{0,\alpha},$$

for which the γ^s well-posedness for $s < 1 + \alpha/2$ was proved. This upper bound is sharp.

Subsequently, such a result was extended by Nishitani [11] to the second order equations with coefficients also depending on x, and, finally, by Ohya and Tarama [12] to any *scalar equation of order* m. In the last case, the range of s for γ^s well-posedness is:

$$1 < s < 1 + \frac{\alpha}{m} .$$

The purpose of this paper is investigate the *vector case*, and prove that the same range of well-posedness holds for any $m \times m$ system (1), at least for $m \leq 3$:

Theorem 1. *Let $m = 2, 3$. Assume that $A(t)$ is hyperbolic, i.e., has real eigenvalues $\lambda_j(t)$, and $A(t) \in C^{0,\alpha}([0,T])$, $B(t) \in C^0([0,T])$. Therefore, (1) is well posed in γ^s for all $s < 1 + \alpha/m$, more precisely for*

$$1 < s < 1 + \frac{\alpha}{r} \qquad (r = 2, 3)$$

where r is the maximum multiplicity of the $\lambda_j(t)$.

If $r = 1$, i.e., in the strictly hyperbolic case, we have γ^s well-posedness for

$$1 < s < \frac{1}{1-\alpha} .$$

It should be mentioned that the case $r = 1$ was already proved by Jannelli [6] in full generality, i.e., for a differential system with arbitrary size and x-depending coefficients, and then extended by Cicognani [2] to pseudodifferential systems. We also recall that Kajitani [10] (cf. Yuzawa [13]) proved the γ^s well-posedness for any size m, but with a smaller range of s than in Theorem 1:

$$1 < s < 1 + \min\{\alpha/(r+1), (2-\alpha)/(2r-1)\} .$$

In this paper we also prove a result of well-posedness for a special class of systems with arbitrary size m: the systems (1) where the *square* of the matrix $A(t)$ is Hermitian. Note that, if $A(t)$ is Hermitian, then (1) is a *symmetric system*, hence the Cauchy Problem is well posed in C^∞ no matter how regular the coefficients are. However, A^2 may be Hermitian even if A is not; for instance, A^2 is Hermitian for any 2×2 hyperbolic matrix A *with trace zero.*

Theorem 2. *If $A(t)$ is hyperbolic, $A(t) \in C^{0,\alpha}([0,T])$, $B(t) \in C^0([0,T])$, and*

(2) $$A(t)^2 \text{ is Hermitian,}$$

then (1) *is well posed in γ^s for*

$$1 < s < 1 + \frac{\alpha}{2} .$$

If, in addition, $\lambda_1(t)^2 + \cdots + \lambda_m(t)^2 \neq 0$ for all $t \in [0,T]$, then (1) *is well posed for*

$$1 < s < \frac{1}{1-\alpha} .$$

Remark 1 : By (2), the condition $\lambda_1(t)^2 + \cdots + \lambda_m(t)^2 \neq 0$ is equivalent to $A(t)^2 \neq 0$.

Remark 2 : The case $m = 2$ of Theorem 1 can be easily derived from Theorem 2: indeed, it is not restrictive to assume that the 2×2 matrix $A(t)$ has trace zero (see §2), which implies that $A(t)^2$ is Hermitian. The case $m = 2$ of Theorem 1 is also a special case of the case $m = 3$; indeed, any 2×2 system can be viewed as a 3×3 system with maximum multiplicity $r \leq 2$. However, we prefer to give here a direct proof of Theorem 1 even for $m = 2$.

Remark 3 : The conclusions of Theorems 1 and 2 can easily be extended to spatial dimension $n \geq 2$. Here, for the sake of simplicity, we shall consider only the one dimensional case.

Our proof of Theorem 1 is rather elementary, relying on an appropriate choice of the energy function. To define such an energy, we suitably approximate the characteristic invariants of $A(t)$ and apply the Hamilton-Cayley equation. Due to its simplicity, the case $m = 2$ will be treated in a direct way (see §3), while the case $m = 3$ (see §5) can be better understood in the framework of *quasi-symmetrizers* introduced in [5] (see also [7, 8]).

§2. Preliminaries

In order to prove Theorem 1, we can assume that the matrix $A(t)$ satisfies

(3) $$\operatorname{tr}(A(t)) = 0, \qquad \forall\, t \in [0,T].$$

Indeed, if we put $U(t,x) = \widetilde{U}\big(t,\, x + \int_0^t \operatorname{tr}(A(\tau))d\tau/m\big)$, we can reduce (1) to

$$\begin{cases} \partial_t \widetilde{U} = & \widetilde{A}(t)\partial_x \widetilde{U} + B(t)\widetilde{U} \\ \widetilde{U}(0,x) = & U_0(x), \end{cases}$$

where the matrix $\widetilde{A}(t) \equiv A(t) - \{\operatorname{tr}(A(t))/m\}I$ is traceless. Note that, if $\widetilde{U}$ belongs to $C^1\big([0,T],[\gamma^s]^m\big)$, then also $U \in C^1\big([0,T],[\gamma^s]^m\big)$.

By a standard argument based on Holmgren uniqueness theorem and on Paley-Wiener theorem (see for instance [4], or [3]), the γ^s well-posedness of (1) follows from the *a priori* estimate in $\widehat{\gamma^s}$ of $\widehat{U}(t,\xi)$, the Fourier transform w.r. t. x of a smooth solution $U(t,x)$ with compact support in $\mathbf{R}_x$ for each t.

Now, by Fourier transform (1) yields

(4) $$\begin{cases} V' = & i\xi A(t)V + B(t)V \\ V(0,\xi) = V_0(\xi) \end{cases}$$

where $V = \hat{U}(t,\xi)$, and a compactly supported function $f(x)$ belongs to $\gamma^s(\mathbf{R})$ if and only if, for some $C, \delta > 0$, one has

$$|\widehat{f}(\xi)| \le \ Ce^{-\delta|\xi|^{1/s}} \qquad \text{for } \ |\xi| \ge 1\,.$$

Thus, to conclude that $U(t,x) \in C^1([0,T],[\gamma^s]^m)$ for all $s < \sigma$, it will be sufficient to prove that there are some ν and C for which

(5) $$|V(t,\xi)| \ \le \ |\xi|^\nu\, |V_0(\xi)|\, e^{C|\xi|^{1/\sigma}} \qquad \text{for } \ |\xi| \ge 1\,.$$

Given a non-negative function $\varphi \in C_0^\infty(\mathbf{R})$ with $\int_{-\infty}^{\infty} \varphi(\tau)d\tau = 1$, and $0 < \varepsilon \le 1$, we extend $A(t)$ as a Hölder function on $\mathbf{R}$, constant outside of $]0,T[$, and define the mollified matrix

(6) $$A_\varepsilon(t) \ = \int_{-\infty}^{\infty} A(t - \varepsilon\tau)\varphi(\tau)d\tau.$$

Since $A(t) \in C^{0,\alpha}$, we can find a constant M for which

(7) $$\| A_\varepsilon(t) \| \le \ M, \qquad \| A_\varepsilon'(t) \| \le \ M\varepsilon^{\alpha-1}, \qquad \| A_\varepsilon(t) - A(t) \| \le \ M\varepsilon^\alpha,$$

for all $t \in [0,T]$, where $\| \cdot \|$ denotes the matrix norm.

§3. Proof of Theorem 1 in the case $m = 2$

For the sake of brevity, we shall limit ourselves to assuming $B(t) \equiv 0$, the general case requires only minor changes. We put

$$h_A(t) = -\det(A(t)), \qquad h_{A_\varepsilon}(t) = -\det(A_\varepsilon(t)), \qquad h_\varepsilon(t) = \Re h_{A_\varepsilon}(t).$$

Note that $h_A(t) \geq 0$, by (3), whereas $h_{A_\varepsilon}(t)$ is only complex valued. The characteristic equation and the Hamilton-Cayley equality have, respectively, the forms:

$$\lambda^2 - h_A(t) = 0, \qquad A(t)^2 - h_A(t)I = 0.$$

Since $\mathrm{tr}\,(A_\varepsilon(t)) = \mathrm{tr}\,(A(t)) = 0$, we also get

$$A_\varepsilon(t)^2 - h_{A_\varepsilon}(t)I = 0. \tag{8}$$

From (7) we obtain, for possibly a larger constant M,

$$|h'_{A_\varepsilon}(t)| \leq M\varepsilon^{\alpha-1}, \qquad |h_{A_\varepsilon}(t) - h_A(t)| \leq M\varepsilon^\alpha,$$

hence

$$|\,h'_\varepsilon(t)| \leq M\varepsilon^{\alpha-1}, \qquad |\,h_\varepsilon(t) - h_A(t)| \leq M\varepsilon^\alpha, \qquad |\Im h_{A_\varepsilon}(t)| \leq M\varepsilon^\alpha. \tag{9}$$

Now, having fixed a constant M which fulfills (7) and (9), we define, for any solution $V(t,\xi)$ of (4) and for any ε, the energy

$$E(t,\xi) = |A_\varepsilon(t)V|^2 + \big\{h_\varepsilon(t) + 2M\varepsilon^\alpha\big\}|V|^2. \tag{10}$$

From (9) we have, observing that $h_A(t) \geq c > 0$ in the strictly hyperbolic case,

$$h_\varepsilon(t) + 2M\varepsilon^\alpha \;\geq\; h_A(t) + M\varepsilon^\alpha \;\geq \begin{cases} c & \text{if } \; r = 1, \\ M\varepsilon^\alpha & \text{if } \; r = 2, \end{cases}$$

hence

$$C(M)\,|V|^2 \;\geq\; E(t,\xi) \;\geq \begin{cases} |A_\varepsilon(t)V|^2 + c\,|V|^2 & \text{if } \; r = 1, \\ |A_\varepsilon(t)V|^2 + M\varepsilon^\alpha|V|^2 & \text{if } \; r = 2. \end{cases} \tag{11}$$

Differentiating the energy w.r.t. time, and using (4), we find the equality

$$
\begin{aligned}
E'(t,\xi) = \;& 2\Re\big(A_\varepsilon V', A_\varepsilon V\big) + 2\Re\big(A'_\varepsilon V, A_\varepsilon V\big) + h'_\varepsilon |V|^2 + 2\big\{h_\varepsilon + 2M\varepsilon^\alpha\big\}\Re\big(V', V\big) \\
= \;& -2\xi\,\Im\big(A_\varepsilon^2 V, A_\varepsilon V\big) - 2\xi\,\Im\big(A_\varepsilon\{A - A_\varepsilon\}V, A_\varepsilon V\big) + 2\Re\big(A'_\varepsilon V, A_\varepsilon V\big) + h'_\varepsilon|V|^2 \\
& - 2\big\{h_\varepsilon + 2M\varepsilon^\alpha\big\}\xi\,\Im\big(A_\varepsilon V, V\big) - 2\big\{h_\varepsilon + 2M\varepsilon^\alpha\big\}\xi\,\Im\big(\{A - A_\varepsilon\}V, V\big) \\
\equiv \;& I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
$$

Recalling that $\Re h_{A_\varepsilon} = h_\varepsilon$ we see, by (8), that

$$
\Im\big(A_\varepsilon^2 V, A_\varepsilon V\big) \;=\; h_\varepsilon \Im\big(V, A_\varepsilon V\big) + \Im h_{A_\varepsilon}\,\Re\big(V, A_\varepsilon V\big),
$$

hence, by (7) and (10), we find

$$
\begin{aligned}
I_1 + I_5 \;&=\; -2\xi\,\Im h_{A_\varepsilon}\,\Re\big(V, A_\varepsilon V\big) - 4M\varepsilon^\alpha\xi\,\Im\big(A_\varepsilon V, V\big) \le\; 6M\varepsilon^\alpha|\xi||V||A_\varepsilon V| \\
I_2 \;&\le\; 2\,|\xi| \parallel A_\varepsilon \parallel \parallel A - A_\varepsilon \parallel |V||A_\varepsilon V| \le\; 2M^2\varepsilon^\alpha|\xi||V||A_\varepsilon V| \\
I_3 \;&\le\; 2 \parallel A'_\varepsilon \parallel |V||A_\varepsilon V| \le\; 2M\varepsilon^{\alpha-1}|V||A_\varepsilon V| \\
I_4 \;&\le\; |h'_\varepsilon||V|^2 \le\; M\varepsilon^{\alpha-1}|V|^2 \\
I_6 \;&\le\; 2\,|\xi| \parallel A - A_\varepsilon \parallel \big\{h_\varepsilon + 2M\varepsilon^\alpha\big\}|V|^2 \le\; 2M\varepsilon^\alpha|\xi|E(t,\xi).
\end{aligned}
$$

Thus, choosing

$$
\varepsilon \;=\; \begin{cases} |\xi|^{-1} & \text{if } \; r = 1, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } \; r = 2, \end{cases}
$$

and recalling (11), we find a constant $C = C(M)$ such that, for all $|\xi| \ge 1$,

$$
E'(t,\xi) \;\le\; \begin{cases} CE(t,\xi)\big\{\varepsilon^\alpha|\xi| \,+\, \varepsilon^{\alpha-1}\big\} \;\le\; 2\,CE(t,\xi)|\xi|^{1-\alpha} & \text{if } \; r = 1, \\ CE(t,\xi)\big\{\varepsilon^{\alpha/2}|\xi| + \varepsilon^{-1}\,\big\} \;\le\; 2\,CE(t,\xi)|\xi|^{1/(1+\alpha/2)} & \text{if } \; r = 2. \end{cases}
$$

Gronwall's inequality and (11) yield the estimate (5) with $\sigma = 1/(1-\alpha)$ or $\sigma = 1 + \alpha/2$ respectively. This concludes the proof of Theorem 1 for $m = 2$. □

§4. Proof of Theorem 2

Theorem 2 can be proved in a similar way to the proof of Theorem 1 for $m = 2$, but we do not need to suppose (3). We still assume $B \equiv 0$.

Let us first observe that $\parallel A_\varepsilon^2 - A^2 \parallel \le (\parallel A_\varepsilon \parallel + \parallel A \parallel) \parallel A_\varepsilon - A \parallel$, thus recalling that $A^2 = (A^2)^*$, we can choose a constant M large enough to satisfy, besides (7),

(12) $$\parallel A_\varepsilon(t)^2 - A(t)^2 \parallel \;\le\; M\varepsilon^\alpha, \qquad \parallel A_\varepsilon(t)^2 - (A_\varepsilon(t)^2)^* \parallel \;\le\; M\varepsilon^\alpha.$$

Then we define, instead of (10), the following energy:

$$E(t,\xi) \;=\; |A_\varepsilon(t)V|^2 + \Re\big(\{A_\varepsilon(t)^2 + 2M\varepsilon^\alpha\}V,V\big).$$

By the first inequality in (12) we derive:

$$\Re\big(\{A_\varepsilon(t)^2 + 2M\varepsilon^\alpha\}V,V\big) \geq \;\big(A(t)^2V,V\big) + M\varepsilon^\alpha|V|^2.$$

But the Hermitian matrix A^2 has eigenvalues $\lambda_j^2 \;\geq\; 0$, hence we see that $\big(A^2V,V\big) \geq 0$, while $\big(A^2V,V\big)|V|^{-2} \geq c > 0$ when $\lambda_1(t)^2 + \cdots + \lambda_m(t)^2 \neq 0$. Thus, we obtain the estimates

$$C(M)|V|^2 \;\geq\; E(t,\xi) \;\geq\; \begin{cases} |A_\varepsilon(t)V|^2 + c\,|V|^2 & \text{if } \;\lambda_1^2 + \cdots + \lambda_m^2 \neq 0, \\ |A_\varepsilon(t)V|^2 + M\varepsilon^\alpha|V|^2 & \text{if } \;\lambda_1^2 + \cdots + \lambda_m^2 \geq 0. \end{cases} \tag{13}$$

We differentiate the energy and use (2) and (4) to get the equality

$$\begin{aligned} E'(t,\xi) \;&=\; 2\Re\big(A_\varepsilon V', A_\varepsilon V\big) + 2\Re\big(A_\varepsilon' V, A_\varepsilon V\big) + \Re\big(\{A_\varepsilon^2\}'V,V\big) + \Re\big(\{A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha\}V',V\big) \\ &=\; -2\xi\,\Im\big(A_\varepsilon^2V, A_\varepsilon V\big) - 2\xi\,\Im\big(A_\varepsilon\{A - A_\varepsilon\}V, A_\varepsilon V\big) + 2\Re\big(A_\varepsilon'V, A_\varepsilon V\big) + \Re\big(\{A_\varepsilon^2\}'V,V\big) \\ &\quad - \xi\,\Im\big(\{A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha\}A_\varepsilon V,\; V\big) - \xi\,\Im\big(\{A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha\}(A - A_\varepsilon)V,\; V\big) \\ &\equiv\; I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned}$$

Using (7) and the second inequality in (12), we find a constant $C = C(M)$ for which

$$\begin{aligned} I_1 + I_5 \;&=\; -\xi\Im\Big[2\left(A_\varepsilon^2V, A_\varepsilon V\right) + \left(\{A_\varepsilon^2 + A_\varepsilon^{2^*}\}A_\varepsilon V, V\right)\Big] - 4M\varepsilon^\alpha\xi\,\Im\big(A_\varepsilon V, V\big) \\ &=\; -\xi\Im\Big[\left(\{A_\varepsilon^2 - A_\varepsilon^{2^*}\}V, A_\varepsilon V\right)\Big] - 4M\varepsilon^\alpha\xi\,\Im\big(A_\varepsilon V, V\big) \leq\; C\varepsilon^\alpha|\xi||V||A_\varepsilon V|, \\ I_2 \;&\leq\; C\varepsilon^\alpha|\xi||V||A_\varepsilon V|, \qquad I_3 \leq C\varepsilon^{\alpha-1}|V||A_\varepsilon V|, \qquad I_4 \leq C\varepsilon^{\alpha-1}|V|^2, \\ I_6 \;&\leq\; |\xi|\,\big\|A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha\big\|^{1/2}\big\|A - A_\varepsilon\big\|\,|V|\sqrt{2E(t)} \;\leq\; C\varepsilon^\alpha|\xi|\,|V|\,\sqrt{E(t)}. \end{aligned}$$

Note that, to estimate I_6, we have applied the Schwarz's inequality for the scalar product (TV,V) where $T \equiv T^* = A_\varepsilon^2 + A_\varepsilon^{2^*} + 4M\varepsilon^\alpha \;\geq 0$, to get

$$|(TSV,V)| \;\leq\; (TSV,SV)^{1/2}(TV,V)^{1/2} \;\leq\; \|T\|^{1/2}\|S\||V|(TV,V)^{1/2},$$

where $S = A - A_\varepsilon$. Also note that $E(t) = |A_\varepsilon V|^2 + (TV,V)/2$.

In conclusion, recalling (13) and choosing

$$\varepsilon = \begin{cases} |\xi|^{-1} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \neq 0, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \geq 0, \end{cases}$$

we have the following estimate for $|\xi| \geq 1$:

$$E'(t,\xi) \leq \begin{cases} CE(t,\xi)\left[\varepsilon^{\alpha}|\xi| + \varepsilon^{\alpha-1}\right] \leq 2\,CE(t,\xi)|\xi|^{1-\alpha} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \neq 0, \\ CE(t,\xi)\left[\varepsilon^{\alpha/2}|\xi| + \varepsilon^{-1}\right] \leq 2\,CE(t,\xi)|\xi|^{1/(1+\alpha/2)} & \text{if } \lambda_1^2 + \cdots + \lambda_m^2 \geq 0. \end{cases}$$

This yields (5) with $\sigma = 1/(1-\alpha)$, or $\sigma = 1 + \alpha/2$, respectively. Hence, the conclusion of Theorem 2 follows. □

§5. Proof of Theorem 1 in the case $m = 3$

We now define:

$$h_A(t) = \det(A(t)) = \lambda_1(t)\lambda_2(t)\lambda_3(t)$$

$$k_A(t) = \sum_{1\leq i,j\leq 3} \left\{a_{ij}(t)a_{ji}(t) - a_{ii}(t)a_{jj}(t)\right\} = \frac{1}{2}\sum_{j=1}^{3} \lambda_j(t)^2,$$

thus, by (3), the characteristic equation and the Hamilton-Cayley equality are

$$\lambda^3 - k_A(t)\lambda - h_A(t) = 0, \qquad A(t)^3 - k_A(t)A(t) - h_A(t)I = 0.$$

By the assumption of hyperbolicity, we see that $k_A(t)$ is a non-negative function, and, in particular, $k_A(t) \geq c > 0$ when $r \leq 2$. Moreover we have

$$\triangle_A(t) \equiv \prod_{1\leq i<j\leq 3} (\lambda_i(t) - \lambda_j(t))^2 = 4k_A(t)^3 - 27h_A(t)^2 \geq 0$$

Similarly as case $m = 2$, since $\mathrm{tr}\,(A_\varepsilon(t)) = \mathrm{tr}\,(A(t)) = 0$, the regularized matrix (6) satisfies the equality

$$A_\varepsilon(t)^3 - k_{A_\varepsilon}(t)A_\varepsilon(t) - h_{A_\varepsilon}(t)I = 0. \tag{14}$$

However, the eigenvalues of $A_\varepsilon(t)$ may be non real, thus $k_{A_\varepsilon}(t)$ and $h_{A_\varepsilon}(t)$ are complex valued. To overcome this difficulty, we introduce the real functions

$$h_\varepsilon(t) = \Re h_{A_\varepsilon}(t), \qquad k_\varepsilon(t) = \left\{\left\{\Re k_{A_\varepsilon}(t) + M\varepsilon^{\alpha}\right\}^{3/2} + 12\,M^{3/2}\varepsilon^{\alpha}\right\}^{2/3}. \tag{15}$$

Here M is a constant ≥ 1, which is chosen large enough to satisfy, besides (7), the following inequalities on $[0,T]$:

$$
\begin{cases} |\, h_\varepsilon(t) - h_A(t)| \,\leq\, M\varepsilon^\alpha, \quad |\Im h_{A_\varepsilon}(t)| \leq M\varepsilon^\alpha, \quad |\, h'_\varepsilon(t)| \leq M\varepsilon^{\alpha-1}, \\ |\, k_{A_\varepsilon}(t)| \,\leq\, M, \quad |\, k_{A_\varepsilon}(t) - k_A(t)| \,\leq\, M\varepsilon^\alpha, \quad |\, k'_{A_\varepsilon}(t)| \,\leq\, M\varepsilon^{\alpha-1}, \end{cases} \tag{16}
$$

which imply, in particular,

$$
|\,\Re k'_{A_\varepsilon}(t)| \leq\, M\varepsilon^{\alpha-1}, \qquad |\,\Re k_{A_\varepsilon}(t) - k_A(t)| \leq\, M\varepsilon^\alpha, \qquad |\Im k_{A_\varepsilon}(t)| \leq\, M\varepsilon^\alpha. \tag{17}
$$

We also define

$$
\triangle_\varepsilon(t) \;=\; 4k_\varepsilon(t)^3 - 27\, h_\varepsilon(t)^2. \tag{18}
$$

Next we show that $\triangle_\varepsilon(t) \,\geq\, 0$, thus $z^3 \,-\, k_\varepsilon(t)z \,+\, h_\varepsilon(t)$ is a *hyperbolic polynomial*, and we also prove some crucial estimates on $k_\varepsilon(t)$:

Lemma 1. *We have for $C = C(M)$ and $c > 0$*

$$
k_\varepsilon(t) \;\geq\; \begin{cases} c & \text{if} \;\; r = 1, 2, \\ M\varepsilon^{2\alpha/3} & \text{if} \;\; r = 3, \end{cases} \tag{19}
$$

$$
|k'_\varepsilon(t)| \;\leq\; C\varepsilon^{\alpha-1}, \qquad |k_\varepsilon(t) - k_{A_\varepsilon}(t)| \;\leq\; C\varepsilon^\alpha k_\varepsilon(t)^{-1/2}, \tag{20}
$$

$$
\triangle_\varepsilon(t) \;\geq\; \begin{cases} c & \text{if} \;\; r = 1, \\ M^{3/2}\, \varepsilon^\alpha k_\varepsilon(t)^{3/2} & \text{if} \;\; r = 2, 3, \end{cases} \tag{21}
$$

$$
|h_\varepsilon(t)| \;\leq\; \sqrt{\frac{4}{27}}\; k_\varepsilon(t)^{3/2}. \tag{22}
$$

Proof : We write for brevity (15) in the form

$$
k_\varepsilon(t) \;=\; \big\{\widetilde{k}_\varepsilon(t)^{3/2} + 12M^{3/2}\varepsilon^\alpha\big\}^{2/3}, \quad \text{where} \quad \widetilde{k}_\varepsilon(t) \;=\; \Re k_{A_\varepsilon}(t) + M\varepsilon^\alpha,
$$

and observe that, by (17), we have

$$
\widetilde{k}_\varepsilon(t) \;=\; \big\{\Re k_{A_\varepsilon}(t) - k_A(t)\big\} + k_A(t) \,+ M\varepsilon^\alpha \;\geq\; k_A(t) \;\geq\; \begin{cases} c & \text{if} \;\; r = 1, 2, \\ 0 & \text{if} \;\; r = 3. \end{cases}
$$

This yields (19). Let us now prove (20). From (15) and (17) it follows that

$$
|\, k'_\varepsilon| \;\; = |\, \widetilde{k}'_\varepsilon \,|\; \widetilde{k}_\varepsilon^{1/2}\; \{\widetilde{k}_\varepsilon^{3/2} + 12M^{3/2}\varepsilon^\alpha\}^{-1/3} \leq\; |\,\widetilde{k}'_\varepsilon| \;=\; |\,\Re k'_{A_\varepsilon}|\; \leq\; M\varepsilon^{\alpha-1}.
$$

Moreover we get, since $k_\varepsilon(t) \geq \widetilde{k}_\varepsilon(t)$,

$$| \, k_\varepsilon - \widetilde{k}_\varepsilon| = \frac{\big\{k_\varepsilon^{3/2} - \widetilde{k}_\varepsilon^{3/2}\big\}\big\{k_\varepsilon^{3/2} + \widetilde{k}_\varepsilon^{3/2}\big\}}{k_\varepsilon^2 + k_\varepsilon\widetilde{k}_\varepsilon + \widetilde{k}_\varepsilon^2} \leq \frac{12M^{3/2}\varepsilon^\alpha \cdot 2k_\varepsilon^{3/2}}{k_\varepsilon^2} = 24\, M^{3/2}\varepsilon^\alpha k_\varepsilon^{-1/2} \ ,$$

and hence, using again (17),

$$| \, k_\varepsilon - k_{A_\varepsilon}| \ \leq \ \big| k_\varepsilon(t) - \widetilde{k}_\varepsilon(t)\big| + \big| \widetilde{k}_\varepsilon(t) - \Re k_{A_\varepsilon}(t)\big| + | \, \Im k_{A_\varepsilon}(t)| \ \leq \ C\varepsilon^\alpha k_\varepsilon^{-1/2}.$$

This completes the proof of (20).

To prove (21), we first derive the following estimate by (16) and (17), recalling that $\widetilde{k}_\varepsilon(t) \geq k_A(t)$,

$$\begin{aligned} (23) \qquad \big| \, \widetilde{k}_\varepsilon^{3/2} - k_A^{3/2} \big| = \ & \big| \, \widetilde{k}_\varepsilon - k_A \big| \cdot \frac{\widetilde{k}_\varepsilon + \widetilde{k}_\varepsilon^{1/2} k_A^{1/2} + k_A}{\widetilde{k}_\varepsilon^{1/2} + k_A^{1/2}} \ \leq \Big\{ |\Re k_{A_\varepsilon} - k_A| + M\varepsilon^\alpha \Big\} \cdot \frac{3\,\widetilde{k}_\varepsilon}{\widetilde{k}_\varepsilon^{1/2}} \\ & \leq \ 2M\varepsilon^\alpha \cdot 3\,\widetilde{k}_\varepsilon^{1/2} \ \leq \ 2M\varepsilon^\alpha \cdot 3\,(|\Re k_{A_\varepsilon}| + M\varepsilon^\alpha)^{1/2} \ \leq \ 6\sqrt{2}\, M^{3/2}\varepsilon^\alpha, \end{aligned}$$

Then, we write

$$(24) \qquad \triangle_\varepsilon \ = \ 4\,\big\{2k_\varepsilon^{3/2} + \sqrt{27}\, h_\varepsilon\big\}\big\{2k_\varepsilon^{3/2} - \sqrt{27}\, h_\varepsilon\big\}.$$

We know that

$$\big\{2k_A^{3/2} + \sqrt{27}\, h_A\big\}\big\{2k_A^{3/2} - \sqrt{27}\, h_A\big\} \ = \ \triangle_A(t) \ \geq \ 0, \qquad \text{and} \qquad k_A(t) \geq 0\,,$$

thus

$$(25) \qquad \big\{2k_A(t)^{3/2} \pm \sqrt{27}\, h_A(t)\big\} \ \geq \ 0\,.$$

For each fixed $t \in [0, T]$, we have either $h_\varepsilon(t) \geq 0$ or $h_\varepsilon(t) \leq 0$. In the first case, we have $\big\{2k_\varepsilon(t)^{3/2} + \sqrt{27}\, h_\varepsilon(t)\big\} \geq k_\varepsilon(t)^{3/2}$, while, by (16), (23) and (25), we obtain

$$\begin{aligned} \big\{2k_\varepsilon(t)^{3/2} - \sqrt{2}7\, h_\varepsilon(t)\big\} \ = \ & 24\, M^{3/2}\varepsilon^\alpha + \big\{2\widetilde{k}_\varepsilon^{3/2} - \sqrt{27}\, h_\varepsilon\big\} \\ = \ & 24\, M^{3/2}\varepsilon^\alpha + 2\big\{\widetilde{k}_\varepsilon^{3/2} - k_A^{3/2}\big\} + \big\{2k_A^{3/2} - \sqrt{27}\, h_A\big\} + \sqrt{27}\,(h_A - h_\varepsilon) \\ \geq \ & 24\, M^{3/2}\varepsilon^\alpha - 2\,\big|\,\widetilde{k}_A^{3/2} - k_\varepsilon^{3/2}\,\big| + \big\{2k_A^{3/2} - \sqrt{27}\, h_A\big\} - \sqrt{27}\,|\, h_A - h_\varepsilon| \\ \geq \ & \big[24 - 12\sqrt{2} - \sqrt{27}\,\big] M^{3/2}\varepsilon^\alpha + \big\{2k_A^{3/2} - \sqrt{27}h_A\big\} \\ \geq \ & M^{3/2}\varepsilon^\alpha. \end{aligned}$$

In the same way, when $h_\varepsilon(t) \leq 0$ we obtain

$$\big\{2k_\varepsilon^{3/2} - \sqrt{27}\, h_\varepsilon(t)\big\} \ \geq \ k_\varepsilon(t)^{3/2}, \qquad \big\{2k_\varepsilon(t)^{3/2} + \sqrt{27}\, h_\varepsilon(t)\big\} \ \geq \ M^{3/2}\varepsilon^\alpha.$$

Thus, in both the cases we get by (24)

$$\triangle_\varepsilon(t) \;\geq\; 4M^{3/2}\,\varepsilon^\alpha\, k_\varepsilon(t)^{3/2}.$$

In the special case when $r = 1$, the discriminant $\triangle_A(t)$ is strictly positive, hence both the inequalities in (25) are strict, and we conclude that $\triangle_\varepsilon(t) \;\geq c > 0$.

Finally, (22) follows directly from (21) and the definition (18) of $\triangle_\varepsilon(t)$. □

In the following Lemma, we exhibit an exact (but possibly non-coercive) symmetrizer $Q_\varepsilon(t)$ for the 3×3 Sylvester matrix whose characteristic polynomial is the polynomial $z^3 - k_\varepsilon(t)z + h_\varepsilon(t)$. We also give a lower estimate for such a symmetrizer $Q_\varepsilon(t)$, which will be decisive in our proof.

Lemma 2. *Let us define*

$$A_\varepsilon^\sharp(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ h_\varepsilon(t) & k_\varepsilon(t) & 0 \end{pmatrix}, \quad Q_\varepsilon(t) = \begin{pmatrix} k_\varepsilon(t)^2 & 3h_\varepsilon(t) & -k_\varepsilon(t) \\ 3h_\varepsilon(t) & 2k_\varepsilon(t) & 0 \\ -k_\varepsilon(t) & 0 & 3 \end{pmatrix}. \tag{26}$$

Then, the matrix $Q_\varepsilon(t)$ is Hermitian and satisfies

$$Q_\varepsilon(t)\, A_\varepsilon^\sharp(t) \;=\; A_\varepsilon^\sharp(t)^*\, Q_\varepsilon(t). \tag{27}$$

$$\big(Q_\varepsilon(t)W, W\big) \;\geq\; c\,|L_\varepsilon(t)W|^2 \qquad \text{for all } W \in \mathbf{C}^3, \qquad c > 0, \tag{28}$$

where

$$L_\varepsilon(t) \;=\; \triangle_\varepsilon(t)^{1/2} \begin{pmatrix} k_\varepsilon(t)^{-1/2} & 0 & 0 \\ 0 & k_\varepsilon(t)^{-1} & 0 \\ 0 & 0 & k_\varepsilon(t)^{-3/2} \end{pmatrix}.$$

Proof: (27) follows from the definitions (26). Let us prove (28). Since

$$L_\varepsilon^{-1} \;=\; (L_\varepsilon^{-1})^* \;=\; \triangle_\varepsilon^{-1/2} \begin{pmatrix} k_\varepsilon^{1/2} & 0 & 0 \\ 0 & k_\varepsilon & 0 \\ 0 & 0 & k_\varepsilon^{3/2} \end{pmatrix},$$

we have

$$\big(L_\varepsilon^{-1}\big)^* Q_\varepsilon L_\varepsilon^{-1} = \frac{k_\varepsilon^3}{\triangle_\varepsilon}\,\widetilde{Q}_\varepsilon, \tag{29}$$

where

$$\widetilde{Q}_\varepsilon(t) \;\equiv\; \big[\widetilde{q}_{ij}(t)\big]_{1\leq i,j\leq 3} \;=\; \begin{pmatrix} 1 & 3h_\varepsilon k_\varepsilon^{-3/2} & -1 \\ 3h_\varepsilon k_\varepsilon^{-3/2} & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}.$$

Now, by (22) we see that $\| \widetilde{Q}_\varepsilon(t) \| \le C$ on $[0,T]$. Moreover, by (19) and (20), the determinant and the minor determinants of $\widetilde{Q}_\varepsilon(t)$ satisfy

$$\det \widetilde{Q}_\varepsilon(t) \;=\; 4 - \frac{27h_\varepsilon^2}{k_\varepsilon^3} \;=\; \frac{\triangle_\varepsilon}{k_\varepsilon^3} \;>\; 0$$

$$\widetilde{q}_{11}(t)\widetilde{q}_{22}(t) - \widetilde{q}_{12}(t)\widetilde{q}_{21}(t) = \; 2 - \frac{9h_\varepsilon^2}{k_\varepsilon^3} \;=\; \frac{2}{3} + \frac{\triangle_\varepsilon}{3k_\varepsilon^3} \;>\; 0, \qquad \widetilde{q}_{11}(t) \;=\; 1 \;>\; 0.$$

This implies that the eigenvalues $\mu_1(t), \mu_2(t), \mu_3(t)$ of $\widetilde{Q}_\varepsilon(t)$ are non-negative, and thus we have, for $\{i,j,k\} = \{1,2,3\}$,

$$\mu_i(t) \;=\; \frac{\mu_i(t)\mu_j(t)\mu_k(t)}{\mu_j(t)\mu_k(t)} \;\ge\; \frac{\det(\widetilde{Q}_\varepsilon(t))}{\| \widetilde{Q}_\varepsilon(t) \|^2} \;\ge\; c\,\frac{\triangle_\varepsilon(t)}{k_\varepsilon(t)^3}\,, \qquad c > 0.$$

Hence we get

$$\big(\widetilde{Q}_\varepsilon(t)\widetilde{W}, \widetilde{W}\big) \;\ge\; c\,\frac{\triangle_\varepsilon(t)}{k_\varepsilon(t)^3}\,|\widetilde{W}|^2 \qquad \text{for all } \widetilde{W} \in \mathbf{C}^3,$$

and consequently, taking $\widetilde{W} = L_\varepsilon(t)W$ and recalling (29),

$$\big(Q_\varepsilon(t)W, W\big) \;=\; \frac{k_\varepsilon(t)^3}{\triangle_\varepsilon(t)}\,\big(\widetilde{Q}_\varepsilon(t)\widetilde{W}, \widetilde{W}\big) \;\ge\; c\,|\widetilde{W}|^2 \;=\; c\,|L_\varepsilon(t)W|^2. \qquad \square$$

Lemma 2 also applicable to 9×9 block-matrices whose blocks are 3×3 matrices of scalar type. Indeed, denoting by I the 3×3 identity matrix, we have:

Lemma 3. *Let us define the 9×9 matrices*

(30) $$\mathcal{A}_\varepsilon(t) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ h_\varepsilon(t)I & k_\varepsilon(t)I & 0 \end{pmatrix}, \quad \mathcal{Q}_\varepsilon(t) = \begin{pmatrix} k_\varepsilon(t)^2 I & 3h_\varepsilon(t)I & -k_\varepsilon(t)I \\ 3h_\varepsilon(t)I & 2k_\varepsilon(t)I & 0 \\ -k_\varepsilon(t)I & 0 & 3I \end{pmatrix}.$$

Therefore, $\mathcal{Q}_\varepsilon(t)$ is Hermitian and satisfies

(31) $$\mathcal{Q}_\varepsilon(t)\mathcal{A}_\varepsilon(t) \;=\; \mathcal{A}_\varepsilon(t)^*\mathcal{Q}_\varepsilon(t),$$

(32) $$\big(\mathcal{Q}_\varepsilon(t)\mathcal{W}, \mathcal{W}\big) \;\ge\; c\,|\mathcal{L}_\varepsilon(t)\mathcal{W}|^2 \qquad \text{for all } \mathcal{W} \in \mathbf{C}^9, \qquad c > 0,$$

where

(33) $$\mathcal{L}_\varepsilon(t) \;=\; \triangle_\varepsilon(t)^{1/2} \begin{pmatrix} k_\varepsilon(t)^{-1/2}I & 0 & 0 \\ 0 & k_\varepsilon(t)^{-1}I & 0 \\ 0 & 0 & k_\varepsilon(t)^{-3/2}I \end{pmatrix}.$$

Proof : Since the 3×3 submatrices in $\mathcal{A}_\varepsilon(t)$, $\mathcal{Q}_\varepsilon(t)$ and $\mathcal{L}_\varepsilon(t)$ consist of the 3×3 identity matrix I, (31) and (32) can be easily derived from (27) and (28) respectively. □

Now, we transform the 3×3 system (4) into a 9×9 system whose principal part is the block Sylvester matrix $\mathcal{A}_\varepsilon(t)$ of Lemma 3. We deduce from (4) that

$$\begin{aligned}
(i) \qquad V' &= i\xi AV + BV = i\xi A_\varepsilon V + i\xi(A - A_\varepsilon)V + BV, \\
(ii) \quad (A_\varepsilon V)' &= i\xi A_\varepsilon^2 V + i\xi A_\varepsilon(A - A_\varepsilon)V + A_\varepsilon' V + A_\varepsilon BV, \\
(iii) \quad (A_\varepsilon^2 V)' &= i\xi A_\varepsilon^3 V + i\xi A_\varepsilon^2(A - A_\varepsilon)V + (A_\varepsilon^2)'V + A_\varepsilon^2 BV \\
&= \big[i\xi\, h_\varepsilon V + i\xi k_\varepsilon A_\varepsilon V\big] - \xi\, \Im h_{A_\varepsilon} V + i\xi(k_{A_\varepsilon} - k_\varepsilon)A_\varepsilon V \\
&\qquad + i\xi A_\varepsilon^2(A - A_\varepsilon)V + (A_\varepsilon^2)'V + A_\varepsilon^2 BV\,,
\end{aligned}$$

where, in the last equality, we have used the Hamilton-Cayley equality (14). Putting

$$\mathcal{V} \equiv \mathcal{V}(t,\xi) = \begin{pmatrix} V \\ A_\varepsilon V \\ A_\varepsilon^2 V \end{pmatrix} \in \mathbf{C}^9,$$

we combine together (i), (ii) and (iii) to get the 9×9 system:

(34) $$\mathcal{V}' = i\xi\mathcal{A}_\varepsilon(t)\mathcal{V} + i\xi\mathcal{R}_\varepsilon(t)\mathcal{V} - \xi\mathcal{P}_\varepsilon(t)\mathcal{V} + \mathcal{D}_\varepsilon(t)\mathcal{V} + \mathcal{B}_\varepsilon(t)\mathcal{V},$$

where $\mathcal{A}_\varepsilon(t)$ is defined in (30), and:

$$\mathcal{R}_\varepsilon(t) = \begin{pmatrix} A - A_\varepsilon & 0 & 0 \\ A_\varepsilon(A - A_\varepsilon) & 0 & 0 \\ A_\varepsilon^2(A - A_\varepsilon) & 0 & 0 \end{pmatrix}, \quad \mathcal{P}_\varepsilon(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Im h_{A_\varepsilon} I & -i(k_{A_\varepsilon} - k_\varepsilon)I & 0 \end{pmatrix},$$

$$\mathcal{D}_\varepsilon(t) = \begin{pmatrix} 0 & 0 & 0 \\ A_\varepsilon' & 0 & 0 \\ (A_\varepsilon^2)' & 0 & 0 \end{pmatrix}, \quad \mathcal{B}_\varepsilon(t) = \begin{pmatrix} B & 0 & 0 \\ A_\varepsilon B & 0 & 0 \\ A_\varepsilon^2 B & 0 & 0 \end{pmatrix}.$$

Then, recalling (30), we define the energy:

$$E(t,\xi) = \big(\mathcal{Q}_\varepsilon(t)\mathcal{V}, \mathcal{V}\big).$$

By the definition (33) of $\mathcal{L}_\varepsilon(t)$, using (19) and (21), we see that

(35) $$\big|\mathcal{L}_\varepsilon(t)\mathcal{W}\big|^2 \geq c_1 \triangle_\varepsilon(t) k_\varepsilon(t)^{-1} \big|\mathcal{W}\big|^2 \geq c_2\, \varepsilon^{4\alpha/3} \big|\mathcal{W}\big|^2,$$

hence, remarking that $\|\mathcal{Q}_\varepsilon(t)\| \leq C$, and $\big|V\big|^2 \leq \big|\mathcal{V}\big|^2 \leq C\big|V\big|^2$, we deduce from (32) and (35) :

(36) $$c\, \varepsilon^{4\alpha/3} \big|V\big|^2 \leq E(t,\xi) \leq C\, \big|V\big|^2.$$

By (31) and (34), considering that $\mathcal{Q}_\varepsilon$ is Hermitian, we get the equality

$$\begin{aligned} E'(t,\xi) &= \big(\mathcal{Q}'_\varepsilon\mathcal{V},\mathcal{V}\big)+\big(\mathcal{Q}_\varepsilon\mathcal{V}',\mathcal{V}\big)+\big(\mathcal{Q}_\varepsilon\mathcal{V},\mathcal{V}'\big)\\ &= \big(\mathcal{Q}'_\varepsilon\mathcal{V},\mathcal{V}\big)+i\xi\big(\{\mathcal{Q}_\varepsilon\mathcal{A}_\varepsilon-\mathcal{A}^*_\varepsilon\mathcal{Q}^*_\varepsilon\}\mathcal{V},\mathcal{V}\big)\\ &\quad+\big(\mathcal{Q}_\varepsilon\{i\xi\mathcal{R}_\varepsilon-\xi\mathcal{P}_\varepsilon+\mathcal{D}_\varepsilon+\mathcal{B}_\varepsilon\}\mathcal{V},\mathcal{V}\big)+\overline{\big(\mathcal{Q}_\varepsilon\{i\xi\mathcal{R}_\varepsilon-\xi\mathcal{P}_\varepsilon+\mathcal{D}_\varepsilon+\mathcal{B}_\varepsilon\}\mathcal{V},\mathcal{V}\big)}\\ &= \big(\mathcal{Q}'_\varepsilon\mathcal{V},\mathcal{V}\big)-2\xi\Im\big(\mathcal{Q}_\varepsilon\mathcal{R}_\varepsilon\mathcal{V},\mathcal{V}\big)-2\xi\Re\big(\mathcal{Q}_\varepsilon\mathcal{P}_\varepsilon\mathcal{V},\mathcal{V}\big)+2\Re\big(\mathcal{Q}_\varepsilon\mathcal{D}_\varepsilon\mathcal{V},\mathcal{V}\big)+2\Re\big(\mathcal{Q}_\varepsilon\mathcal{B}_\varepsilon\mathcal{V},\mathcal{V}\big). \end{aligned}$$

In order to prove the energy estimate, we use the following:

Lemma 4. *If $\mathcal{S}$ be a 9×9 matrix, then we have, for all $\mathcal{W}\in\mathbf{C}^9$,*

$$\big(\mathcal{S}\mathcal{W},\mathcal{W}\big)\le C\parallel \mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1}\parallel\big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big), \tag{37}$$

$$\big(\mathcal{Q}_\varepsilon\mathcal{S}\mathcal{W},\mathcal{W}\big)\le C\parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{S}^*\mathcal{Q}_\varepsilon\mathcal{S})\mathcal{L}_\varepsilon^{-1}\parallel^{1/2}\big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big)\,. \tag{38}$$

Proof : (37) follows directly from (32); indeed, noting that $\mathcal{L}^*_\varepsilon=\mathcal{L}_\varepsilon$, we find

$$\begin{aligned}\big(\mathcal{S}\mathcal{W},\mathcal{W}\big) &= \big(\mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1}\mathcal{L}_\varepsilon\mathcal{W},\mathcal{L}^*_\varepsilon\mathcal{W}\big)\ \le\ \parallel \mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1}\parallel|\mathcal{L}_\varepsilon(t)\mathcal{W}|^2\\ &\le \frac{1}{c}\parallel \mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1}\parallel\big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big).\end{aligned}$$

To prove (38), we use the Schwarz's inequality for the scalar product $\langle\mathcal{Y},\mathcal{W}\rangle\equiv\big(\mathcal{Q}_\varepsilon\mathcal{Y},\mathcal{W}\big)$, and (37) with $\mathcal{S}^*\mathcal{Q}_\varepsilon\mathcal{S}$ in place of $\mathcal{S}$. Thus we obtain

$$\begin{aligned}\big(\mathcal{Q}_\varepsilon\mathcal{S}\mathcal{W},\mathcal{W}\big) &= \big(\mathcal{Q}_\varepsilon\mathcal{S}\mathcal{W},\mathcal{S}\mathcal{W}\big)^{1/2}\big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big)^{1/2}\\ &\le C\parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{S}^*\mathcal{Q}_\varepsilon\mathcal{S})\mathcal{L}_\varepsilon^{-1}\parallel^{1/2}\big(\mathcal{Q}_\varepsilon\mathcal{W},\mathcal{W}\big). \qquad\square\end{aligned}$$

By (37) and (38), it follows

$$\begin{aligned}E'(t,\xi)\ \le\ C\,E(t,\xi)\Big\{&\parallel \mathcal{L}_\varepsilon^{-1}\mathcal{Q}'_\varepsilon\mathcal{L}_\varepsilon^{-1}\parallel+|\xi|\parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{R}^*_\varepsilon\mathcal{Q}_\varepsilon\mathcal{R}_\varepsilon)\mathcal{L}_\varepsilon^{-1}\parallel^{1/2}\\ &+|\xi|\parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{P}^*_\varepsilon\mathcal{Q}_\varepsilon\mathcal{P}_\varepsilon)\mathcal{L}_\varepsilon^{-1}\parallel^{1/2}+\parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{D}^*_\varepsilon\mathcal{Q}_\varepsilon\mathcal{D}_\varepsilon)\mathcal{L}_\varepsilon^{-1}\parallel^{1/2}+\parallel \mathcal{L}_\varepsilon^{-1}(\mathcal{B}^*_\varepsilon\mathcal{Q}_\varepsilon\mathcal{B}_\varepsilon)\mathcal{L}_\varepsilon^{-1}\parallel^{1/2}\Big\}.\end{aligned}$$

Now we estimate the five summands on the left hand side. To this end, let us firstly observe that, for any 9×9 block matrix $\mathcal{S}=\big[S_{ij}\big]_{1\le i,j\le 3}$, one has

$$\mathcal{L}_\varepsilon^{-1}\mathcal{S}\mathcal{L}_\varepsilon^{-1}\ =\ \frac{1}{\triangle_\varepsilon}\,\big[k_\varepsilon^{(i+j)/2}S_{ij}\big]_{1\le i,j\le 3}\ . \tag{39}$$

1) Estimate of $\| \mathcal{L}_\varepsilon^{-1} \mathcal{Q}'_\varepsilon \mathcal{L}_\varepsilon^{-1} \|$: By using (39), we see that

$$\mathcal{L}_\varepsilon^{-1} \mathcal{Q}'_\varepsilon \mathcal{L}_\varepsilon^{-1} = \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon} \begin{pmatrix} 2k_\varepsilon^{1/2} k'_\varepsilon I & 3h'_\varepsilon I & -k_\varepsilon^{1/2} k'_\varepsilon I \\ 3h'_\varepsilon I & 2k_\varepsilon^{1/2} k'_\varepsilon I & 0 \\ -k_\varepsilon^{1/2} k'_\varepsilon I & 0 & 0 \end{pmatrix},$$

thus, by (16) and (20), we get

$$\| \mathcal{L}_\varepsilon^{-1} \mathcal{Q}'_\varepsilon \mathcal{L}_\varepsilon^{-1} \| \leq \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon} C \left\{ k_\varepsilon^{1/2} | k'_\varepsilon | + | h'_\varepsilon | \right\} \leq \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon} C_1 \varepsilon^{\alpha-1}. \tag{40}$$

2) Estimate of $\| \mathcal{L}_\varepsilon^{-1} (\mathcal{P}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{P}_\varepsilon) \mathcal{L}_\varepsilon^{-1} \|$: By the equality

$$\begin{pmatrix} 0 & 0 & Y_1^* \\ 0 & 0 & Y_2^* \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k^2 I & 3hI & -I \\ 3hI & 2kI & 0 \\ -kI & 0 & 3I \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y_1 & Y_1 & 0 \end{pmatrix} = 3 \begin{pmatrix} Y_1^* Y_1 & Y_1^* Y_2 & 0 \\ Y_2^* Y_1 & Y_2^* Y_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and by (39), we find

$$\mathcal{L}_\varepsilon^{-1} (\mathcal{P}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{P}_\varepsilon) \mathcal{L}_\varepsilon^{-1} = \frac{3k_\varepsilon}{\triangle_\varepsilon} \begin{pmatrix} (\Im h_{A_\varepsilon})^2 I & -ik_\varepsilon^{1/2} (k_{A_\varepsilon} - k_\varepsilon) \Im h_{A_\varepsilon} I & 0 \\ ik_\varepsilon^{1/2} \overline{(k_{A_\varepsilon} - k_\varepsilon)} \Im h_{A_\varepsilon} I & k_\varepsilon |k_{A_\varepsilon} - k_\varepsilon|^2 I & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, by (16) and (20),

$$\| \mathcal{L}_\varepsilon^{-1} (\mathcal{P}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{P}_\varepsilon) \mathcal{L}_\varepsilon^{-1} \| \leq \frac{k_\varepsilon}{\triangle_\varepsilon} C \Big\{ \varepsilon^{2\alpha} + k_\varepsilon^{1/2} | k_{A_\varepsilon} - k_\varepsilon | \varepsilon^\alpha + k_\varepsilon | k_{A_\varepsilon} - k_\varepsilon |^2 \Big\} \leq \frac{k_\varepsilon}{\triangle_\varepsilon} C_2 \, \varepsilon^{2\alpha}. \tag{41}$$

To compute the products $\mathcal{X}^* \mathcal{Q}_\varepsilon \mathcal{X}$ with $\mathcal{X} = \mathcal{R}_\varepsilon, \mathcal{D}_\varepsilon, \mathcal{B}_\varepsilon$, we note that

$$\begin{pmatrix} X_1^* & X_2^* & X_3^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_\varepsilon^2 I & 3h_\varepsilon I & -k_\varepsilon I \\ 3h_\varepsilon I & 2k_\varepsilon I & 0 \\ -k_\varepsilon I & 0 & 3I \end{pmatrix} \begin{pmatrix} X_1 & 0 & 0 \\ X_2 & 0 & 0 \\ X_3 & 0 & 0 \end{pmatrix} = Z_\varepsilon \, \mathcal{J} \tag{42}$$

where

$$Z_\varepsilon = k_\varepsilon^2 X_1^* X_1 + 3h_\varepsilon (X_1^* X_2 + X_2^* X_1) - k_\varepsilon (X_1^* X_3 + X_3^* X_1 - 2X_2^* X_2) + 3X_3^* X_3$$

and

$$\mathcal{J} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3) Estimate of $\| \mathcal{L}_\varepsilon^{-1}(\mathcal{R}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{R}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \|$: From (42) with $X_j = A_\varepsilon^{j-1}(A - A_\varepsilon)$, $j = 1, 2, 3$, recalling (39), we see that

$$\mathcal{L}_\varepsilon^{-1}(\mathcal{R}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{R}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \;=\; \frac{k_\varepsilon}{\triangle_\varepsilon} F_\varepsilon \, \mathcal{J},$$

where

$$F_\varepsilon = \,(A - A_\varepsilon)^* \Big\{ k_\varepsilon^2 I + 3h_\varepsilon(A_\varepsilon + A_\varepsilon^*) - k_\varepsilon(A_\varepsilon - A_\varepsilon^*)^2 + 3A_\varepsilon^{*2}\, A_\varepsilon^2 \Big\}\, (A - A_\varepsilon).$$

Hence, by using (7), we get

(43) $$\| \mathcal{L}_\varepsilon^{-1}(\mathcal{R}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{R}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \| \le \frac{k_\varepsilon}{\triangle_\varepsilon} C \, \| A - A_\varepsilon \|^2 \le \frac{k_\varepsilon}{\triangle_\varepsilon} C_3 \, \varepsilon^{2\alpha}.$$

4) Estimate of $\| \mathcal{L}_\varepsilon^{-1}(\mathcal{D}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{D}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \|$: From (42) with $X_1 = 0$, $X_2 = A'_\varepsilon$ and $X_3 = (A_\varepsilon^2)'$, by (39) we see that

$$\mathcal{L}_\varepsilon^{-1}(\mathcal{D}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{D}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \;=\; \frac{k_\varepsilon}{\triangle_\varepsilon} G_\varepsilon \, \mathcal{J},$$

where $G_\varepsilon = 2k_\varepsilon {A'_\varepsilon}^* A'_\varepsilon + 3(A_\varepsilon^2)'^* (A_\varepsilon^2)'$. Hence we get, by using (7),

(44) $$\| \mathcal{L}_\varepsilon^{-1}(\mathcal{D}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{D}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \| \le \frac{k_\varepsilon}{\triangle_\varepsilon} C \, \| A'_\varepsilon \|^2 \le \frac{k_\varepsilon}{\triangle_\varepsilon} C_4 \, \varepsilon^{2(\alpha-1)}.$$

5) Estimate of $\| \mathcal{L}_\varepsilon^{-1}(\mathcal{B}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{B}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \|$: From (42) with $X_1 = B$, $X_2 = A_\varepsilon B$, $X_3 = A_\varepsilon^2 B$, and by using (39), we see that

$$\mathcal{L}_\varepsilon^{-1}(\mathcal{B}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{B}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \;=\; \frac{k_\varepsilon}{\triangle_\varepsilon} H_\varepsilon \, \mathcal{J},$$

where

$$H_\varepsilon \;=\; B^* \Big\{ k_\varepsilon^2 + 3h_\varepsilon(A_\varepsilon + A_\varepsilon^*) - k_\varepsilon(A_\varepsilon - A_\varepsilon^*)^2 + 3A_\varepsilon^{*2} A_\varepsilon^2 \Big\} B.$$

Hence

(45) $$\| \mathcal{L}_\varepsilon^{-1}(\mathcal{B}_\varepsilon^* \mathcal{Q}_\varepsilon \mathcal{B}_\varepsilon)\mathcal{L}_\varepsilon^{-1} \| \le \frac{k_\varepsilon}{\triangle_\varepsilon} \, \| H_\varepsilon \| \le C_5 \frac{k_\varepsilon}{\triangle_\varepsilon} \, \| B(t) \|^2 \, .$$

From (40), (41), (43), (44), (45) and (19), (21), recalling that $\|B(t)\| \leq C$, and choosing

$$\varepsilon = \begin{cases} |\xi|^{-1} & \text{if } \; r=1, \\ |\xi|^{-1/(1+\alpha/2)} & \text{if } \; r=2, \\ |\xi|^{-1/(1+\alpha/3)} & \text{if } \; r=3, \end{cases}$$

we have the following estimate, for $|\xi| \geq 1$,

$$\begin{aligned} E'(t,\xi) \; &\leq \; C_6 E(t,\xi) \left[\varepsilon^{\alpha-1} \, \frac{k_\varepsilon^{3/2}}{\triangle_\varepsilon} + \varepsilon^{\alpha} \, \frac{k_\varepsilon^{1/2}}{\triangle_\varepsilon^{1/2}} \, |\xi| + \varepsilon^{\alpha-1} \, \frac{k_\varepsilon^{1/2}}{\triangle_\varepsilon^{1/2}} \right] \\ &\leq \begin{cases} C_7 E(t,\xi) \left[\varepsilon^{\alpha-1} k_\varepsilon^{3/2} + \varepsilon^{\alpha} k_\varepsilon^{1/2} |\xi| + \varepsilon^{\alpha-1} k_\varepsilon^{1/2} \right] & \text{if } \; r=1 \\ C_7 E(t,\xi) \left[\varepsilon^{-1} + \varepsilon^{\alpha/2} k_\varepsilon^{-1/4} |\xi| + \varepsilon^{\alpha/2-1} k_\varepsilon^{-1/4} \right] & \text{if } \; r=2,3 \end{cases} \\ &\leq \begin{cases} C E(t,\xi) \left[\varepsilon^{\alpha} |\xi| + \varepsilon^{\alpha-1} \right] \; \leq \; 2\, C E(t,\xi) \, |\xi|^{1-\alpha} & \text{if } \; r=1, \\ C E(t,\xi) \left[\varepsilon^{\alpha/2} |\xi| + \varepsilon^{-1} \right] \leq \; 2\, C E(t,\xi) \, |\xi|^{1/(1+\alpha/2)} & \text{if } \; r=2, \\ C E(t,\xi) \left[\varepsilon^{\alpha/3} |\xi| + \varepsilon^{-1} \right] \leq \; 2\, C E(t,\xi) \, |\xi|^{1/(1+\alpha/3)} & \text{if } \; r=3, \end{cases} \end{aligned}$$

which gives, by (36), the required a priori estimate (5) with σ equal respectively to $1/(1-\alpha)$, $1+\alpha/2$, or $1+\alpha/3$. This concludes the proof of Theorem 1 for $m=3$. □

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