Derived equivalences and Gorenstein algebras

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Abstract

We introduce a notion of Gorenstein R-algebras over a commutative Gorenstein ring R. Then we provide a necessary and sufficient condition for a tilting complex over a Gorenstein R-algebra A to have a Gorenstein R-algebra B as the endomorphism algebra and a construction of such a tilting complex. Furthermore, we provide an example of a tilting complex over a Gorenstein R-algebra A whose endomorphism algebra is not a Gorenstein R-algebra.

In this note, extending the notion of selfinjective artin algebras to noether algebras, we introduce a notion of Gorenstein algebras. Our main aim is to provide a necessary and sufficient condition for a tilting complex over a Gorenstein algebra to have a Gorenstein algebra as the endomorphism algebra.

Let R be a commutative noetherian ring and A a noether R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. To define the Gorensteinness for A, we assume the base ring R is a Gorenstein ring (see [?]). Then we call A a Gorenstein R-algebra provided that A has Gorenstein dimension zero as an R-module (see [?]) and that DA is a projective generator in the category of right A-modules, where $D = \mathrm{Hom}_R(-, R)$. Assume A is a Gorenstein R-algebra. We will see in Section 3 that A satisfies the Auslander condition (see [?]) and has selfinjective dimension at most $\dim R$ on both sides, where $\dim R$ denotes the Krull dimension of R. In particular, in case A is commutative, A is a Gorenstein ring. Also, in case $\dim R = 0$, A is a selfinjective artin algebra. Furthermore, for any prime ideal $\mathfrak{p}$ of R with $A_\mathfrak{p} \neq 0$ we will see that $A_\mathfrak{p}$ is maximal Cohen-Macaulay as an $R_\mathfrak{p}$-module and has selfinjective dimension equal to $\dim R_\mathfrak{p}$ on both sides. It follows that A is a Gorenstein algebra in the sense of [?] in which the theory of Gorenstein algebras is studied in detail. So we refer to [?] for the relationship of the notion of Gorenstein algebras to the theory of commutative Gorenstein rings. Next, let $P^\bullet$ be a tilting complex (see [?]) over A and denote by B the endomorphism algebra of $P^\bullet$

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in the homotopy category. We will show in Section 4 that B is a Gorenstein R-algebra if and only if $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where $\nu = D \circ \mathrm{Hom}_A(-, A)$, and that if $A \cong DA$ as A-bimodules then B is a Gorenstein R-algebra with $B \cong DB$ as B-bimodules. Furthermore, we will provide an example of A and $P^\bullet$ such that B does not have Gorenstein dimension zero as an R-module. On the other hand, we will show in Section 5 that if $P^\bullet$ is associated with a certain sequence of idempotents in A then the condition $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$ is always satisfied. There is another notion of Gorenstein algebras. Consider the case where R is an artinian Gorenstein ring. Then an R-algebra A is sometimes called a Gorenstein algebra if inj dim ${}_A A$ = inj dim $A_A < \infty$ (see e.g. [?]). It follows by [?, Proposition 1.6] that an R-algebra A is a Gorenstein algebra in this sense if and only if $D({}_A A)$ is a tilting module. We will extend this fact to the case where R is a Gorenstein ring with dim $R < \infty$.

For a ring A, we denote by Mod-A the category of right A-modules and mod-A the full subcategory of Mod-A consisting of finitely presented modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_A X$) to stress that the module X considered is a right (resp., left) A-module. In particular, we denote by inj dim X_A (resp., inj dim ${}_A X$) the injective dimension of a right (resp., left) A-module X. A similar notation is used for projective and flat dimensions. In this note, complexes are cochain complexes of modules and as usual modules are considered as complexes concentrated in degree zero. For a complex $X^\bullet$ and an integer $n \in \mathbb{Z}$, we denote by $\mathrm{B}^n(X^\bullet)$, $\mathrm{Z}^n(X^\bullet)$, $\mathrm{B}'^n(X^\bullet)$, $\mathrm{Z}'^n(X^\bullet)$ and $\mathrm{H}^n(X^\bullet)$ the n-th boundary, the n-th cycle, the n-th coboundary, the n-th cocycle and the n-th cohomology of $X^\bullet$, respectively. We denote by $\mathcal{K}(A)$ (resp., $\mathcal{D}(A)$) the homotopy (resp., derived) category of complexes of right A-modules and by $\mathcal{K}^+(A)$, $\mathcal{K}^-(A)$, $\mathcal{K}^{\mathrm{b}}(A)$ (resp., $\mathcal{D}^+(A)$, $\mathcal{D}^-(A)$, $\mathcal{D}^{\mathrm{b}}(A)$) the full triangulated subcategories of $\mathcal{K}(A)$ (resp., $\mathcal{D}(A)$) consisting of bounded below complexes, bounded above complexes and bounded complexes, respectively. We denote by $\mathcal{P}_A$ the full subcategory of mod-A consisting of finitely generated projective modules and by $\mathcal{K}^*(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}^*(A)$ consisting of complexes whose terms belong to $\mathcal{P}_A$, where $* = +, -, \mathrm{b}$ or nothing. We use the notation $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$) to denote the single complex associated with the double hom (resp., tensor) complex. Finally, for an object X in an additive category $\mathfrak{A}$ we denote by $\mathrm{add}(X)$ the full additive subcategory of $\mathfrak{A}$ whose objects are direct summands of finite direct sums of copies of X and by $\overset{n}{\bigoplus} X$ the direct sum of n copies of X.

We refer to [?], [?], [?] for basic results in the theory of derived categories and to [?] for definitions and basic properties of derived equivalences and tilting complexes. Also, we refer to [?] for standard homological algebra in module categories and to [?] for standard commutative ring theory.

1 Preliminaries

Throughout this note, R is a commutative ring and A is an R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A. We assume further that R is a noetherian ring and A is a noether R-algebra, i.e., A is finitely generated as an R-module. Note that A is a left and right noetherian ring. In particular, mod-A is abelian and consists of all finitely generated right A-modules. We set $D = \mathrm{Hom}_R(-, R)$. Note that for any $X \in \mathrm{Mod}\text{-}A$ we have a functorial isomorphism in $\mathrm{Mod}\text{-}A^{\mathrm{op}}$

$$DX \xrightarrow{\sim} \mathrm{Hom}_A(X, DA), h \mapsto (x \mapsto (a \mapsto h(xa))).$$

For R-algebras A, B we identify an $(A^{\mathrm{op}} \otimes_R B)$-module X with an A-B-bimodule X such that $rx = xr$ for all $r \in R$ and $x \in X$. Also, for an R-algebra A we set $A^{\mathrm{e}} = A^{\mathrm{op}} \otimes_R A$. We identify $(A^{\mathrm{op}})^{\mathrm{op}}$ with A and $(A^{\mathrm{e}})^{\mathrm{op}}$ with A^{e}.

In this section, we recall several definitions and basic facts which we need in later sections.

Definition 1.1. A module $X \in \mathrm{Mod}\text{-}R$ is said to be reflexive if the canonical homomorphism

$$\varepsilon_X : X \to D^2X, x \mapsto (h \mapsto h(x))$$

is an isomorphism, where $D^2X = D(DX)$.

Definition 1.2 ([?]). A module $X \in \mathrm{mod}\text{-}R$ is said to have Gorenstein dimension zero if X is reflexive, $\mathrm{Ext}^i_R(X, R) = 0$ for $i > 0$ and $\mathrm{Ext}^i_R(DX, R) = 0$ for $i > 0$. We denote by $\mathcal{G}_R$ the full additive subcategory of mod-R consisting of modules which have Gorenstein dimension zero.

Lemma 1.3 ([?, Lemma 3.10]). *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in* mod-R. *Then the following hold.*

(1) If $Y, Z \in \mathcal{G}_R$, then $X \in \mathcal{G}_R$.

(2) Assume $\mathrm{Ext}^1_R(Z, R) = 0$. If $X, Y \in \mathcal{G}_R$, then $Z \in \mathcal{G}_R$.

Proof. See the proof of [?, Lemma 3.10]. □

Lemma 1.4. *For any $X^\bullet \in \mathcal{K}(R)$ we have a functorial homomorphism*

$$\xi_{X^\bullet} : \mathrm{H}^0(DX^\bullet) \to D\mathrm{H}^0(X^\bullet)$$

and the following hold.

(1) If $\mathrm{B}^0(DX^\bullet) \xrightarrow{\sim} D\mathrm{B}'^0(X^\bullet)$ canonically, then $\xi_{X^\bullet}$ is monic.

(2) If $\mathrm{B}^0(DX^\bullet) \xrightarrow{\sim} D\mathrm{B}'^0(X^\bullet)$ canonically and $\mathrm{Ext}^1_R(\mathrm{B}'^0(X^\bullet), R) = 0$, then $\xi_{X^\bullet}$ is an isomorphism.

Proof. We have functorial commutative diagrams in Mod-R with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{B}^0(DX^\bullet) \arrow[r] \arrow[d, "\eta_{X^\bullet}"'] & DX^0 \arrow[r] \arrow[d, equal] & \mathrm{Z}'^0(DX^\bullet) \arrow[r] \arrow[d, "\zeta_{X^\bullet}"] & 0 \\
0 \arrow[r] & D\mathrm{B}'^0(X^\bullet) \arrow[r] & DX^0 \arrow[r] & D\mathrm{Z}^0(X^\bullet), &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(DX^\bullet) \arrow[r] \arrow[d, "\xi_{X^\bullet}"'] & \mathrm{Z}'^0(DX^\bullet) \arrow[r] \arrow[d, "\zeta_{X^\bullet}"] & DX^{-1} \arrow[d, equal] \\
0 \arrow[r] & D\mathrm{H}^0(X^\bullet) \arrow[r] & D\mathrm{Z}^0(X^\bullet) \arrow[r] & DX^{-1}.
\end{tikzcd}$$

Assume $\eta_{X^\bullet}$ is an isomorphism. Then $\zeta_{X^\bullet}$ is monic and so is $\xi_{X^\bullet}$. Furthermore, if $\mathrm{Ext}^1_R(\mathrm{B}'^0(X^\bullet), R) = 0$, then $DX^0 \to D\mathrm{Z}^0(X^\bullet)$ is epic, so that $\zeta_{X^\bullet}$ and hence $\xi_{X^\bullet}$ are isomorphisms. $\square$

Recall that rings A, B are said to be derived equivalent if $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ are equivalent as triangulated categories (see [**?**] for details). Since A is a noether R-algebra, every ring B derived equivalent to A is also a noether R-algebra ([**?**, Proposition 9.4]).

Lemma 1.5. *Let A, B be derived equivalent R-algebras. Let $F : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be an equivalence of triangulated categories and $F^* : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ a quasi-inverse of F. Set $P^\bullet = F(B) \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet = \mathrm{Hom}^\bullet_B(F^*(A), B) \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}})$. Then for any $i \in \mathbb{Z}$ we have an isomorphism in* Mod-$(B^{\mathrm{op}} \otimes_R A)$

$$\mathrm{Hom}_{\mathcal{K}(A)}(A, P^\bullet[i]) \cong \mathrm{Hom}_{\mathcal{K}(B^{\mathrm{op}})}(B, Q^\bullet[i])$$

and an isomorphism in Mod-$(A^{\mathrm{op}} \otimes_R B)$

$$\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, A[i]) \cong \mathrm{Hom}_{\mathcal{K}(B^{\mathrm{op}})}(Q^\bullet, B[i]).$$

Proof. Set

$$\begin{aligned}
G \ &= F \circ \mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(-, B) \ : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}}) \to \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A), \\
G^* &= \mathrm{Hom}^\bullet_B(-, B) \circ F^* : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \to \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}}).
\end{aligned}$$

Then for any $i \in \mathbb{Z}$ we have a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathcal{K}(A)}(X^\bullet, G(Y^\bullet)[i]) \cong \mathrm{Hom}_{\mathcal{K}(B^{\mathrm{op}})}(Y^\bullet, G^*(X^\bullet)[i])$$

for $X^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Y^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}})$. Since $G(B) \cong P^\bullet$ in $\mathcal{K}(A)$ and $G^*(A) \cong Q^\bullet$ in $\mathcal{K}(B^{\mathrm{op}})$, and since $G^*(P^\bullet) \cong B$ in $\mathcal{K}(B^{\mathrm{op}})$ and $G(Q^\bullet) \cong A$ in $\mathcal{K}(A)$, the assertions follow. $\square$

In several places below, our argument will depend on the term length of a complex. So we truncate redundant terms of complexes. To do so, we need the following.

Remark 1.6. For any $P^\bullet \in \mathcal{K}(\mathcal{P}_A)$ the following hold.

(1) We have a functorial isomorphism of complexes

$$P^\bullet \xrightarrow{\sim} \mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\mathrm{Hom}^\bullet_A(P^\bullet, A), A).$$

(2) If $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ and $\mathrm{H}^i(P^\bullet) = 0$ for all $i \in \mathbb{Z}$, then $P^\bullet = 0$ in $\mathcal{K}(A)$.

(3) If $P^\bullet \in \mathcal{K}^+(\mathcal{P}_A)$ and $\mathrm{H}^i(\mathrm{Hom}^\bullet_A(P^\bullet, A)) = 0$ for all $i \in \mathbb{Z}$, then $P^\bullet = 0$ in $\mathcal{K}(A)$.

Now, for any complex $X^\bullet$ and $n \in \mathbb{Z}$ we define the following truncations:

$$\begin{aligned}
\sigma_{>n}(X^\bullet) &: \cdots \to 0 \to \mathrm{B}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots, \\
\sigma_{\le n}(X^\bullet) &: \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{Z}^n(X^\bullet) \to 0 \to \cdots, \\
\sigma'_{\ge n}(X^\bullet) &: \cdots \to 0 \to \mathrm{Z}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots, \\
\sigma'_{<n}(X^\bullet) &: \cdots \to X^{n-2} \to X^{n-1} \to \mathrm{B}^n(X^\bullet) \to 0 \to \cdots.
\end{aligned}$$

Remark 1.7. For any $P^\bullet \in \mathcal{K}(\mathcal{P}_A)$ and $n \in \mathbb{Z}$ the following hold.

(1) If $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ and $\mathrm{H}^i(P^\bullet) = 0$ for $i > n$, then $\sigma_{\le n}(P^\bullet) \in \mathcal{K}^-(\mathcal{P}_A)$ and $P^\bullet \cong \sigma_{\le n}(P^\bullet)$ in $\mathcal{K}(A)$.

(2) If $P^\bullet \in \mathcal{K}^+(\mathcal{P}_A)$ and $\mathrm{H}^{-i}(\mathrm{Hom}^\bullet_A(P^\bullet, A)) = 0$ for $i < n$, then $\sigma'_{\ge n}(P^\bullet) \in \mathcal{K}^+(\mathcal{P}_A)$ and $P^\bullet \cong \sigma'_{\ge n}(P^\bullet)$ in $\mathcal{K}(A)$.

Proof. (1) It follows by the assumption that $\sigma_{>n}(P^\bullet) = 0$ in $\mathcal{K}(A)$ and $\mathrm{B}'^n(P^\bullet) \in \mathcal{P}_A$. Since the exact sequence $0 \to \mathrm{Z}^n(P^\bullet) \to P^n \to \mathrm{B}'^n(P^\bullet) \to 0$ in Mod-A splits, $\sigma_{\le n}(P^\bullet) \in \mathcal{K}^-(\mathcal{P}_A)$ and $P^\bullet \cong \sigma_{\le n}(P^\bullet) \oplus \sigma_{>n}(P^\bullet)$ as complexes, so that $P^\bullet \cong \sigma_{\le n}(P^\bullet)$ in $\mathcal{K}(A)$.

(2) Set $Q^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, A) \in \mathcal{K}^-(\mathcal{P}_{A^{\mathrm{op}}})$. Since $\mathrm{H}^i(Q^\bullet) = 0$ for $i > -n$, by (1) $\sigma_{\le -n}(Q^\bullet) \in \mathcal{K}^-(\mathcal{P}_{A^{\mathrm{op}}})$ and $Q^\bullet \cong \sigma_{\le -n}(Q^\bullet)$ in $\mathcal{K}(A^{\mathrm{op}})$. Thus we have isomorphisms in $\mathcal{K}(A)$

$$\begin{aligned}
P^\bullet &\cong \mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(Q^\bullet, A) \\
&\cong \mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\sigma_{\le -n}(Q^\bullet), A) \\
&\cong \sigma'_{\ge n}(\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(Q^\bullet, A)) \\
&\cong \sigma'_{\ge n}(P^\bullet).
\end{aligned}$$

□

Definition 1.8. For any $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ with $P^\bullet \ne 0$ in $\mathcal{K}(A)$ we set

$$a(P^\bullet) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i(P^\bullet) \ne 0\}$$

and for any $P^\bullet \in \mathcal{K}^+(\mathcal{P}_A)$ with $P^\bullet \ne 0$ in $\mathcal{K}(A)$ we set

$$b(P^\bullet) = \inf\{i \in \mathbb{Z} \mid \mathrm{H}^{-i}(\mathrm{Hom}^\bullet_A(P^\bullet, A)) \ne 0\}.$$

Then for any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $P^\bullet \ne 0$ in $\mathcal{K}(A)$ we set $l(P^\bullet) = a(P^\bullet) - b(P^\bullet)$.

Recall that an idempotent $e \in A$ is said to be primitive if eA is an indecomposable A-module and to be local if $eAe \cong \mathrm{End}_A(eA)$ is a local ring. Then a ring A is said to be semiperfect if $1 = e_1 + \cdots + e_n$ in A with the e_i orthogonal local idempotents (cf. [?]).

Lemma 1.9. *Assume R is a complete local ring. Then A is semiperfect and the Krull-Schmidt theorem holds in* mod-A*, i.e., for any nonzero $X \in$* mod-A *the following hold.*

(1) X decomposes into a direct sum of indecomposable submodules.

(2) X is indecomposable if and only if $\mathrm{End}_A(X)$ is local.

Proof. This is well known but for the benefit of the reader we include a proof. Let $\mathfrak{m}$ be the maximal ideal of R and I an injective envelope of $R/\mathfrak{m}$ in Mod-R. Since A is right noetherian, $A = e_1A \oplus \cdots \oplus e_nA$ with the e_i orthogonal primitive idempotents. Furthermore, every $\mathrm{Hom}_R(e_iA, I) \in \text{Mod-}A^{\mathrm{op}}$ is indecomposable injective and hence $e_iAe_i \cong \mathrm{End}_A(e_iA) \cong \mathrm{End}_{A^{\mathrm{op}}}(\mathrm{Hom}_R(e_iA, I))^{\mathrm{op}}$ is local. Next, for any nonzero $X \in$ mod-A, $\mathrm{End}_A(X)$ is a noether R-algebra and hence is semiperfect. The last assertion follows. □

2 Nakayama functor

In the following, we set $\nu = D \circ \mathrm{Hom}_A(-, A)$ which we call the Nakayama functor for A. Note that for any $P \in \mathcal{P}_A$ we have a functorial isomorphism in Mod-A

$$P \otimes_A DA \xrightarrow{\sim} \nu P, x \otimes h \mapsto (g \mapsto h(g(x))).$$

Lemma 2.1. *For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathcal{K}(A)$ we have a bifunctorial isomorphism of complexes*

$$D\mathrm{Hom}^\bullet_A(P^\bullet, Q^\bullet) \cong \mathrm{Hom}^\bullet_A(Q^\bullet, \nu P^\bullet).$$

Proof. For any $P \in \mathcal{P}_A$ and $Q \in$ Mod-A, we have a bifunctorial isomorphism

$$Q \otimes_A \mathrm{Hom}_A(P, A) \xrightarrow{\sim} \mathrm{Hom}_A(P, Q), x \otimes h \mapsto (a \mapsto xh(a))$$

and hence bifunctorial isomorphisms

$$\begin{aligned} D\mathrm{Hom}_A(P, Q) &\cong D(Q \otimes_A \mathrm{Hom}_A(P, A)) \\ &\cong \mathrm{Hom}_A(Q, \nu P). \end{aligned}$$

It is obvious that the bifunctorial isomorphism

$$D\mathrm{Hom}_A(P, Q) \cong \mathrm{Hom}_A(Q, \nu P)$$

for $P \in \mathcal{P}_A$ and $Q \in$ Mod-A can be extended to a bifunctorial isomorphism of complexes

$$D\mathrm{Hom}^\bullet_A(P^\bullet, Q^\bullet) \cong \mathrm{Hom}^\bullet_A(Q^\bullet, \nu P^\bullet)$$

for $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathcal{K}(A)$. □

Lemma 2.2. *For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathcal{K}(A)$ we have a bifunctorial homomorphism*

$$\xi_{P^\bullet, Q^\bullet} : \mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, \nu P^\bullet) \to D\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, Q^\bullet).$$

Furthermore, in case $Q^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ and $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i > 0$, the following hold.

(1) $\xi_{P^\bullet, Q^\bullet}$ is monic if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i < a(Q^\bullet) - b(P^\bullet)$.

(2) $\xi_{P^\bullet, Q^\bullet}$ is an isomorphism if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i \leq a(Q^\bullet) - b(P^\bullet)$.

Proof. Set $X^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, Q^\bullet) \in \mathcal{K}(R)$. Then $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, Q^\bullet) \cong \mathrm{H}^0(X^\bullet)$ and by Lemma **??** $\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, \nu P^\bullet) \cong \mathrm{H}^0(DX^\bullet)$. Thus the functorial homomorphism $\xi_{X^\bullet} : \mathrm{H}^0(DX^\bullet) \to D\mathrm{H}^0(X^\bullet)$ in Lemma **??** yields a desired bifunctorial homomorphism. Next, assume $Q^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ and $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i > 0$. Set $l = a(Q^\bullet) - b(P^\bullet)$. By Remark **??**, we may assume $X^i = 0$ for $i > l$. In case $l \leq 0$, we have $\mathrm{B}'^0(X^\bullet) = 0$ and $\mathrm{B}^0(DX^\bullet) = 0$. Assume $l \geq 1$. Then, since $\mathrm{H}^i(X^\bullet) = 0$ for $i > 0$, we have an exact sequence

$$0 \to \mathrm{B}'^0(X^\bullet) \to X^1 \to \cdots \to X^l \to 0$$

with $X^i \in \mathrm{add}(A_R)$ for all $1 \leq i \leq l$. Thus, if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i < l$, then $\mathrm{B}^0(DX^\bullet) \xrightarrow{\sim} D\mathrm{B}'^0(X^\bullet)$ canonically. Furthermore, if $\mathrm{Ext}^i_R(A, R) = 0$ for $1 \leq i \leq l$, then $\mathrm{Ext}^1_R(\mathrm{B}'^0(X^\bullet), R) = 0$. The last assertions now follow by Lemma **??**. □

Corollary 2.3. *Assume $\mathrm{Ext}^i_A(A, R) = 0$ for $i > 0$. Then for any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[i]) = 0$ for $i > 0$ we have $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, \nu P^\bullet[i]) = 0$ for $i < 0$.*

Proof. For any $i < 0$, since $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[-i+j]) = 0$ for $j > 0$, by applying Lemma **??**(2) to $Q^\bullet = P^\bullet[-i]$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, \nu P^\bullet[i]) &\cong \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet[-i], \nu P^\bullet) \\ &\cong D\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[-i]) \\ &= 0. \end{aligned}$$

□

In the following, for a complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we always define $\mathrm{add}(P^\bullet)$ as a full subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$. Note however that the canonical functor $\mathcal{K}(A) \to \mathcal{D}(A)$ induces an equivalence between $\mathrm{add}(P^\bullet)$ defined in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $\mathrm{add}(P^\bullet)$ defined in $\mathcal{D}(A)$ (cf. [**?**, Remark 1.7]).

Definition 2.4 ([?]). A complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is said to be a tilting complex if the following conditions are satisfied:

(1) $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$; and

(2) $\operatorname{add}(P^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category, i.e., a full triangulated subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ coincides with $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ if it contains $\operatorname{add}(P^\bullet)$ and is closed under isomorphisms.

Remark 2.5 *([**?**, Proposition 5.4]).* Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\operatorname{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Then $P^\bullet$ is a tilting complex if and only if for any $X^\bullet \in \mathcal{D}^-(A)$ with $\operatorname{Hom}_{\mathcal{D}(A)}(P^\bullet, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$ we have $X^\bullet = 0$ in $\mathcal{D}(A)$.

Definition 2.6. For any $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we denote by $\mathcal{S}(P^\bullet)$ the full subcategory of $\mathcal{D}^-(A)$ consisting of $X^\bullet \in \mathcal{D}^-(A)$ with $\operatorname{Hom}_{\mathcal{D}(A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$.

Proposition 2.7 ([?]). *Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex and $B = \operatorname{End}_{\mathcal{K}(A)}(P^\bullet)$. Then there exists an equivalence of triangulated categories*

$$F^* : \mathcal{D}^-(A) \xrightarrow{\sim} \mathcal{D}^-(B)$$

such that $F^*(X^\bullet) \cong \operatorname{Hom}_{\mathcal{D}(A)}(P^\bullet, X^\bullet)$ in $\mathcal{D}(B)$ for all $X^\bullet \in \mathcal{S}(P^\bullet)$. In particular, we have an equivalence

$$\operatorname{Hom}_{\mathcal{D}(A)}(P^\bullet, -) : \mathcal{S}(P^\bullet) \xrightarrow{\sim} \text{Mod-}B$$

Proof. See [**?**, Section 4] for the first assertion. Then, since $F^*(P^\bullet) \cong B$ in $\mathcal{D}(B)$, F^* induces an equivalence $\mathcal{S}(P^\bullet) \xrightarrow{\sim} \mathcal{S}(B)$. Note also that we have an equivalence $\text{Mod-}B \xrightarrow{\sim} \mathcal{S}(B)$. Thus the last assertion follows (cf. [**?**, Theorem 1.3(3)]). □

In the following, we use the notation A_A (resp., ${}_AA$) to stress that A is considered as a right (resp., left) A-module. Then the notation $D(A_A)$ (resp., $D({}_AA)$) is used to stress that DA is considered as a left (resp., right) A-module. Note that $\nu(A_A) \cong D({}_AA)$ and $\mathcal{P}_A = \operatorname{add}(A_A)$.

Lemma 2.8. *Assume A is reflexive as an R-module and $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$. Then we have an equivalence $\nu : \mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_A$. In particular, for any tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, $\nu P^\bullet$ is also a tilting complex and the following are equivalent.*

(1) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ and $P^\bullet \in \mathcal{S}(\nu P^\bullet)$.

(2) $\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.

Proof. We have an anti-equivalence $\operatorname{Hom}_A(-, A) : \mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_{A^{\mathrm{op}}}$. Also, since A is reflexive as an R-module, we have an anti-equivalence $D : \mathcal{P}_{A^{\mathrm{op}}} \xrightarrow{\sim} \operatorname{add}(D({}_AA))$. Thus, since $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$, we have an equivalence $\nu : \mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_A$ which is extended to an equivalence of triangulated categories $\nu : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, so that $\nu P^\bullet$ is a tilting complex.

(1) $\Rightarrow$ (2). We have $\operatorname{Hom}_{\mathcal{K}(A)}(P^\bullet \oplus \nu P^\bullet, (P^\bullet \oplus \nu P^\bullet)[i]) = 0$ for $i \neq 0$ and hence by [**?**, Lemma 1.8] $\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.

(2) $\Rightarrow$ (1). Obvious. □

Lemma 2.9. *Assume $A \cong DA$ in $\text{Mod-}A^{\mathrm{e}}$. Then the following hold.*

(1) For any $P^\bullet \in \mathcal{K}(\mathcal{P}_A)$ we have a functorial isomorphism of complexes $P^\bullet \cong \nu P^\bullet$.

(2) $A \in \mathcal{G}_R$ as an R-module if and only if $\mathrm{Ext}_R^i(A, R) = 0$ for $i > 0$.

Proof. (1) Fix an isomorphism $A \xrightarrow{\sim} DA$ in Mod-A^{e}. Then we have functorial isomorphisms of complexes $P^\bullet \cong P^\bullet \otimes_A^\bullet A \cong P^\bullet \otimes_A^\bullet DA \cong \nu P^\bullet$.

(2) For any $X, Y \in$ Mod-A^{e} we have a bifunctorial isomorphism

$$\theta_{X,Y} : \mathrm{Hom}_{A^{\mathrm{e}}}(X, DY) \xrightarrow{\sim} \mathrm{Hom}_{A^{\mathrm{e}}}(Y, DX), h \mapsto Dh \circ \varepsilon_Y.$$

We claim that $\theta_{A,A} = \mathrm{id}_{\mathrm{Hom}_{A^{\mathrm{e}}}(A,DA)}$. Let $h \in \mathrm{Hom}_{A^{\mathrm{e}}}(A, DA)$ and $a, b \in A$. Then $h(a)(b) = (h(1)a)(b) = h(1)(ab)$ and $h(b)(a) = (bh(1))(a) = h(1)(ab)$, so that $(\theta_{A,A}(h)(a))(b) = \varepsilon_A(a)(h(b)) = h(b)(a) = h(a)(b)$. It follows that $\theta_{A,A}(h) = h$. Since $Dh \circ \varepsilon_A = h$, if h is an isomorphism, so is ε_A. Thus A is reflexive as an R-module and the assertion follows. □

Proposition 2.10. *Assume $A \cong DA$ in* Mod-A^{e} *and $A \in \mathcal{G}_R$ as an R-module. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $B = \mathrm{End}_{\mathcal{K}(A)}(P^\bullet)$. Then $B \cong DB$ in* Mod-B^{e}.

Proof. By Lemmas **??**(2), **??**(1) we have isomorphisms in Mod-B^{e}

$$\begin{aligned} DB &= D\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, \nu P^\bullet) \\ &\cong \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet) \\ &= B. \end{aligned}$$

□

3 Gorenstein algebras

In this section, we introduce the notion of Gorenstein R-algebras over a Gorenstein ring R. We refer to [**?**] for the definition and basic properties of commutative Gorenstein rings.

We denote by $\dim R$ the Krull dimension of R, by $\mathrm{Spec}(R)$ the set of prime ideals in R and by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p} \in \mathrm{Spec}(R)$. Also, for a module $X \in$ Mod-R we denote by $\mathrm{Supp}(X)$ the subset of $\mathrm{Spec}(R)$ consisting of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $X_\mathfrak{p} \neq 0$. Note that we do not exclude the case where $\mathrm{Supp}(A) \neq \mathrm{Spec}(R)$, i.e., the kernel of the structure ring homomorphism $R \to A$ may not be nilpotent.

Definition 3.1. Assume R is a Gorenstein ring. Then A is said to be a Gorenstein R-algebra if $A \in \mathcal{G}_R$ as an R-module and $\mathrm{add}(D({}_A A)) = \mathcal{P}_A$.

In the rest of this section, we provide several basic properties of Gorenstein R-algebras. Especially, we will see that our Gorenstein R-algebras are Gorenstein algebras in the sense of [**?**]. However, unless otherwise stated, R is assumed to be an arbitrary commutative noetherian ring.

Remark 3.2. Assume A is reflexive as an R-module. Then the following hold.

(1) $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$ if and only if $\mathrm{add}(D(A_A)) = \mathcal{P}_{A^{\mathrm{op}}}$.

(2) In case R is a complete local ring, $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$ if either $A_A \in \mathrm{add}(D({}_AA))$ or $D({}_AA) \in \mathcal{P}_A$.

Proof. (1) Obvious.

(2) It follows by Lemma **??** that $A = e_1A \oplus \cdots \oplus e_nA$ with the e_i orthogonal local idempotents and every indecomposable module in $\mathcal{P}_A$ is isomorphic to some e_iA. In particular, $\mathcal{P}_A$ contains only a finite number of nonisomorphic indecomposable modules. Also, as remarked in the proof of Lemma **??**, we have an equivalence $\nu : \mathcal{P}_A \xrightarrow{\sim} \mathrm{add}(D({}_AA))$. Thus $\mathcal{P}_A$ and $\mathrm{add}(D({}_AA))$ contain the same number of nonisomorphic indecomposable modules and the assertion follows. □

Lemma 3.3. *The following hold.*

(1) If $I \in \mathrm{Mod}\text{-}R$ is injective, so is $\mathrm{Hom}_R(A, I) \in \mathrm{Mod}\text{-}A$.

(2) Let $\mathfrak{p} \in \mathrm{Supp}(A)$ and $X \in \mathrm{Mod}\text{-}A_{\mathfrak{p}}$. Then $X \in \mathrm{Mod}\text{-}A_{\mathfrak{p}}$ is flat if and only if so is $X \in \mathrm{Mod}\text{-}A$.

Proof. (1) Obvious.

(2) The "only if" part follows by the flatness of $A_{\mathfrak{p}}$ as an A-module and the "if" part follows by the fact that $A_{\mathfrak{p}} \otimes_A A_{\mathfrak{p}} \xrightarrow{\sim} A_{\mathfrak{p}}$ canonically. □

Lemma 3.4. *Assume $\mathrm{Ext}^i_R(A, R) = 0$ for $i > 0$. Then the following hold.*

(1) For an injective resolution $R \to I^\bullet$ in $\mathrm{Mod}\text{-}R$, we have an injective resolution $DA \to \mathrm{Hom}^\bullet_R(A, I^\bullet)$ in $\mathrm{Mod}\text{-}A$. In particular, we have $\mathrm{inj\ dim}\ D({}_AA) \leq \mathrm{inj\ dim}\ R_R$.

(2) For any $X \in \mathrm{Mod}\text{-}A$, we have $\mathrm{Ext}^i_A(X, DA) \cong \mathrm{Ext}^i_R(X, R)$ for all $i \geq 0$.

(3) If R is a Gorenstein ring, then for any $X \in \mathrm{mod}\text{-}A$, $X \in \mathcal{G}_R$ as an R-module if and only if $\mathrm{Ext}^i_A(X, DA) = 0$ for $i > 0$.

(4) If R is a Gorenstein ring with $\dim R = \dim R_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p} \in \mathrm{Spec}(R)$, then $\mathrm{inj\ dim}\ D({}_AA) = \dim R$.

Proof. (1) follows by Lemma **??**(1).

(2) Take an injective resolution $R \to I^\bullet$ in $\mathrm{Mod}\text{-}R$. Then by (1) for any $i \geq 0$ we have

$$\begin{aligned}\mathrm{Ext}^i_A(X, DA) &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_A(X, \mathrm{Hom}^\bullet_R(A, I^\bullet))) \\ &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_R(X, I^\bullet)) \\ &\cong \mathrm{Ext}^i_R(X, R).\end{aligned}$$

(3) The "only if" part follows by (2). Assume $\mathrm{Ext}^i_A(X, DA) = 0$ for $i > 0$. Then by (2) $\mathrm{Ext}^i_R(X, R) = 0$ for $i > 0$. Take a projective resolution $P^\bullet \to X$ in mod-R and set $Q^\bullet = \mathrm{Hom}^\bullet_R(P^\bullet, R) \in \mathcal{K}^+(\mathcal{P}_R)$. We have only to show that $\mathrm{Ext}^i_R(\mathrm{Z}'^1(Q^\bullet), R) = 0$ for $i > 0$ (see [?, Proposition 3.8]). Note that $\mathrm{H}^i(Q^\bullet) = 0$ for $i > 0$. Thus for any $i > 0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$ we have

$$\begin{aligned}\mathrm{Ext}^i_R(\mathrm{Z}'^1(Q^\bullet), R)_\mathfrak{p} &\cong \mathrm{Ext}^{i+j}_R(\mathrm{Z}'^{1+j}(Q^\bullet), R)_\mathfrak{p}\\ &\cong \mathrm{Ext}^{i+j}_{R_\mathfrak{p}}(\mathrm{Z}'^{1+j}(Q^\bullet)_\mathfrak{p}, R_\mathfrak{p})\\ &= 0\end{aligned}$$

for $j \geq \dim R_\mathfrak{p}$. Thus $\mathrm{Ext}^i_R(\mathrm{Z}'^1(Q^\bullet), R) = 0$ for $i > 0$.

(4) Take a maximal ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ with $R/\mathfrak{p} \otimes_R A \neq 0$. Let $d = \dim R_\mathfrak{p} = \dim R$. Note that $d < \infty$. Then, since $R/\mathfrak{p} \otimes_R A$ is a finite direct sum of copies of $R/\mathfrak{p}$ in Mod-R, and since $\mathrm{Ext}^d_R(R/\mathfrak{p}, R) \neq 0$, we have $\mathrm{Ext}^d_R(R/\mathfrak{p} \otimes_R A, R) \neq 0$ and hence by (2) $\mathrm{Ext}^d_A(R/\mathfrak{p} \otimes_R A, DA) \neq 0$. The assertion follows by (1). □

Definition 3.5 (cf. [?]). A left and right noetherian ring A is said to satisfy the Auslander condition if it admits an injective resolution $A \to E^\bullet$ in Mod-A such that flat dim $E^n \leq n$ for all $n \geq 0$.

Proposition 3.6. *Assume R is a Gorenstein ring, $A \in \mathcal{G}_R$ as an R-module and ${}_AA \in \mathrm{add}(D(A_A))$. Then the following hold.*

(1) inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} \leq \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Supp}(A)$.

(2) For any $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$ with $P^\bullet \neq 0$ in $\mathcal{K}(A)$ we have $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, A[i]) \neq 0$ for some $i \in \mathbb{Z}$.

(3) A satisfies the Auslander condition.

Proof. (1) Note that $\mathrm{Ext}^i_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p}) \cong \mathrm{Ext}^i_R(A, R)_\mathfrak{p} = 0$ for $i > 0$ and $D(A_A)_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})$ in Mod-$A_\mathfrak{p}^{\mathrm{op}}$. Thus we can apply Lemma **??**(1) to $A_\mathfrak{p}^{\mathrm{op}}$ to conclude that inj dim $D(A_A)_\mathfrak{p} \leq \dim R_\mathfrak{p}$ as a left $A_\mathfrak{p}$-module. Then, since ${}_AA \in \mathrm{add}(D(A_A))$, we have ${}_{A_\mathfrak{p}}A_\mathfrak{p} \in \mathrm{add}(D(A_A)_\mathfrak{p})$ and hence inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} \leq \dim R_\mathfrak{p}$.

(2) Let $P^\bullet \in \mathcal{K}^-(\mathcal{P}_A)$. Set $Q^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, A) \in \mathcal{K}^+(\mathcal{P}_{A^{\mathrm{op}}})$ and assume $\mathrm{H}^i(Q^\bullet) \cong \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, A[i]) = 0$ for all $i \in \mathbb{Z}$. We claim that $P^\bullet = 0$ in $\mathcal{K}(A)$. It suffices to show that $\mathrm{H}^i(P^\bullet)_\mathfrak{p} = 0$ for all $i \in \mathbb{Z}$ and $\mathfrak{p} \in \mathrm{Spec}(R)$. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. For any $X \in$ mod-A^{op} we have a functorial isomorphism

$$\mathrm{Hom}_{A^{\mathrm{op}}}(X, A)_\mathfrak{p} \xrightarrow{\sim} \mathrm{Hom}_{A_\mathfrak{p}^{\mathrm{op}}}(X_\mathfrak{p}, A_\mathfrak{p}).$$

Thus for any $i \in \mathbb{Z}$ we have

$$\begin{aligned}\mathrm{H}^i(P^\bullet)_\mathfrak{p} &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(Q^\bullet, A))_\mathfrak{p}\\ &\cong \mathrm{Hom}_{\mathcal{K}(A^{\mathrm{op}})}(Q^\bullet, A[i])_\mathfrak{p}\\ &\cong \mathrm{Ext}^j_{A^{\mathrm{op}}}(\mathrm{Z}'^{-i+j}(Q^\bullet), A)_\mathfrak{p}\\ &\cong \mathrm{Ext}^j_{A_\mathfrak{p}^{\mathrm{op}}}(\mathrm{Z}'^{-i+j}(Q^\bullet)_\mathfrak{p}, A_\mathfrak{p})\end{aligned}$$

for all $j > 0$. It follows by (1) that $\mathrm{H}^i(P^\bullet)_\mathfrak{p} = 0$.

(3) By Lemma **??**(1), it suffices to show that flat dim $\mathrm{Hom}_R(A, E(R/\mathfrak{p}))_A \le$ dim $R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, where $E(R/\mathfrak{p})$ denotes an injective envelope of $R/\mathfrak{p}$ in Mod-R. Note that $E(R/\mathfrak{p}) \in$ Mod-$R_\mathfrak{p}$ and hence $\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, E(R/\mathfrak{p})) \cong \mathrm{Hom}_R(A, E(R/\mathfrak{p}))$ in Mod-$A_\mathfrak{p}$. Thus we may assume $\mathfrak{p} \in \mathrm{Supp}(A)$ and by Lemma **??**(2) we have

$$\text{flat dim } \mathrm{Hom}_R(A, E(R/\mathfrak{p}))_A = \text{flat dim } \mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, E(R/\mathfrak{p}))_{A_\mathfrak{p}}.$$

On the other hand, since by (1) inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} \le$ dim $R_\mathfrak{p}$, for any $i >$ dim $R_\mathfrak{p}$ and $X \in$ mod-$A_\mathfrak{p}^{\mathrm{op}}$ we have

$$\begin{aligned}\mathrm{Tor}_i^{A_\mathfrak{p}}(\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, E(R/\mathfrak{p})), X) &\cong \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{Ext}^i_{A_\mathfrak{p}^{\mathrm{op}}}(X, A_\mathfrak{p}), E(R/\mathfrak{p})) \\ &= 0\end{aligned}$$

and hence flat dim $\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, E(R/\mathfrak{p}))_{A_\mathfrak{p}} \le$ dim $R_\mathfrak{p}$. □

Proposition 3.7. *Assume R is a Gorenstein ring and A is a Gorenstein R-algebra. Then for any $\mathfrak{p} \in \mathrm{Supp}(A)$ the following hold.*

(1) $A_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-algebra.

(2) $A_\mathfrak{p}$ is maximal Cohen-Macaulay as an $R_\mathfrak{p}$-module.

(3) inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} =$ inj dim ${A_\mathfrak{p}}_{A_\mathfrak{p}} =$ dim $R_\mathfrak{p}$.

Proof. (1) Note that $D({}_AA)_\mathfrak{p} \cong \mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})$ in Mod-$A_\mathfrak{p}$. Thus we have $\mathrm{add}(\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})_{A_\mathfrak{p}}) = \mathcal{P}_{A_\mathfrak{p}}$. Also, $\mathrm{Ext}^i_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p}) \cong \mathrm{Ext}^i_R(A, R)_\mathfrak{p} = 0$ for $i > 0$. Thus by Lemma **??**(3) $A_\mathfrak{p} \in \mathcal{G}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module.

(2) Note that by (1) $A_\mathfrak{p} \in \mathcal{G}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module. Take a projective resolution $P^\bullet \to \mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})$ in mod-$R_\mathfrak{p}$ and set $Q^\bullet = \mathrm{Hom}^\bullet_{R_\mathfrak{p}}(P^\bullet, R_\mathfrak{p}) \in \mathcal{K}^+(\mathcal{P}_{R_\mathfrak{p}})$. Then we have an exact sequence in mod-$R_\mathfrak{p}$

$$0 \to A_\mathfrak{p} \to Q^0 \to Q^1 \to \cdots$$

and the assretion follows.

(3) By Lemma **??**(4) inj dim $\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})_{A_\mathfrak{p}} =$ dim $R_\mathfrak{p}$. Thus, since by (1) $\mathrm{add}(\mathrm{Hom}_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p})_{A_\mathfrak{p}}) = \mathcal{P}_{A_\mathfrak{p}}$, inj dim ${A_\mathfrak{p}}_{A_\mathfrak{p}} =$ dim $R_\mathfrak{p}$. By symmetry, we also have inj dim ${}_{A_\mathfrak{p}}A_\mathfrak{p} =$ dim $R_\mathfrak{p}$. □

Assume R is a Gorenstein ring and A is a Gorenstein R-algebra. It then follows by (2), (3) of Proposition **??** that A is a Gorenstein algebra in the sense of [**?**]. So we refer to [**?**] for further properties enjoyed by A and for the relationship of the notion of Gorenstein algebras to the theory of commutative Gorenstein rings. Also, in case R is a semilocal ring with dim $R =$ dim $R_\mathfrak{p}$ for all maximal ideals $\mathfrak{p} \in \mathrm{Spec}(R)$ and $A \cong DA$ in Mod-A^{e}, it follows by Proposition **??**(2) that A is a Gorenstein R-order in the sense of [**?**].

There is another notion of Gorenstein algebras. Consider the case where

R is an artinian Gorenstein ring. Then an R-algebra A is sometimes called a Gorenstein algebra if inj dim ${}_AA =$ inj dim $A_A < \infty$ (see e.g. [?]). It follows by [?, Proposition 1.6] that an R-algebra A is a Gorenstein algebra in this sense if and only if $D({}_AA)$ is a tilting module. In the following, we will extend this fact to the case where R is a Gorenstein ring with dim $R < \infty$.

Definition 3.8. A module $T \in \text{Mod-}A$ is said to be a tilting module if there exists a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $P^\bullet \cong T$ in $\mathcal{D}(A)$, i.e., $\mathrm{H}^i(P^\bullet) = 0$ for $i \neq 0$ and $\mathrm{H}^0(P^\bullet) \cong T$ in Mod-A.

Proposition 3.9 (cf. [?]). *A module $T \in \text{Mod-}A$ is a tilting module if and only if the following conditions are satisfied:*

(1) $\mathrm{Ext}^i_A(T, T) = 0$ *for* $i > 0$;

(2) there exists an exact sequence $0 \to P^{-l} \to \cdots \to P^0 \to T \to 0$ *in* Mod-A *with* $P^{-i} \in \mathcal{P}_A$ *for all* $0 \le i \le l$; *and*

(3) there exists an exact sequence $0 \to A \to T^0 \to \cdots \to T^m \to 0$ *in* Mod-A *with* $T^i \in \mathrm{add}(T)$ *for all* $0 \le i \le m$.

Proof. This is well known but for the benefit of the reader we include a proof.

"If" part. By the condition (2) we have a projective resolution $P^\bullet \to T$ in Mod-A with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$. Then $P^\bullet \cong T$ in $\mathcal{D}(A)$ and by the condition (1) $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Finally, for any $X^\bullet \in \mathcal{D}^-(A)$ with $\mathrm{Hom}_{\mathcal{D}(A)}(P^\bullet, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$, by the condition (3) we have $\mathrm{H}^i(X^\bullet) \cong \mathrm{Hom}_{\mathcal{D}(A)}(A, X^\bullet[i]) = 0$ for all $i \in \mathbb{Z}$ and hence $X^\bullet = 0$ in $\mathcal{D}(A)$. Thus by Remark **??** $P^\bullet$ is a tilting complex.

"Only if" part. According to Remark **??**, we have a projective resolution $P^\bullet \to T$ in Mod-A with $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ a tilting complex. Thus the conditions (1), (2) are satisfied. Let $B = \mathrm{End}_A(T)$. Then $\mathrm{End}_{\mathcal{K}(A)}(P^\bullet) \cong B$ and there exists an equivalence of triangulated categories $F : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $F(B) \cong P^\bullet$. Let $F^* : \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ be a quasi-inverse of F. Then $Q^\bullet = \mathrm{Hom}^\bullet_B(F^*(A), B) \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_{B^{\mathrm{op}}})$ is a tilting complex with $\mathrm{End}_{\mathcal{K}(B^{\mathrm{op}})}(Q^\bullet) \cong A^{\mathrm{op}}$. Also, by Lemma **??** $Q^\bullet$ is a projective resolution of T in Mod-B^{op}. Thus $\mathrm{End}_{B^{\mathrm{op}}}(T) \cong A^{\mathrm{op}}$ and we have a right resolution $A \to \mathrm{Hom}^\bullet_{B^{\mathrm{op}}}(Q^\bullet, T)$ in Mod-A. Since every $\mathrm{Hom}_{B^{\mathrm{op}}}(Q^i, T)$ belongs to $\mathrm{add}(T_A)$, the condition (3) is satisfied. □

Proposition 3.10. *Assume R is a Gorenstein ring with* dim $R < \infty$ *and $A \in \mathcal{G}_R$ as an R-module. Then the following hold.*

(1) proj dim $D({}_AA) < \infty$ *if and only if* inj dim ${}_AA < \infty$.

(2) $D({}_AA)$ is a tilting module if and only if inj dim ${}_AA =$ inj dim $A_A < \infty$.

(3) If $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$, *then* inj dim ${}_AA =$ inj dim $A_A \le$ dim R.

Proof. (1) "If" part. For any injective $I \in \text{Mod-}R$ and any $X \in \text{mod-}A^{\text{op}}$ we have

$$\text{Tor}_i^A(\text{Hom}_R(A, I), X) \cong \text{Hom}_R(\text{Ext}^i_{A^{\text{op}}}(X, A), I)$$

for all $i \geq 0$ and hence flat dim $\text{Hom}_R(A, I)_A < \infty$. Then by Lemma **??**(1) flat dim $D({}_AA) < \infty$. Finally, since $D({}_AA) \in \text{mod-}A$, flat dim $D({}_AA) =$ proj dim $D({}_AA)$.

"Only if" part. Take a projective resolution $P^\bullet \to DA$ in Mod-A with $P^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{P}_A)$. Then we have a right resolution $A \to DP^\bullet$ in Mod-A^{op}. Since by applying Lemma **??**(1) to A^{op} we have inj dim $D(A_A) < \infty$, and since every term of $DP^\bullet$ belongs to $\text{add}(D(A_A))$, it follows that inj dim ${}_AA < \infty$.

(2) "If" part. By applying (1) to both A and A^{op} we have proj dim $D({}_AA) < \infty$ and proj dim $D(A_A) < \infty$. Also, by applying Lemma **??**(2) to both A and A^{op} we have $\text{Ext}^i_A(DA, DA) = \text{Ext}^i_{A^{\text{op}}}(DA, DA) = 0$ for $i > 0$. Since $A \xrightarrow{\sim} \text{End}_A(DA)$ and $A \xrightarrow{\sim} \text{End}_{A^{\text{op}}}(DA)^{\text{op}}$ canonically, the assertion follows by [**?**, Proposition 1.6].

"Only if" part. Since $A \xrightarrow{\sim} \text{End}_A(DA)$ canonically, it follows by [**?**, Theorem 1.5] that $D(A_A)$ is also a tilting module. Thus by applying (1) to both A and A^{op} we have inj dim ${}_AA < \infty$ and inj dim $A_A < \infty$. The assertion follows by [**?**, Lemma A].

(3) By Lemma **??**(1) inj dim $D({}_AA) \leq \dim R$ and, since $A_A \in \text{add}(D({}_AA))$, inj dim $A_A \leq \dim R$. By symmetry, we also have inj dim ${}_AA \leq \dim R$. The assertion follows by [**?**, Lemma A]. □

4 Derived equivalences in Gorenstein algebras

In this section, for a tilting complex $P^\bullet$ over a Gorenstein R-algebra A we ask when $B = \text{End}_{\mathcal{K}(A)}(P^\bullet)$ is also a Gorenstein R-algebra. This question does not seem to depend on the base ring R. So, unless otherwise stated, we assume R is an arbitrary commutative noetherian ring.

We fix a complex $P^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{P}_A)$ such that $P^\bullet \neq 0$ in $\mathcal{K}(A)$ and $\text{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Set $B = \text{End}_{\mathcal{K}(A)}(P^\bullet)$ and $X^\bullet = \text{Hom}^\bullet_A(P^\bullet, P^\bullet) \in \mathcal{K}^{\text{b}}(R)$. Note that $X^i \in \text{add}(A_R)$ for all $i \in \mathbb{Z}$. Since $\text{H}^i(X^\bullet) = 0$ for $i \neq 0$, we have exact sequences in mod-R of the form

$$(*) \qquad 0 \to \text{Z}^0(X^\bullet) \to X^0 \to \cdots \to X^l \to 0,$$

$$(**) \quad 0 \to X^{-l} \to \cdots \to X^{-1} \to \text{Z}^0(X^\bullet) \to B \to 0.$$

Lemma 4.1. *The following hold.*

(1) Assume $\text{Ext}^i_R(A, R) = 0$ *for* $i > 0$. *Then* $\text{Ext}^i_R(B, R) = 0$ *for* $i > 0$ *if and only if* $\nu P^\bullet \in \mathcal{S}(P^\bullet)$.

(2) Assume $A \in \mathcal{G}_R$ *as an* R*-module. Then* $B \in \mathcal{G}_R$ *as an* R*-module if and only if* $\nu P^\bullet \in \mathcal{S}(P^\bullet)$.

(3) Assume $A \in \mathcal{P}_R$ as an R-module. Then $B \in \mathcal{P}_R$ as an R-module if and only if $\nu P^\bullet \in \mathcal{S}(P^\bullet)$.

Proof. The "only if" parts of (2), (3) follow by (1).

(1) Apply D to $(*)$. Then $DX^0 \to D\mathrm{Z}^0(X^\bullet)$ is epic and $\mathrm{Ext}^i_R(\mathrm{Z}^0(X^\bullet), R) = 0$ for $i > 0$. Next, apply D to $(**)$. Then

$$\begin{aligned} \mathrm{Ext}^1_R(B, R) &\cong \mathrm{Cok}(D\mathrm{Z}^0(X^\bullet) \to D\mathrm{B}^0(X^\bullet)) \\ &\cong \mathrm{Cok}(DX^0 \to D\mathrm{B}^0(X^\bullet)) \\ &\cong \mathrm{H}^1(DX^\bullet) \end{aligned}$$

and $\mathrm{Ext}^i_R(B, R) \cong \mathrm{Ext}^{i-1}_R(\mathrm{B}^0(X^\bullet), R) \cong \mathrm{H}^i(DX^\bullet)$ for $i > 1$. Since by Lemma **??**

$$\begin{aligned} \mathrm{H}^i(DX^\bullet) &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_A(P^\bullet, \nu P^\bullet)) \\ &\cong \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, \nu P^\bullet[i]) \end{aligned}$$

for all $i \in \mathbb{Z}$, and since by Corollary **??** $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, \nu P^\bullet[i]) = 0$ for $i < 0$, the assertion follows.

(2) "If" part. Note that $X^i \in \mathcal{G}_R$ for all $i \in \mathbb{Z}$. Applying Lemma **??**(1) successively to $(*)$, we conclude that $\mathrm{Z}^0(X^\bullet) \in \mathcal{G}_R$. Next, since by (1) $\mathrm{Ext}^i_R(B, R) = 0$ for $i > 0$, by applying Lemma **??**(2) successively to $(**)$, we conclude that $B \in \mathcal{G}_R$ as an R-module.

(3) "If" part. By $(*)$ we have $\mathrm{Z}^0(X^\bullet) \in \mathcal{P}_R$. Since by (1) $\mathrm{Ext}^i_R(B, R) = 0$ for $i > 0$, it follows by $(**)$ that $B \in \mathcal{P}_R$ as an R-module. □

Lemma 4.2. *For any $\mathfrak{p} \in \mathrm{Supp}(A)$ with $A_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module the following are equivalent.*

(1) $B_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module.

(2) $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, \nu P^\bullet[i])_\mathfrak{p} = 0$ for $i \neq 0$, this is the case if $\nu P^\bullet \in \mathcal{S}(P^\bullet)$.

Proof. For any $X \in \text{mod-}A$ and $Y \in \text{Mod-}A$ we have a bifunctorial isomorphism

$$\mathrm{Hom}_A(X, Y)_\mathfrak{p} \xrightarrow{\sim} \mathrm{Hom}_{A_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}).$$

Also, for any $X \in \text{mod-}A$ we have functorial isomorphisms in $\text{Mod-}A_\mathfrak{p}$

$$\begin{aligned} (\nu X)_\mathfrak{p} &\cong \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{Hom}_A(X, A)_\mathfrak{p}, R_\mathfrak{p}) \\ &\cong \mathrm{Hom}_{R_\mathfrak{p}}(\mathrm{Hom}_{A_\mathfrak{p}}(X_\mathfrak{p}, A_\mathfrak{p}), R_\mathfrak{p}). \end{aligned}$$

Thus we can apply Lemma **??**(3) to $P^\bullet \otimes^\bullet_R R_\mathfrak{p} \in \mathcal{K}^\mathrm{b}(\mathcal{P}_{A_\mathfrak{p}})$ (cf. [**?**, Theorem 2.1]). □

Theorem 4.3. *Assume $A \cong DA$ in $\text{Mod-}A^\mathrm{e}$ and $A \in \mathcal{G}_R$ as an R-module. Then the following hold.*

(1) $B \cong DB$ in $\text{Mod-}B^\mathrm{e}$ and $B \in \mathcal{G}_R$ as an R-module.

(2) *If* $A \in \mathcal{P}_R$ *as an* R*-module, then* $B \in \mathcal{P}_R$ *as an* R*-module.*

(3) *For any* $\mathfrak{p} \in \operatorname{Supp}(A)$*, if* $A_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ *as an* $R_\mathfrak{p}$*-module, then* $B_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ *as an* $R_\mathfrak{p}$*-module.*

Proof. By Proposition **??** $B \cong DB$ in Mod-B^{e}. Also, by Lemma **??**(1) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$. The assertions follow by Lemmas **??**(2), **??**(3) and **??**, respectively. □

Throughout the rest of this section, we assume $P^\bullet$ is a tilting complex. Then A, B are derived equivalent and hence there exists a tilting complex $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ such that $A \cong \operatorname{End}_{\mathcal{K}(B)}(Q^\bullet)$.

Remark 4.4. We have $\operatorname{Supp}(A) = \operatorname{Supp}(B)$.

Proof. It follows by $(*)$, $(**)$ that for any $\mathfrak{p} \in \operatorname{Spec}(R)$ with $A_\mathfrak{p} = 0$ we have $B_\mathfrak{p} = 0$. By symmentry, the assertion follows. □

Theorem 4.5. *Assume* $A \in \mathcal{G}_R$ *as an* R*-module and* $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$*. Then the following are equivalent.*

(1) $B \in \mathcal{G}_R$ *as an* R*-module and* $\operatorname{add}(D({}_BB)) = \mathcal{P}_B$.

(2) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ *and* $P^\bullet \in \mathcal{S}(\nu P^\bullet)$.

(3) $\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.

Proof. By Proposition **??** we have an equivalence

$$\operatorname{Hom}_{\mathcal{D}(A)}(P^\bullet, -) : \mathcal{S}(P^\bullet) \xrightarrow{\sim} \text{Mod-}B.$$

Also, by Lemma **??**(2) $\operatorname{Hom}_{\mathcal{D}(A)}(P^\bullet, \nu P^\bullet) \cong DB$ in Mod-B. The assertion follows by Lemmas **??**, **??**(2). □

According to Lemma **??**(3), we can replace $\mathcal{G}_R$ by $\mathcal{P}_R$ in Theorem **??**.

Corollary 4.6. *Assume* $A \in \mathcal{P}_R$ *as an* R*-module and* $\operatorname{add}(D({}_AA)) = \mathcal{P}_A$*. Then the following are equivalent.*

(1) $B \in \mathcal{P}_R$ *as an* R*-module and* $\operatorname{add}(D({}_BB)) = \mathcal{P}_B$.

(2) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ *and* $P^\bullet \in \mathcal{S}(\nu P^\bullet)$.

(3) $\operatorname{add}(P^\bullet) = \operatorname{add}(\nu P^\bullet)$.

Example 4.7. Assume R contains a regular element c which is not a unit. Let

$$A = \begin{pmatrix} R & R \\ cR & R \end{pmatrix}$$

be an R-algebra which is free of rank 4 as an R-module. We construct a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $\nu P^\bullet \notin \mathcal{S}(P^\bullet)$. Set

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $\nu(e_1A) \cong e_2A$ and $\nu(e_2A) \cong e_1A$. In particular, $D({}_AA) \cong A_A$, so that A is a Gorenstein R-algebra if R is a Gorenstein ring. Set $P_1^\bullet = e_1A[1]$ and let $P_2^\bullet$ be the mapping cone of $h : e_1A \to e_2A, x \mapsto ax$. Then $\mathrm{Cok}\ h \cong R/cR$ in Mod-R and $\mathrm{Hom}_R(\mathrm{Cok}\ h, e_1A) = 0$. Thus $\mathrm{Hom}_A(\mathrm{Cok}\ h, e_1A) = 0$ and by [?, Proposition 1.2] $P^\bullet = P_1^\bullet \oplus P_2^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex. On the other hand, $\nu P_2^\bullet$ is isomorphic to the mapping cone of $e_2A \to e_1A, x \mapsto bx$, and hence $\mathrm{Hom}_{\mathcal{K}(A)}(P_1^\bullet, \nu P_2^\bullet[1]) \neq 0$. Thus $\nu P^\bullet \notin \mathcal{S}(P^\bullet)$ and by Lemma **??**(1) $\mathrm{Ext}_R^1(B, R) \neq 0$, where $B = \mathrm{End}_{\mathcal{K}(A)}(P^\bullet)$. More precisely, we have an R-algebra isomorphism

$$B \cong \begin{pmatrix} R & R/cR \\ 0 & R/cR \end{pmatrix}.$$

At present, we do not have any example of tilting complexes $P^\bullet$ over a Gorenstein R-algebra A such that $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ and $\mathrm{add}(P^\bullet) \neq \mathrm{add}(\nu P^\bullet)$. In case R is an artinian Gorenstein ring, it follows by the exactness of D that for any tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ we have $\nu P^\bullet \in \mathcal{S}(P^\bullet)$ (cf. [?, Lemma 3.1]).

Proposition 4.8. *Assume $A, B \in \mathcal{G}_R$ as R-modules. Then the following hold.*

(1) $A \in \mathcal{P}_R$ as an R-module if and only if $B \in \mathcal{P}_R$ as an R-module.

(2) For any $\mathfrak{p} \in \mathrm{Supp}(A)$, $A_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module if and only if $B_\mathfrak{p} \in \mathcal{P}_{R_\mathfrak{p}}$ as an $R_\mathfrak{p}$-module.

(3) If $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$, then $D({}_BB)$ is a tilting module.

Proof. (1) follows by (2), (3) of Lemma **??** and (2) follows by Lemmas **??**(2), **??**.

(3) By Lemma **??** $\nu P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex and by Lemma **??**(2) $\nu P^\bullet \in \mathcal{S}(P^\bullet)$. Let $F^* : \mathcal{D}^-(A) \xrightarrow{\sim} \mathcal{D}^-(B)$ be the equivalence of triangulated categories stated in Proposition **??**. Then $F^*(\nu P^\bullet) \cong \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, \nu P^\bullet)$ in $\mathcal{D}(B)$. Since by Lemma **??**(2) $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, \nu P^\bullet) \cong DB$ in Mod-B, the assertion follows. □

Proposition 4.9. *Assume R is a Gorenstein ring with* $\dim R < \infty$ *and $A, B \in \mathcal{G}_R$ as R-modules. Then $D({}_AA)$ is a tilting module if and only if so is $D({}_BB)$.*

Proof. By [?, Proposition 1.7(2)], $\mathrm{inj\ dim}\ A_A < \infty$ if and only if $\mathrm{inj\ dim}\ B_B < \infty$. Note also that A^{op}, B^{op} are derived equivalent ([?, Proposition 9.1]). Thus $\mathrm{inj\ dim}\ {}_AA < \infty$ if and only if $\mathrm{inj\ dim}\ {}_BB < \infty$. According to [?, Lemma A], the assertion follows by Proposition **??**(2). □

5 Suitable tilting complexes

In this section, R is an arbitrary commutative noetherian ring. Following [?], we provide a way to construct tilting complexes $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $\mathrm{add}(T^\bullet) = \mathrm{add}(\nu T^\bullet)$.

We start by formulating the argument in [?, Lemma of 1.2] as follows.

Lemma 5.1. *Let $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $P^\bullet \neq 0$ in $\mathcal{K}(A)$ and with $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and form a distinguished triangle in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$*

$$Q^\bullet \to \bigoplus^n P^\bullet \xrightarrow{f} T^\bullet \to$$

such that $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, f)$ is epic. Then $Q^\bullet \oplus P^\bullet$ is a tilting complex if the following conditions are satisfied:

(1) $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, T^\bullet[i]) = 0$ *unless* $-1 \leq i \leq 0$*;*

(2) $\mathrm{Hom}_{\mathcal{K}(A)}(T^\bullet, P^\bullet[i]) = 0$ *for* $i > 1$*;*

(3) $P^\bullet \in \mathrm{add}(\nu P^\bullet)$*; and*

(4) $\mathrm{Ext}^i_R(A, R) = 0$ *for* $1 \leq i < a(Q^\bullet) - b(P^\bullet) - 1$.

Proof. Note first that such a homomorphism f exists. Since $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, T^\bullet) \cong \mathrm{H}^0(\mathrm{Hom}^\bullet_A(P^\bullet, T^\bullet)) \in \text{mod-}R$, it follows that $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, T^\bullet)$ is finitely generated over $\mathrm{End}_{\mathcal{K}(A)}(P^\bullet)$. Let $f_1, \cdots, f_n \in \mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, T^\bullet)$ be generators over $\mathrm{End}_{\mathcal{K}(A)}(P^\bullet)$ and set

$$f = (f_1, \cdots, f_n) : \bigoplus^n P^\bullet \to T^\bullet.$$

Then $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, f)$ is epic.

Obviously, $\mathrm{add}(Q^\bullet \oplus P^\bullet)$ generates $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as a triangulated category.

Claim. The following hold.

(1) $\mathrm{Hom}_{\mathcal{K}(A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i \neq 0$.

(2) $\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$.

(3) $\mathrm{Hom}_{\mathcal{K}(A)}(T^\bullet, Q^\bullet[i]) = 0$ for $i > 1$.

(4) $\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, T^\bullet[i]) = 0$ for $i < -1$.

Proof. (1), (3) and (4) follow by the construction.

(2) Let $i > 0$. By the construction, $\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, P^\bullet[i]) = 0$. Next, since

$$\begin{aligned} a(Q^\bullet[i]) - b(P^\bullet) &= a(Q^\bullet) - i - b(P^\bullet) \\ &\leq a(Q^\bullet) - b(P^\bullet) - 1, \end{aligned}$$

by (1) and Lemma **??**(1) we have $\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet[i], \nu P^\bullet) = 0$. It then follows that $\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet[i], P^\bullet) = 0$.

Now, by (1), (3) of Claim we have $\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, Q^\bullet[i]) = 0$ for $i > 0$ and by (2), (4) of Claim we have $\mathrm{Hom}_{\mathcal{K}(A)}(Q^\bullet, Q^\bullet[i]) = 0$ for $i < 0$. $\square$

Throughout the rest of this section, we fix a sequence of idempotents $e_0, e_1, \cdots$ in A such that $\mathrm{add}(e_0A_A) = \mathcal{P}_A$ and $e_{i+1} \in e_iAe_i$ for all $i \geq 0$. We will construct inductively a sequence of complexes $T_0^\bullet$, $T_1^\bullet$, $\cdots$ in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ as follows. Set $T_0^\bullet = e_0A$. Let $k \geq 1$ and assume $T_0^\bullet$, $T_1^\bullet$, $\cdots$, $T_{k-1}^\bullet$ have been constructed. Then we form a distinguished triangle in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$

$$Q_k^\bullet \to \bigoplus^{n_k} e_kA \xrightarrow{f_k} T_{k-1}^\bullet \to$$

such that $\mathrm{Hom}_{\mathcal{K}(A)}(e_kA, f_k)$ is epic and set $T_k^\bullet = Q_k^\bullet \oplus e_kA$.

Lemma 5.2. *For any $l \geq 0$ the following hold.*

(1) $T_l^i = 0$ *unless* $0 \leq i \leq l$.

(2) $T_l^i \in \mathrm{add}(e_{l-i}A_A)$ *for* $0 \leq i \leq l$.

(3) $\mathrm{Hom}_{\mathcal{K}(A)}(e_lA, T_l^\bullet[i]) = 0$ *for* $i > 0$.

(4) $\mathrm{add}(T_l^\bullet)$ *generates* $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ *as a triangulated category.*

Proof. By induction on $l \geq 0$. □

Lemma 5.3. *For any $l \geq 1$ the following hold.*

(1) $\mathrm{H}^j(T_l^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_{l-i}A)$ *for* $0 \leq i < j \leq l$.

(2) If $D(e_iA_A) \in \mathrm{add}({}_AAe_i)$ *for* $1 \leq i \leq l$, *then* $\mathrm{H}^j(\nu T_l^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_{l-i}A)$ *for* $0 \leq i < j \leq l$.

Proof. (1) We have $\mathrm{H}^j(T_l^\bullet) = \mathrm{H}^j(Q_l^\bullet) \cong \mathrm{H}^{j-1}(T_{l-1}^\bullet) \cong \cdots \cong \mathrm{H}^1(T_{l-j+1}^\bullet)$. Also, by Lemma **??**(3)

$$\begin{aligned}
\mathrm{H}^1(T_{l-j+1}^\bullet) \otimes_A Ae_{l-j+1} &\cong \mathrm{H}^1(T_{l-j+1}^\bullet \otimes_A^\bullet Ae_{l-j+1}) \\
&\cong \mathrm{H}^1(\mathrm{Hom}_A^\bullet(e_{l-j+1}A, T_{l-j+1}^\bullet)) \\
&\cong \mathrm{Hom}_{\mathcal{K}(A)}(e_{l-j+1}A, T_{l-j+1}^\bullet[1]) \\
&= 0.
\end{aligned}$$

Thus, since $l - i \geq l - j + 1$, it follows that $\mathrm{H}^j(T_l^\bullet) \otimes_A Ae_{l-i} = 0$.

(2) Since by (1) $\mathrm{H}^j(T_l^\bullet \otimes_A^\bullet Ae_{l-i}) \cong \mathrm{H}^j(T_l^\bullet) \otimes_A Ae_{l-i} = 0$, we have

$$\begin{aligned}
\mathrm{H}^j(\nu T_l^\bullet) \otimes_A Ae_{l-i} &\cong \mathrm{H}^j(\nu T_l^\bullet \otimes_A^\bullet Ae_{l-i}) \\
&\cong \mathrm{H}^j(T_l^\bullet \otimes_A^\bullet DA \otimes_A^\bullet Ae_{l-i}) \\
&\cong \mathrm{H}^j(T_l^\bullet \otimes_A^\bullet D(e_{l-i}A)) \\
&= 0.
\end{aligned}$$

□

Lemma 5.4 ([?, Remark 2.3]). *Let $l \geq 0$. For any $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$,* $\mathrm{add}(T^\bullet)$ *is uniquely determined if the following conditions are satisfied:*

(1) $T^i = 0$ *unless* $0 \le i \le l$*;*

(2) $T^i \in \mathrm{add}(e_{l-i}A_A)$ *for* $0 \le i \le l$*;*

(3) $\mathrm{H}^j(T^\bullet) \in \mathrm{Mod}\text{-}(A/Ae_{l-i}A)$ *for* $0 \le i < j \le l$*; and*

(4) $\mathrm{add}(T^\bullet)$ *generates* $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ *as a triangulated category.*

Proof. We can apply [**?**, Remark 2.3] to $P^\bullet = T^\bullet[l]$. □

Theorem 5.5. *Let* $l \ge 1$ *and assume* $\mathrm{Ext}^i_R(A, R) = 0$ *for* $1 \le i < l - 1$*. Then the following hold.*

(1) If $e_iA_A \in \mathrm{add}(D({}_AAe_i))$ *for* $1 \le i \le l$*, then* $T_l^\bullet$ *is a tilting complex.*

(2) If A *is reflexive as an* R*-module and* $\mathrm{add}(e_iA_A) = \mathrm{add}(D({}_AAe_i))$ *for* $0 \le i \le l$*, then* $\mathrm{add}(T_l^\bullet) = \mathrm{add}(\nu T_l^\bullet)$*.*

Proof. (1) It is obvious that $T_0^\bullet$ is a tilting complex. Thus by Lemmas **??**, **??** we can make use of induction to prove that $T_k^\bullet$ is a tilting complex for $0 \le k \le l$.

(2) By (1) $T_l^\bullet$ is a tilting complex. Then, since $\mathrm{add}(e_0A_A) = \mathcal{P}_A$, we have $\mathrm{add}(D({}_AA)) = \mathcal{P}_A$ and hence by Lemma **??** $\nu T_l^\bullet$ is also a tilting complex. Thus by Lemmas **??**, **??** both $T_l^\bullet$ and $\nu T_l^\bullet$ satisfy the conditions (1)–(4) of Lemma **??** and hence $\mathrm{add}(T_l^\bullet) = \mathrm{add}(\nu T_l^\bullet)$. □

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