On derived equivalences for selfinjective algebras

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Abstract

We show that if A is a representation-finite selfinjective artin algebra then every $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$ is a direct summand of a tilting complex, and that if A, B are derived equivalent representation-finite selfinjective artin algebras then there exists a sequence of selfinjective artin algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1.

Let A be an artin algebra. Rickard [16, Proposition 9.3] showed that for any tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ the number of nonisomorphic indecomposabele direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$, the Grothendieck group of A, which generalizes earlier results [6, Proposition 3.2] and [15, Theorem 1.19]. He raised a question whether a complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ is a tilting complex or not if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$ (see also [15]). In case $P^\bullet$ is a projective resolution of a module $T \in \text{mod-}A$ with proj dim $T_A \leq 1$, Bongartz [4, Lemma of 2.1] has settled the question affirmatively. More precisely, he showed that every $T \in \text{mod-}A$ with proj dim $T_A \leq 1$ and $\mathrm{Ext}^1_A(T, T) = 0$ is a direct summand of a classical tilting module, i.e., a tilting module of projective dimension ≤ 1. Unfortunetely, this is not true in general (see [16, Section 8]). Our first aim of this note is to show that if A is a representation-finite selfinjective artin algebra then every $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where ν is the Nakayama functor, is a direct summand of a tilting complex (Theorem 3.6).

Rickard [17, Theorem 4.2] showed that the Brauer tree algebras over a field with the same numerical invariants are derived equivalent to each other. Subsequently, Okuyama pointed out that for any Brauer tree algebras A, B with the same numerical invariants there exists a sequence of Brauer tree algebras

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$A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \le i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1. These facts can be formulated as follows. For any tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ associated with a certain sequence of idempotents in a ring A, there exists a sequence of rings $A = B_0, B_1, \cdots, B_m = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ such that, for any $0 \le i < m$, B_{i+1} is the endomorphism ring of a tilting complex for B_i of length ≤ 1 determined by an idempotent (see [11, Proposition 3.2]). We refer to [7], [14] for other examples of derived equivalences which are iterations of derived equivalences induced by tilting complexes of length ≤ 1. Our second aim of this note is to show that for any derived equivalent representation-finite selfinjective artin algebras A, B there exists a sequence of selfinjective artin algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \le i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1 (Theorem 3.7).

For a ring A, we denote by Mod-A the category of right A-modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. For an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X and by $X^{(n)}$ the direct sum of n copies of X. For a cochain complex $X^\bullet$ over an abelian category $\mathcal{A}$, we denote by $\mathrm{Z}^n(X^\bullet)$, $\mathrm{Z}'^n(X^\bullet)$ and $\mathrm{H}^n(X^\bullet)$ the n-th cycle, the n-th cocycle and the n-th cohomology of $X^\bullet$, respectively. For an additive category $\mathcal{B}$, we denote by $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{K}^+(\mathcal{B})$, $\mathsf{K}^-(\mathcal{B})$, $\mathsf{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{B}$. As usual, we consider objects of $\mathcal{B}$ as complexes over $\mathcal{B}$ concentrated in degree zero. For an abelian category $\mathcal{A}$, we denote by $\mathsf{D}(\mathcal{A})$ (resp., $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$, $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{A}$. We always consider $\mathsf{K}^*(\mathcal{B})$ (resp., $\mathsf{D}^*(\mathcal{A})$) as a full triangulated subcategory of $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{D}(\mathcal{A})$), where $* = +,\ -$ or b. We denote by $\mathrm{Hom}^\bullet(-,-)$ the associated single complex of the double hom complex.

We refer to [3], [8], [19] for basic results in the theory of derived categories and to [16], [18] for definitions and basic properties of derived equivalences and tilting complexes.

1 Preliminaries

Throughout this note, R is a commutative artinian ring with the Jacobson radical $\mathfrak{m}$ and A is an artin R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and is finitely generated as an R-module.

For any artin R-algebra A, we denote by mod-A the full subcategory of Mod-A consisting of finitely generated modules and by $\mathcal{P}_A$ (resp., $\mathcal{I}_A$) the full subcategory of mod-A consisting of projective (resp., injective) modules. Also, we set $D = \mathrm{Hom}_R(-, E(R/\mathfrak{m}))$, where $E(R/\mathfrak{m})$ is an injective envelope of $R/\mathfrak{m}$

in Mod-R, and $\nu = D \circ \mathrm{Hom}_A(-, A)$, which is called the Nakayama functor.

Remark 1.1. The Krull-Schmidt theorem holds in mod-A, i.e., for any nonzero module $X \in$ mod-A the following hold.

(1) X decomposes into a direct sum of indecomposable submodules.

(2) X is indecomposable if and only if $\mathrm{End}_A(X)$ is local.

Remark 1.2. The following hold.

(1) $X \xrightarrow{\sim} D^2X, x \mapsto (h \mapsto h(x))$, for all $X \in$ mod-R.

(2) D : mod-$A \to$ mod-A^{op} is an anti-equivalence and induces anti-equivalences $\mathcal{P}_A \xrightarrow{\sim} \mathcal{I}_{A^{\mathrm{op}}}$ and $\mathcal{I}_A \xrightarrow{\sim} \mathcal{P}_{A^{\mathrm{op}}}$.

(3) ν : mod-$A \to$ mod-A induces an equivalence $\mathcal{P}_A \xrightarrow{\sim} \mathcal{I}_A$.

Lemma 1.3. *For any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ the following are equivalent.*

(1) $P^\bullet \in \mathrm{add}(\nu P^\bullet)$.

(2) $\nu P^\bullet \in \mathrm{add}(P^\bullet)$.

(3) $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

Proof. Note that every idempotent splits in $\mathsf{K}(\text{Mod-}A)$ (see [3, Proposition 3.2]). Thus, since we have an isomorphism of artin R-algebras

$$\mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet) \simeq \mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(\nu P^\bullet),$$

it follows that $P^\bullet$ and $\nu P^\bullet$ have the same number of nonisomorphic indecomposable direct summands. □

Recall that A is said to be selfinjective if the equivalent conditions of Lemma 1.3 are satisfied for $P^\bullet = A$.

Remark 1.4. If A is selfinjective, then ν : mod-$A \to$ mod-A is an equivalence and induces an equivalence $\mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_A$.

Lemma 1.5 ([10, Lemma 3.1]). *For any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $X^\bullet \in \mathsf{K}(\text{Mod-}A)$ we have a bifunctorial isomorphism*

$$\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(X^\bullet, \nu P^\bullet) \simeq D\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, X^\bullet).$$

Definition 1.6. For any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ we denote by $\mathcal{C}(P^\bullet)$ the full subcategory of $\mathsf{D}^-(\text{Mod-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{Hom}_{\mathsf{D}(\text{Mod-}A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$.

Lemma 1.7. *Assume A is selfinjective. Then for any tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ the following are equivalent.*

(1) $\mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet)$ *is selfinjective.*

(2) $P^\bullet \in \mathcal{C}(\nu P^\bullet)$.

(3) $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

Proof. Set $B = \mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet)$. Note that by Lemma 1.5 $\nu P^\bullet \in \mathcal{C}(P^\bullet)$ and $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet) \simeq D({}_B B)$.

(1) $\Leftrightarrow$ (3). Note first that we have an equivalence (see [16, Section 4])

$$\mathrm{Hom}_{\mathsf{D}(\text{Mod-}A)}(P^\bullet, -) : \mathcal{C}(P^\bullet) \xrightarrow{\sim} \text{Mod-}B.$$

We may consider $\mathrm{add}(P^\bullet)$ and $\mathrm{add}(\nu P^\bullet)$ as full subcategories of $\mathcal{C}(P^\bullet)$ via the canonical functor $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A) \to \mathsf{D}^-(\text{Mod-}A)$. Then $\mathrm{add}(P^\bullet)$ and $\mathrm{add}(\nu P^\bullet)$ are closed under direct summands because every idempotent splits in $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ (see [3, Proposition 3.4]). Thus the equivalence above induces equivalences $\mathrm{add}(P^\bullet) \xrightarrow{\sim} \mathcal{P}_B$ and $\mathrm{add}(\nu P^\bullet) \xrightarrow{\sim} \mathcal{I}_B$.

(2) $\Rightarrow$ (3). We have $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet \oplus \nu P^\bullet, (P^\bullet \oplus \nu P^\bullet)[i]) = 0$ for $i \neq 0$ and hence by [11, Lemma 1.8] $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$.

(3) $\Rightarrow$ (2). Obvious. □

In case A, B are finite dimentional selfinjective algebras over a field and $F : \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\mathcal{P}_B)$ is an equivalence of triangulated categories, it was pointed out in [1, Section 2] that for any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ there exists an object-wise isomorphism $F(\nu P^\bullet) \simeq \nu F(P^\bullet)$. We need to extend this fact to the case of artin algebras.

Lemma 1.8. *Let A, B be derived equivalent selfinjective artin R-algebras and $F : \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\mathcal{P}_B)$ an equivalence of triangulated categories. Then for any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ we have a functorial isomorphism $\nu F(P^\bullet) \simeq F(\nu P^\bullet)$.*

Proof. Let $G : \mathsf{K}^{\mathrm{b}}(\mathcal{P}_B) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a quasi-inverse of F. Then for any $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_B)$, by Lemma 1.5 we have bifunctorial isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{K}(\text{Mod-}B)}(Q^\bullet, \nu F(P^\bullet)) &\simeq D\mathrm{Hom}_{\mathsf{K}(\text{Mod-}B)}(F(P^\bullet), Q^\bullet) \\
&\simeq D\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, G(Q^\bullet)) \\
&\simeq \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(G(Q^\bullet), \nu P^\bullet) \\
&\simeq \mathrm{Hom}_{\mathsf{K}(\text{Mod-}B)}(Q^\bullet, F(\nu P^\bullet)).
\end{aligned}$$

The assertion follows by Yoneda lemma. □

Definition 1.9. For any nonzero $P^\bullet \in \mathsf{K}^-(\mathcal{P}_A)$ we set

$$a(P^\bullet) = \max\{i \in \mathbb{Z} \mid \mathrm{H}^i(P^\bullet) \neq 0\},$$

and for any nonzero $P^\bullet \in \mathsf{K}^+(\mathcal{P}_A)$ we set

$$b(P^\bullet) = \min\{i \in \mathbb{Z} \mid \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet[i], A) \neq 0\}.$$

Then for any nonzero $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ we set $l(P^\bullet) = a(P^\bullet) - b(P^\bullet)$ and call it the length of $P^\bullet$. For the sake of convenience, we set $l(P^\bullet) = 0$ for $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $P^\bullet \simeq 0$.

Remark 1.10 *([6]).* For any complex $X^\bullet$ and $n \in \mathbb{Z}$ we define truncations

$$\sigma_{\le n}(X^\bullet) : \cdots \to X^{n-2} \to X^{n-1} \to \mathsf{Z}^n(X^\bullet) \to 0 \to \cdots ,$$
$$\sigma'_{\ge n}(X^\bullet) : \cdots \to 0 \to \mathsf{Z}'^n(X^\bullet) \to X^{n+1} \to X^{n+2} \to \cdots .$$

Then $P^\bullet \simeq \sigma_{\le a}(P^\bullet)$ for any nonzero $P^\bullet \in \mathsf{K}^-(\mathcal{P}_A)$, where $a = a(P^\bullet)$, and $P^\bullet \simeq \sigma'_{\ge b}(P^\bullet)$ for any nonzero $P^\bullet \in \mathsf{K}^+(\mathcal{P}_A)$, where $b = b(P^\bullet)$.

2 Torsion theories

We need to recall several definitions and basic results on torsion theories.

Definition 2.1 ([3]). A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories $\mathcal{T}$, $\mathcal{F}$ in an abelian category $\mathcal{A}$ is said to be a torsion theory for $\mathcal{A}$ if the following conditions are satisfied:

(1) $\mathcal{T} \cap \mathcal{F} = \{0\}$;

(2) $\mathcal{T}$ is closed under factor objects;

(3) $\mathcal{F}$ is closed under subobjects; and

(4) for any $X \in \mathcal{A}$ there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$.

Definition 2.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$. Then we denote by ${}^\perp\mathcal{C}$ (resp., $\mathcal{C}^\perp$) the full subcategory of $\mathcal{A}$ consisting of objects X with $\mathrm{Hom}_\mathcal{A}(X, \mathcal{C}) = 0$ (resp., $\mathrm{Hom}_\mathcal{A}(\mathcal{C}, X) = 0$). For an object $Y \in \mathcal{A}$, we use the notation ${}^\perp Y$ (resp., $Y^\perp$) instead of ${}^\perp\mathrm{add}(Y)$ (resp., $\mathrm{add}(Y)^\perp$).

Remark 2.3. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for an abelian category $\mathcal{A}$. Then the following hold.

(1) $\mathcal{F} = \mathcal{T}^\perp$ and $\mathcal{T} = {}^\perp\mathcal{F}$.

(2) $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions.

(3) There exists a subfunctor t of the identity functor $\mathbf{1}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$, called the associated torsion radical, such that $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$ for all $X \in \mathcal{A}$.

Proof. (1) By the conditions (1)–(3), $\mathcal{F} \subset \mathcal{T}^\perp$ and $\mathcal{T} \subset {}^\perp\mathcal{F}$. On the other hand, by the condition (4), $\mathcal{T}^\perp \subset \mathcal{F}$ and ${}^\perp\mathcal{F} \subset \mathcal{T}$.

(2) Immediate by (1).

(3) For each $X \in \mathcal{A}$, take an exact sequence

$$0 \to X' \xrightarrow{\iota_X} X \xrightarrow{\pi_X} X'' \to 0$$

with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$. For any $Z \in \mathcal{T}$, since $\mathrm{Hom}_\mathcal{A}(Z, X'') = 0$, $\mathrm{Hom}_\mathcal{A}(Z, \iota_X)$ is an isomorphism. It follows that X' is maximum in the collection of subobjects of X belonging to $\mathcal{T}$. We set $t(X) = X'$. Next, let

$f : X \to Y$ be a morphism. Since $\mathrm{Hom}_{\mathcal{A}}(X', Y'') = 0$, $\pi_Y \circ f \circ \iota_X = 0$ and there exists a unique morphism $f' : X' \to Y'$ such that $f \circ \iota_X = \iota_Y \circ f'$. We set $t(f) = f'$. Then for any $X \in \mathcal{A}$ we have $\mathrm{id}_X \circ \iota_X = \iota_X \circ \mathrm{id}_{t(X)}$ and hence $t(\mathrm{id}_X) = \mathrm{id}_{t(X)}$. Also, for any consecutive morphisms $f : X \to Y$ and $g : Y \to Z$, since $f \circ \iota_X = \iota_Y \circ t(f)$ and $g \circ \iota_Y = \iota_Z \circ t(g)$, we have $g \circ f \circ \iota_X = \iota_Z \circ t(g) \circ t(f)$ and hence $t(g \circ f) = t(g) \circ t(f)$. □

Although the next lemma is well-known, we include a proof because it will play an indispensable role in the next section.

Lemma 2.4. *For any $Y \in \text{mod-}A$, by setting $\mathcal{T} = {}^{\perp}Y$ and $\mathcal{F} = \mathcal{T}^{\perp}$, we have a torsion theory $(\mathcal{T}, \mathcal{F})$ for* mod-A.

Proof. It is obvious that the conditions (1)–(3) of Definition 2.1 are satisfied. Let $X \in \text{mod-}A$. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be the set of submodules of X belonging to $\mathcal{T}$ and set $X' = \bigcup_{\lambda \in \Lambda} X_\lambda$. Note that $\mathcal{T}$ is closed under extensions and finite direct sums. In particular, Λ is directed, where $\lambda \le \mu$ if and only if $X_\lambda \subset X_\mu$, and X' is a submodule of X. Thus we have an epimorphism $\bigoplus_{\lambda \in \Lambda} X_\lambda \to X'$ in Mod-A and, since $\mathrm{Hom}_A(\bigoplus_{\lambda \in \Lambda} X_\lambda, Y) \simeq \prod_{\lambda \in \Lambda} \mathrm{Hom}_A(X_\lambda, Y) = 0$, it follows that $X' \in \mathcal{T}$. Next, we claim that $X/X' \in \mathcal{F}$. Let $Z \in \mathcal{T}$ and $f \in \mathrm{Hom}_A(Z, X/X')$. Take a pull-back of f along with the canonical epimorphism $X \to X/X'$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \longrightarrow & W & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X/X' & \longrightarrow & 0. \end{array}$$

Then, since $W \in \mathcal{T}$, $\mathrm{Im}\, g \subset X'$ and $f = 0$. □

Definition 2.5. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$ closed under extensions. Then an object $X \in \mathcal{C}$ is said to be Ext-projective (resp., Ext-injective) if $\mathrm{Ext}^1_{\mathcal{A}}(X, \mathcal{C}) = 0$ (resp., $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{C}, X) = 0$).

Lemma 2.6. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for* mod-A*. Then a module $X \in \mathcal{T}$ is* Ext*-injective if and only if $X = t(E)$ with E an injective envelope of X.*

Proof. "If" part. Let $E \in \text{mod-}A$ be an injective module and take an exact sequence

$$0 \to t(E) \xrightarrow{\mu} Y \xrightarrow{\varepsilon} Z \to 0$$

with $Z \in \mathcal{T}$. We claim that μ is a split monomorphism. Denote by $\iota : t(E) \to E$ the inclusion. By the injectivity of E, $\iota = \phi \circ \mu$ for some $\phi : Y \to E$. Note that by Remark 2.3(2) $Y \in \mathcal{T}$. Thus $\phi(Y) \subset t(E)$ and $\phi = \iota \circ \phi'$ for some $\phi' : Y \to t(E)$. Then $\iota = \iota \circ \phi' \circ \mu$ and $\mathrm{id}_{t(E)} = \phi' \circ \mu$. It follows that $t(E)$ is Ext-injective.

"Only if" part. Let $X \in \mathcal{T}$ and E an injective envelope of X. We consider X as a submodule of E. Then $X \subset t(E)$ and we have an exact sequence

$$0 \to X \xrightarrow{\iota} t(E) \to t(E)/X \to 0.$$

Since $t(E)/X \in \mathcal{T}$, and since X is Ext-injective, the inclusion $\iota : X \to t(E)$ has to be a split monomorphism. On the other hand, E and hence $t(E)$ are essential extensions of X. It follows that $X = t(E)$. □

We refer to [2, Chapter V, Sections 1 and 2] for the following Definitions 2.7, 2.8 and Lemmas 2.9, 2.11.

Definition 2.7. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$. Let $f : X \to Y$ be a morphism with $X, Y \in \mathcal{C}$. Then f is said to be right (resp., left) almost split in $\mathcal{C}$ if f is not a split epimorphism (resp., monomorphism) and if every morphism $h : Z \to Y$ (resp., $h : X \to Z$) with $Z \in \mathcal{C}$ factors through f unless h is a split epimorphism (resp., monomorphism).

Definition 2.8. Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$ closed under extensions. Then a nonsplit exact sequence

$$0 \to Z \xrightarrow{g} Y \xrightarrow{f} X \to 0$$

with $X, Z \in \mathcal{C}$ is said to be an almost split sequence in $\mathcal{C}$ if the following conditions are satisfied:

(1) $\mathrm{End}_{\mathcal{A}}(X)$ and $\mathrm{End}_{\mathcal{A}}(Z)$ are local; and

(2) f (resp., g) is right (resp., left) almost split in $\mathcal{C}$.

Lemma 2.9. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$ closed under extensions. Let*

$$0 \to Z_1 \to Y_1 \to X_1 \to 0, \qquad 0 \to Z_2 \to Y_2 \to X_2 \to 0$$

be almost split sequences in $\mathcal{C}$. Then $X_1 \simeq X_2$ if and only if $Z_1 \simeq Z_2$.

Definition 2.10. For each indecomposable module $X \in \text{mod-}A$, we take a minimal projective resolution $P_X^\bullet \to X$ and set $\tau X = \mathrm{Z}^{-1}(\nu P_X^\bullet)$.

Lemma 2.11. *Let $X \in \text{mod-}A$ be an indecomposable nonprojective module. Then $\mathrm{Ext}_A^1(X, \tau X) \neq 0$ and the following hold.*

(1) As a right module over $\mathrm{End}_A(X)$, $\mathrm{Ext}_A^1(X, \tau X)$ is embedded in $D\mathrm{End}_A(X)$ and hence has a simple socle.

(2) A nonsplit exact sequence

$$0 \to \tau X \to Y \to X \to 0$$

representing a nonzero element of the socle of $\mathrm{Ext}_A^1(X, \tau X)$ is an almost split sequence in $\text{mod-}A$.

Lemma 2.12 ([9, Lemma 2]). *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory for $\text{mod-}A$ and $X \in \mathcal{T}$ an indecomposable module. Then the following hold.*

(1) X is Ext*-projective if and only if $\tau X \in \mathcal{F}$.*

(2) Assume X is not Ext*-projective and let $0 \to \tau X \to Y \to X \to 0$ be an almost split sequence in* mod*-A. Then the induced sequence*

$$0 \to t(\tau X) \to t(Y) \to X \to 0$$

is an almost split sequence in $\mathcal{T}$.

Definition 2.13. Assume A is selfinjective and let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A. Then there exists a permutation ρ of the set $I = \{1, \cdots, n\}$, called the Nakayama permutation, such that $\nu(e_i A) \simeq e_{\rho(i)} A$ for all $i \in I$.

Proposition 2.14. *Assume A is selfinjective and has a cyclic Nakayama permutation. Then for any tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ selfinjective we have $l(P^\bullet) = 0$.*

Proof. Set $l = l(P^\bullet)$. We may assume $P^i = 0$ unless $0 \leq i \leq l$. Suppose to the contrary that $l \geq 1$. Set $X = \mathrm{H}^l(P^\bullet)$ and $Y = \mathrm{H}^0(P^\bullet)$. Since by Lemma 1.7 $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, we have $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu^k P^\bullet)$ for all $k \geq 0$. Thus for any $k \geq 0$, since $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[-l]) = 0$, and since $\nu^k P^\bullet \in \mathrm{add}(P^\bullet)$, we have

$$\begin{aligned}\mathrm{Hom}_A(\nu^k X, Y) &\simeq \mathrm{Hom}_A(\mathrm{H}^l(\nu^k P^\bullet), \mathrm{H}^0(P^\bullet)) \\ &\simeq \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(\nu^k P^\bullet, P^\bullet[-l]) \\ &= 0.\end{aligned}$$

By Lemma 2.4 there exists a torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-A such that $\mathcal{T} = {}^\perp Y$ and $\mathcal{F} = \mathcal{T}^\perp$. Let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A and set $S_i = e_i A / e_i J$ for $1 \leq i \leq n$, where J is the Jacobson radical of A. Note that $\nu S_i \simeq S_{\rho(i)}$ for all $1 \leq i \leq n$. Let $S \in$ mod-A be a simple module which is a factor module of X. For any $k \geq 0$, since $\nu^k X \in \mathcal{T}$, and since $\nu^k S$ is a facotr module of $\nu^k X$, we have $\nu^k S \in \mathcal{T}$. Note that $S \simeq S_i$ for some $1 \leq i \leq n$. Then $\nu^k S \simeq S_{\rho^k(i)}$ for all $k \geq 0$. Since ρ is cyclic, it follows that $S_i \in \mathcal{T}$ for all $1 \leq i \leq n$. Thus $\mathcal{F}$ does not contain any simple module and $\mathcal{F} = \{0\}$. On the other hand, by the construction we have $0 \neq Y \in \mathcal{F}$, a contradiction. □

3 Main results

To begin with, we modify [4, Lemma of 2.1] as follows.

Lemma 3.1. *Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Assume there exists a tilting complex $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, T^\bullet[i]) = 0$ unless $-1 \leq i \leq 0$. Form a distinguished triangle in $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$*

$$Q^\bullet \to P^{\bullet(n)} \xrightarrow{f} T^\bullet \to$$

such that $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, f)$ is epic. Then $Q^\bullet \oplus P^\bullet$ is a tilting complex.

Proof. Note first that such a homomorphism f exists. To see this, set $X^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, T^\bullet) \in \mathsf{K}^\mathrm{b}(\text{mod-}R)$. Then $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, T^\bullet) \simeq \mathrm{H}^0(X^\bullet) \in \text{mod-}R$, i.e., $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, T^\bullet)$ is finitely generated over R. It then follows that $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, T^\bullet)$ is finitely generated over $\mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet)$. Take a set of generators $f_1, \cdots, f_n \in \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, T^\bullet)$ over $\mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet)$ and set

$$f = (f_1, \cdots, f_n) : P^{\bullet(n)} \to T^\bullet.$$

It then follows by the construction that $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, f)$ is epic.

Obviously, $\mathrm{add}(Q^\bullet \oplus P^\bullet)$ generates $\mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ as a triangulated category. Note also that by Lemma 1.5 $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) = 0$ unless $0 \le i \le 1$.

Claim: The following hold.

(1) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, Q^\bullet[i]) = 0$ for $i \neq 0$.

(2) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(Q^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$.

(3) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, Q^\bullet[i]) = 0$ for $i > 1$.

(4) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(Q^\bullet, T^\bullet[i]) = 0$ for $i < -1$.

Proof. (1), (3) and (4) follow by the construction and (2) follows by (1) and Lemma 1.5. □

Now, by (1), (3) of Claim $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(Q^\bullet, Q^\bullet[i]) = 0$ for $i > 0$ and by (2), (4) of Claim $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(Q^\bullet, Q^\bullet[i]) = 0$ for $i < 0$. This finishes the proof of Lemma 3.1 □

Corollary 3.2. *Assume A is selfinjective. Let $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ be a complex with $P^i = 0$ unless $0 \le i \le 1$. Assume $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists some $Q^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ such that $Q^\bullet \oplus P^\bullet$ is a tilting complex. In particular, if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of the Grothendieck group $K_0(A)$, then $P^\bullet$ is a tilting complex.*

Proof. Applying Lemma 3.1 to $T^\bullet = A$, the first assertion follows. The last assertion follows by [16, Proposition 9.3]. □

Recall that A is said to be representation-finite if there exist only a finite number of nonisomorphic indecomposable modules in mod-A.

Remark 3.3 *([13] and [17]).* Let A, B be derived equivalent selfinjective artin R-algebras. Then A is representation-finite if and only if so is B.

Proof. This follows by the fact that A, B are stably equivalent (see [13, Theorem 3.8] and [17, Corollary 2.2]). □

Lemma 3.4. *Assume A is selfinjective and representation-finite. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex of length ≥ 1 with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists a tilting complex $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ of length 1 such that*

(1) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, P^\bullet[i]) = 0$ *for $i \geq l(P^\bullet)$,*

(2) $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet[i], T^\bullet) = 0$ *for $i < 0$, and*

(3) $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet)$ *is a selfinjective artin R-algebra whose Nakayama permutation coincides with that of A.*

Proof. Set $l = l(P^\bullet)$. We may assume $P^i = 0$ unless $0 \leq i \leq l$. Note that $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$ implies $\mathrm{add}(\mathrm{H}^0(P^\bullet)) = \mathrm{add}(\mathrm{H}^0(\nu P^\bullet))$. Also, by Lemma 2.4 there exists a torsion theory $(\mathcal{T}, \mathcal{F})$ for mod-A such that $\mathcal{T} = {}^\perp\mathrm{H}^0(P^\bullet) = {}^\perp\mathrm{H}^0(\nu P^\bullet)$ and $\mathcal{F} = \mathcal{T}^\perp$. We denote by t the associated torsion radical.

Claim 1: $\mathrm{H}^l(P^\bullet) \in \mathcal{T}$ and $\mathrm{H}^0(P^\bullet), \mathrm{H}^0(\nu P^\bullet) \in \mathcal{F}$.

Proof. By the construction $\mathrm{H}^0(P^\bullet)$, $\mathrm{H}^0(\nu P^\bullet) \in \mathcal{F}$. Also, by Lemma 1.5

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{H}^l(P^\bullet), \mathrm{H}^0(\nu P^\bullet)) &\simeq \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, \nu P^\bullet[-l]) \\ &\simeq D\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[l]) \\ &= 0\end{aligned}$$

and $\mathrm{H}^l(P^\bullet) \in \mathcal{T}$. □

Claim 2: $\nu : \mathrm{mod}\text{-}A \xrightarrow{\sim} \mathrm{mod}\text{-}A$ induces $\mathcal{T} \xrightarrow{\sim} \mathcal{T}$ and $\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. In particular, $\nu(t(X)) = t(\nu X)$ for all $X \in \mathrm{mod}\text{-}A$.

Proof. We have $\nu\mathcal{T} = {}^\perp(\nu\mathrm{H}^0(P^\bullet)) = {}^\perp\mathrm{H}^0(\nu P^\bullet) = \mathcal{T}$ and then $\nu\mathcal{F} = (\nu\mathcal{T})^\perp = \mathcal{T}^\perp = \mathcal{F}$. □

Let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A. Set $I = \{1, \cdots, n\}$, $I_1 = \{i \in I \mid e_iA \in \mathcal{T}\}$, $I_2 = \{i \in I \mid e_iA \in \mathcal{F}\}$ and $I_3 = I \setminus I_1 \cup I_2$. For each $i \in I$, we define a complex $T_i^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ as follows. Set $T_i^\bullet = e_iA[-1]$ if $i \in I_1$, and set $T_i^\bullet = e_iA$ if $i \in I_2$. Assume $i \in I_3$. Since e_iA is indecomposable injective, $t(e_iA)$ is indecomposable. Also, by Lemma 2.6 $t(e_iA)$ is Ext-injective. To this module $t(e_iA)$, we associate an indecomposable Ext-projective module $X_i \in \mathcal{T}$ as follows. Set $Y_1 = t(e_iA)$ and for $k \geq 1$ set $Y_{k+1} = t(\tau Y_k)$ unless Y_k is Ext-projective. Then, according to Lemma 2.9, Y_m has to be Ext-projective for some $m \geq 1$ because $\mathcal{T}$ contains only a finite number of nonisomorphic indecomposable modules. We set $X_i = Y_m$ and define $T_i^\bullet$ as the (-1)-shift of a minimal projective presentation of X_i. Now, we set $T^\bullet = \bigoplus_{i \in I} T_i^\bullet$ (cf. [12, Theorem 5.8]). Also, we denote by ρ the Nakayama permutation of A.

Claim 3: $\nu T_i^\bullet \simeq T_{\rho(i)}^\bullet$ for all $i \in I$. In particular, $\nu T^\bullet \simeq T^\bullet$ and $\mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet)$ is a selfinjective artin R-algebra with ρ the Nakayama permutation.

Proof. By Claim 2 the sets I_i are ρ-stable. Thus $\nu T_i^\bullet \simeq T_{\rho(i)}^\bullet$ for $i \in I_1 \cup I_2$. Let $i \in I_3$. Then by Claim 2 $\nu(t(e_iA)) \simeq t(\nu(e_iA)) \simeq t(e_{\rho(i)}A)$ and hence $\nu X_i \simeq X_{\rho(i)}$. Thus $\nu T_i^\bullet \simeq T_{\rho(i)}^\bullet$. Now, for any $i \in I$, by Lemma 1.5

$$\begin{aligned} D\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T_i^\bullet, T^\bullet) &\simeq \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, \nu T_i^\bullet) \\ &\simeq \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, T_{\rho(i)}^\bullet). \end{aligned}$$

□

Claim 4: $\mathrm{H}^1(T^\bullet) \in \mathcal{T}$ and $\mathrm{H}^0(T^\bullet), \mathrm{H}^0(\nu T^\bullet) \in \mathcal{F}$.

Proof. By the construction $H^1(T^\bullet) \in \mathcal{T}$. Also, by Lemma 2.12(1) $\mathrm{H}^0(\nu T_i^\bullet) \simeq \tau X_i \in \mathcal{F}$ for all $i \in I_3$ and hence $\mathrm{H}^0(\nu T^\bullet) \in \mathcal{F}$. It then follows by Claim 3 that $\mathrm{H}^0(T^\bullet) \in \mathcal{F}$. □

Claim 5: $T^\bullet$ is a tilting complex.

Proof. By Claim 4 $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, T^\bullet[-1]) \simeq \mathrm{Hom}_A(\mathrm{H}^1(T^\bullet), \mathrm{H}^0(T^\bullet)) = 0$. Then by Lemma 1.5 and Claim 3

$$\begin{aligned} \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, T^\bullet[1]) &\simeq D\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, \nu T^\bullet[-1]) \\ &\simeq D\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, T^\bullet[-1]) \\ &= 0. \end{aligned}$$

Thus by Claim 3 we can apply the last part of Corollary 3.2. □

Claim 6: $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, P^\bullet[i]) = 0$ for $i \geq l$ and $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet[i], T^\bullet) = 0$ for $i < 0$.

Proof. For any $i > l$ we have $a(P^\bullet[i]) < b(T^\bullet)$ and $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, P^\bullet[i]) = 0$. Similarly, for any $i < -1$ we have $a(T^\bullet) < b(P^\bullet[i])$ and $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet[i], T^\bullet) = 0$. Also, by Lemma 1.5 and Claims 1, 4

$$\begin{aligned} \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, P^\bullet[l]) &\simeq D\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, \nu T^\bullet[-l]) \\ &\simeq D\mathrm{Hom}_A(\mathrm{H}^l(P^\bullet), \mathrm{H}^0(\nu T^\bullet)) \\ &= 0, \end{aligned}$$

$$\begin{aligned} \mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet[-1], T^\bullet) &\simeq D\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet, \nu P^\bullet[-1]) \\ &\simeq D\mathrm{Hom}_A(\mathrm{H}^1(T^\bullet), \mathrm{H}^0(\nu P^\bullet)) \\ &= 0. \end{aligned}$$

□

This finishes the proof of Lemma 3.4. □

Remark 3.5. Consider the case where $l(P^\bullet) = 1$ in the above lemma. Then $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(T^\bullet \oplus P^\bullet, (T^\bullet \oplus P^\bullet)[i]) = 0$ for $i \neq 0$ and by [11, Lemma 1.8] we have $P^\bullet \in \mathrm{add}(T^\bullet)$.

Theorem 3.6. *Assume A is selfinjective and representation-finite. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex with $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists some $Q^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $Q^\bullet \oplus P^\bullet$ is a tilting complex. In particular, if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of the Grothendieck group $K_0(A)$, then $P^\bullet$ is a tilting complex.*

Proof. Set $l = l(P^\bullet)$. We may assume $P^i = 0$ unless $0 \leq i \leq l$. In case $l \leq 1$, this is a special case of Corollary 3.2. Assume $l \geq 2$. Let $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex constructed in Lemma 3.4 and set $B = \mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet)$. There exists an equivalence of triangulated categories $F : \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\mathcal{P}_B)$ which sends $T^\bullet$ to B. Denote by $G : \mathsf{K}^{\mathrm{b}}(\mathcal{P}_B) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ a quasi-inverse of F. Set $\bar{P}^\bullet = F(P^\bullet)$. Then $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}B)}(\bar{P}^\bullet, \bar{P}^\bullet[i]) \simeq \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$. Also, by Lemma 1.8 $\nu\bar{P}^\bullet \simeq F(\nu P^\bullet)$ and hence $\mathrm{add}(\bar{P}^\bullet) = \mathrm{add}(\nu\bar{P}^\bullet)$. Furthermore,

$$\begin{aligned} \mathrm{H}^i(\bar{P}^\bullet) &\simeq \mathrm{Hom}_{\mathsf{K}(\text{Mod-}B)}(B, \bar{P}^\bullet[i]) \\ &\simeq \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) \\ &= 0 \end{aligned}$$

for $i \geq l$ and $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}B)}(\bar{P}^\bullet[i], B) \simeq \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet[i], T^\bullet) = 0$ for $i < 0$, so that $l(\bar{P}^\bullet) \leq l-1$. Thus by induction hypothesis there exists some $\bar{Q}^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_B)$ such that $\bar{Q}^\bullet \oplus \bar{P}^\bullet$ is a tilting complex. Then, by setting $Q^\bullet = G(\bar{Q}^\bullet)$, $Q^\bullet \oplus P^\bullet$ is a tilting complex. □

Theorem 3.7. *Assume A is selfinjective and representation-finite. Then for any selfinjective artin R-algebra B derived equivalent to A the following hold.*

(1) There exists a sequence of selfinjective artin R-algebras $A = B_0, B_1, \cdots, B_m = B$ such that for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1.

(2) The Nakayama permutation of B coincides with that of A.

Proof. (1) Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex with $B \simeq \mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet)$. Set $l = l(P^\bullet)$. In case $l \leq 1$, we have nothing to prove. Assume $l \geq 2$. Let $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex constructed in Lemma 3.4. Set $B_1 = \mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet)$ and let $F : \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A) \to \mathsf{K}^{\mathrm{b}}(\mathcal{P}_{B_1})$ be an equivalence of triangulated categories which sends $T^\bullet$ to B_1. Note that B_1 is selfinjective and representation-finite, and that $P_1^\bullet = F(P^\bullet)$ is a tilting complex with $B \simeq \mathrm{End}_{\mathsf{K}(\text{Mod-}B_1)}(P_1^\bullet)$. Also, as in the proof of Theorem 3.6, we have $l(P_1^\bullet) \leq l-1$. The assertion now follows by induction.

(2) By (1) and Lemma 3.4. □

References

[1] S. Al-Nofayee and J. Rickard, Rigidity of tilting complexes and derived equivalence for self-injective algebras, Preprint.

[2] M. Auslander, I. Reiten and S. O. Smalø, *Representation theory of artin algebras*, Cambridge studies in advanced mathematics., 36, Cambridge University Press, 1995.

[3] M. Bökstedt and A. Neeman, Homotopy limits in triangulated categories, Compositio Math. **86** (1993), no. 2, 209–234.

[4] K. Bongartz, Tilted algebras, in: *Representations of algebras (Puebla, 1980)*, 26–38, Lecture Notes in Math., 903, Springer, Berlin.

[5] S. E. Dickson, A torsion theory for Abelian categories, Trans. Amer. Math. Soc. **121** (1966), 223–235.

[6] D. Happel and C. M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. **274** (1982), no. 2, 399–443.

[7] D. Happel, J. Rickard and A. Schofield, Piecewise hereditary algebras, Bull. London Math. Soc. **20** (1988), no. 1, 23–28.

[8] R. Hartshorne, *Residues and duality*, Lecture Notes in Math., 20, Springer, Berlin, 1966.

[9] M. Hoshino, On splitting torsion theories induced by tilting modules, Comm. Algebra **11** (1983), no. 4, 427–439.

[10] M. Hoshino and Y. Kato, Tilting complexes defined by idempotents, Comm. Algebra **30** (2002), no. 1, 83–100.

[11] M. Hoshino and Y. Kato, Tilting complexes associated with a sequence of idempotents, J. Pure Appl. Algebra **183** (2003), no. 1-3, 105–124.

[12] M. Hoshino, Y. Kato and J.-I. Miyachi, On t-structures and torsion theories induced by compact objects, J. Pure Appl. Algebra **167** (2002), no. 1, 15–35.

[13] Y. Kato, On derived equivalent coherent rings, Comm. Algebra **30** (2002), no. 9, 4437–4454.

[14] S. König and A. Zimmermann, Tilting selfinjective algebras and Gorenstein orders, Quart. J. Math. Oxford Ser. (2) **48** (1997), no. 191, 351–361.

[15] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z. **193** (1986), no. 1, 113–146.

[16] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.

[17] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra **61** (1989), 303–317.

[18] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. (2) **43** (1991), no. 1, 37–48.

[19] J. L. Verdier, Catégories dérivées, état 0, in: *Cohomologie étale*, 262–311, Lecture Notes in Math., 569, Springer, Berlin, 1977.

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