Topological entropy of piecewise embedding maps on regular curves

著者別名	加藤 久男
journal or publication title	Ergodic theory and dynamical systems
volume	26
number	04
page range	1115-1125
year	2006-08
権利	(C)2006 Cambridge University Press
URL	http://hdl.handle.net/2241/104375

doi: 10.1017/S0143385706000174

Ergod. Th. & Dynam. Sys. (2006), **26**, 1115–1125
doi:10.1017/S0143385706000174

Topological entropy of piecewise embedding maps on regular curves

HISAO KATO

Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
(e-mail: hisakato@sakura.cc.tsukuba.ac.jp)

(*Received* 7 *September* 2005 *and accepted in revised form* 23 *March* 2006)

Abstract. It is well known that in the dynamics of a piecewise strictly monotone (that is, piecewise embedding) map f on an interval, the topological entropy can be expressed in terms of the growth of the number (that is, the lap number) of strictly monotone intervals for f^n. Recently, there has been an increase in the importance of fractal sets in the sciences, and many geometric and dynamical properties of fractal sets have been studied. In the present paper, we shall study topological entropy of some maps on regular curves, which are contained in the class of fractal sets. We generalize the theorem of Misiurewicz–Szlenk and Young to the cases of regular curves and dendrites.

1. *Introduction*

Recently, there has been an increase in the importance of fractal sets in the sciences and many geometric and dynamical properties of fractal sets have been studied. In the present paper, we shall study topological entropy of some maps on regular curves, which are contained in the class of fractal sets.

In [**11**], Seidler proved that the topological entropy of every homeomorphism on a regular curve is zero. In [**3**], Efremova and Makhrova proved that the topological entropy of every onto monotone map on a dendrite which satisfies some special condition is zero. In [**5**], we investigated the topological entropy of confluent maps on regular curves. As a corollary, the topological entropy of every onto monotone map on any regular curve is zero. In [**6**], we evaluated the topological entropy of general maps f on regular curves X in terms of the growth of the number of components of $f^{-n}(x)$ $(x \in X)$. It is well known that in the dynamics of a piecewise strictly monotone (that is, piecewise embedding) map f on an interval, the topological entropy can be expressed in terms of the growth of the number (that is, the lap number) of strictly monotone intervals for f^n (see Misiurewicz and Szlenk [**8**] and Young [**13**], and also see [**7**, Theorem 7.1]). In this paper, we generalize the theorem of Misiurewicz–Szlenk and Young [**8, 13**] to the cases of regular curves and dendrites.

All spaces considered in this paper are assumed to be separable metric spaces. *Maps* are continuous functions. For a space X, let $\mathrm{Comp}(X)$ be the set of all components of X.

By a *compactum* X we mean a compact metric space. A *continuum* is a non-empty connected compactum. For a set A, $|A|$ denotes the cardinality of the set A. A map $f : X \to Y$ of compacta is an *embedding map* if $f : X \to f(X)$ is a homeomorphism. A map $f : X \to Y$ of compacta is *monotone* if for each $y \in f(X)$, $f^{-1}(y)$ is connected. It is well known that if $f : X \to Y$ is a monotone map, then $f^{-1}(C)$ is also a continuum for any subcontinuum C of $f(X)$.

A continuum X is a *regular continuum* (that is, *regular curve*) if for each $x \in X$ and each open neighborhood V of x in X, there is an open neighborhood U of x in X such that $U \subset V$ and the boundary set $Bd(U)$ of U is a finite set. Clearly, each regular curve is a *Peano curve* (that is, one-dimensional locally connected continuum). For each $p \in X$, we define the cardinal number $ls_X(p)$ of p as follows: $ls_X(p) \leq \alpha$ (α is a cardinal number) if and only if for any neighborhood V of p there is a neighborhood $U \subset V$ of p in X such that $|\mathrm{Comp}(U - \{p\})| \leq \alpha$, and $ls_X(p) = \alpha$ if and only if $ls_X(p) \leq \alpha$ and the inequality $ls_X(p) \leq \beta$ for $\beta < \alpha$ does not hold. We define $ls(X) < \infty$ if $ls_X(p) < \infty$ for each $p \in X$.

A continuum X is a *dendrite* (that is, one-dimensional compact absolute retract (AR)) if X is a locally connected continuum which contains no simple closed curve. It is well known that each local dendrite (that is, one-dimensional compact absolute neighborhood retract (ANR)) is a regular curve. Note that each graph (that is, one-dimensional finite polyhedron) is a local dendrite. There are many regular curves which are not local dendrites. Many *fractal sets* (see [**2, 4**]) are regular curves which are not local dendrites. For example, the Sierpinski triangle S is a well-known regular curve with $ls_S(p) \leq 2$ for each $p \in S$. The Menger universal curve and the Sierpinski carpet are not regular curves.

Let X be a regular continuum. A finite closed covering $\mathcal{A}$ of a regular curve X is a *regular partition* of X provided that if $A, A' \in \mathcal{A}$ and $A \neq A'$, then $\mathrm{Int}(A) \neq \phi$, $A \cap A' = Bd(A) \cap Bd(A')$, and $Bd(A)$ is a finite set. We can easily see that if X is a regular curve and $\epsilon > 0$, then there is a regular partition $\mathcal{A}$ of X such that mesh $\mathcal{A} < \epsilon$, that is, diam $A < \epsilon$ for each $A \in \mathcal{A}$.

Let $\mathcal{A}$ be a regular partition of X. Moreover, $\mathcal{A}$ is called a *strongly regular partition* if $ls_X(a) < \infty$ for each $a \in \bigcup\{Bd(A) |\ A \in \mathcal{A}\}$.

Let X be a regular curve and $\mathcal{A}$ a regular partition of X. A map $f : X \to X$ is a *piecewise embedding map* with respect to $\mathcal{A}$ if the restriction $f|A : A \to X$ is an embedding (that is, injective) map for each $A \in \mathcal{A}$. A map $f : X \to X$ is a *piecewise monotone map* with respect to $\mathcal{A}$ if the restriction $f|A : A \to f(A)$ is a monotone map for each $A \in \mathcal{A}$.

2. *Topological entropy and piecewise embedding maps on regular curves*

Let $f : X \to X$ be a map of a compactum X and let $K \subset X$ be a closed subset of X. We define the topological entropy $h(f, K)$ of f with respect to K as follows (see [**1, 7, 12**]). Let n be a natural number and $\epsilon > 0$. A subset F of K is an (n, ϵ)-*spanning set* for f with respect to K if for each $x \in K$, there is $y \in F$ such that

$$\max\{d(f^i(x), f^i(y)) |\ 0 \leq i \leq n-1\} < \epsilon.$$

A subset E of K is an (n, ϵ)-*separated set* for f with respect to K if for each $x, y \in E$ with $x \neq y$, there is $0 \leq j \leq n-1$ such that

$$d(f^j(x), f^j(y)) > \epsilon.$$

Let $r_n(\epsilon, K)$ be the smallest cardinality of all (n, ϵ)-spanning sets for f with respect to K. Also, let $s_n(\epsilon, K)$ be the maximal cardinality of all (n, ϵ)-separated sets for f with respect to K. Put

$$r(\epsilon, K) = \limsup_{n\to\infty} (1/n) \log r_n(\epsilon, K)$$

and

$$s(\epsilon, K) = \limsup_{n\to\infty} (1/n) \log s_n(\epsilon, K).$$

Also, put

$$h(f, K) = \lim_{\epsilon\to 0} r(\epsilon, K).$$

Then it is well known that $h(f, K) = \lim_{\epsilon\to 0} s(\epsilon, K)$. Finally, put

$$h(f) = h(f, X).$$

It is well known that $h(f)$ is equal to the topological entropy which was defined by Adler *et al.* (see [**1**]).

The following theorem of Misiurewicz–Szlenk and Young is well known (see [**8, 13**] and [**7**, Theorem 7.1]).

THEOREM 2.1. (Misiurewicz–Szlenk [**8**] and Young [**13**]) *If $f : I = [0, 1] \to I$ is a piecewise embedding map (i.e. there is a finite sequence $c_1, c_2, \ldots, c_k$ of I such that $c_0 = 0 < c_1 < c_2 < \cdots < c_k = 1$, each restriction $f|[c_i, c_{i+1}] : [c_i, c_{i+1}] \to I$ is an embedding (that is, strictly monotone) map and each c_i $(i = 1, 2, \ldots, k-1)$ is a turning point of f), then*

$$h(f) = \lim_{n\to\infty} (1/n) \log l(f^n),$$

where $l(f^n)$ denotes the lap number of f^n.

Let $f : X \to X$ be a map of a regular curve X and let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a regular partition of X. For each $n \geq 0$, consider the itinerary set $It(f, n; \mathcal{A})$ for f and n defined by

$$It(f, n; \mathcal{A}) = \left\{ (x_0, x_1, \ldots, x_{n-1}) \,\middle|\, x_i \in \{1, 2, \ldots, m\} \quad \text{and} \quad \bigcap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(A_{x_i})) \neq \phi \right\}.$$

Put $I(f, n; \mathcal{A}) = |It(f, n; \mathcal{A})|$. Note that $I(f, n+m; \mathcal{A}) \leq I(f, n; \mathcal{A}) \cdot I(f, m; \mathcal{A})$. Hence, we see that the limit $\lim_{n\to\infty}(1/n) \log I(f, n; \mathcal{A})$ exists (see [**12**, Theorem 4.9, p. 87]). Note that if $f : I \to I$ is a piecewise embedding map of the unit interval I as in Theorem 2.1, then $l(f^{n-1}) = I(f, n; \mathcal{A})$, where $\mathcal{A} = \{[c_i, c_{i+1}] \mid i = 0, 1, \ldots, k-1\}$.

First, we generalize the theorem of Misiurewicz–Szlenk and Young to the case of piecewise embedding maps with respect to strongly regular partitions of regular curves.

THEOREM 2.2. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$h(f) = \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

We need the following Bowen's result (see [**7**, Theorem 7.1]).

PROPOSITION 2.3. (Bowen) *Let X and Y be compacta, and let $f : X \to X$, $g : Y \to Y$ be maps. If $\pi : X \to Y$ is an onto map such that $\pi \cdot f = g \cdot \pi$, then*

$$h(g) \le h(f) \le h(g) + \sup_{y\in Y} h(f, \pi^{-1}(y)).$$

LEMMA 2.4. *Let $f : X \to X$ be a map of a regular curve X and let C be a subcontinuum of X such that $f^n|C : C \to f^n(C)$ is a monotone map for each $n \ge 1$. Then $h(f, C) = 0$.*

Proof. Let $\epsilon > 0$ and n a natural number. Since X is a regular curve, we can choose a regular partition $\mathcal{A}_\epsilon$ of X with mesh $\mathcal{A}_\epsilon < \epsilon$. Recall that $\mathcal{A}_\epsilon$ is a finite closed cover of X such that if $A, A' \in \mathcal{A}_\epsilon$ and $A \ne A'$, then $\mathrm{Int}(A) \ne \phi$, $A \cap A' = Bd(A) \cap Bd(A')$, $Bd(A)$ is a finite set, and diam $A < \epsilon$. Put

$$B_\epsilon = \bigcup\{Bd(A) |\ A \in \mathcal{A}_\epsilon\}.$$

Let $L_\epsilon = |B_\epsilon| < \infty$. Suppose that $B_\epsilon \cap f^i(C) \ne \phi$ for some $0 \le i \le n-1$. Note that for each $b \in B_\epsilon \cap f^i(C)$, $f^{-i}(b) \cap C = (f^i|C)^{-1}(b)$ is connected. If $f^{-i}(b) \cap C$ contains no element of B_ϵ, we choose a point $c = c(b, i) \in f^{-i}(b) \cap C$. If $f^{-i}(b) \cap C$ contains an element of B_ϵ, we choose a point $c = c(b, i) \in B_\epsilon \cap f^{-i}(b) \cap C$. Consider the set

$$F = \{c(b, i) | b \in B_\epsilon \cap f^i(C),\ 0 \le i \le n-1\}.$$

If $F = \phi$, then $\{c\}$ is an (n, ϵ)-spanning set for f with respect to C for any point $c \in C$. We may assume that $F \ne \phi$. Note that $|F| \le n \cdot L_\epsilon$. By the proof of [**6**, Theorem 2.1], we see that F is an (n, ϵ)-spanning set for f with respect to C. Hence, we see that $h(f, C) = 0$.

COROLLARY 2.5. *Let $f : X \to X$ be a map of a regular curve X. If C is a subcontinuum of X such that $f^n|C : C \to X$ is an embedding map for each $n \ge 1$, then $h(f, C) = 0$.*

The following proposition is well known as the boundary bumping theorem in continuum theory.

PROPOSITION 2.6. [**9**, p. 75] *Let X be a continuum and U a non-empty open set of X. If D is a component of U, then $\overline{D} \cap Bd(U) \ne \phi$.*

Proof of Theorem 2.2. Let $\mathcal{A} = \{A_i \mid i = 1, 2, \ldots, m\}$. Put

$$B = \bigcup\{Bd(A) \mid A \in \mathcal{A}\}.$$

First, we consider the following set:

$$\sum(f, \mathcal{A}) = \left\{ (x_i)_{i=0}^{\infty} \;\middle|\; A_{x_i} \in \mathcal{A} \text{ and } \bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i})) \ne \phi \text{ for all } n = 0, 1, 2, \ldots \right\}.$$

Then we see that $\sum(f, \mathcal{A})$ is a closed subset of the Cantor set $\{1, 2, \dots, m\}^N$, where $N = \{1, 2, 3, \dots\}$. Put

$$\sum_X(f, \mathcal{A}) = \left\{ (x, (x_i)_{i=0}^{\infty}) \in X \times \sum(f, \mathcal{A}) \,\middle|\, x \in \bigcap_{n=0}^{\infty} \overline{\bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i}))} \right\}.$$

Then we show that $\sum_X(f, \mathcal{A})$ is a closed subset of the compact set $X \times \sum(f, \mathcal{A})$. Suppose that $(x^j, (x_i^j)_{i=0}^{\infty}) \in \sum_X(f, \mathcal{A})$ $(j = 1, 2, \dots)$ and $\lim_{j\to\infty}(x^j, (x_i^j)_{i=0}^{\infty}) = (x, (x_i)_{i=0}^{\infty}) \in X \times \sum(f, \mathcal{A})$. Let n be an arbitrary natural number. Then there is j_0 such that if $j \geq j_0$, then $x^j \in \overline{\bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i}))}$. Hence, we see that $x \in \overline{\bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i}))}$. This implies that $(x, (x_i)_{i=0}^{\infty}) \in \sum_X(f, \mathcal{A})$. Hence, $\sum_X(f, \mathcal{A})$ is a closed subset of $X \times \sum(f, \mathcal{A})$.

Note that if $(x, (x_i)_{i=0}^{\infty}) \in \sum_X(f, \mathcal{A})$, then $(f(x), (x_{i+1})_{i=0}^{\infty}) \in \sum_X(f, \mathcal{A})$. Define maps $\sigma_1 : \sum_X(f, \mathcal{A}) \to \sum_X(f, \mathcal{A})$, $\sigma_2 : \sum(f, \mathcal{A}) \to \sum(f, \mathcal{A})$ by

$$\sigma_1(x, (x_i)_{i=0}^{\infty}) = (f(x), (x_{i+1})_{i=0}^{\infty}) \quad \text{and} \quad \sigma_2((x_i)_{i=0}^{\infty}) = (x_{i+1})_{i=0}^{\infty}.$$

Then we have the following commutative diagram

$$\begin{tikzcd}
\sum(f, \mathcal{A}) \arrow[r, "\sigma_2"] & \sum(f, \mathcal{A}) \\
\sum_X(f, \mathcal{A}) \arrow[r, "\sigma_1"] \arrow[u, "\pi_2"] \arrow[d, "\pi_1"'] & \sum_X(f, \mathcal{A}) \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \\
X \arrow[r, "f"] & X
\end{tikzcd}$$

where π_1 and π_2 are the natural projections. Since f is a piecewise embedding map with respect to $\mathcal{A}$ and B is a finite set, we see that π_1 is surjective.

We will show that if $x \in X$, then

$$|\pi_1^{-1}(x)| \leq \max\{ls_X(a) \mid a \in B\}.$$

Let $x \in X$ and $(x, (x_i)_{i=0}^{\infty}) \in \sum_X(f, \mathcal{A})$. If $f^i(x) \in \mathrm{Int}(A_{x_i})$ for each $i \geq 0$, then $\pi_1^{-1}(x) = \{(x, (x_i)_{i=0}^{\infty})\}$ and hence $|\pi_1^{-1}(x)| = 1$. Otherwise, there is a natural number $i_0 \geq 0$ such that $f^{i_0}(x) \in B$ and $f^i(x) \in \mathrm{Int}(A_{x_i})$ for $0 \leq i < i_0$. By induction, we choose the sequence $(C_i)_{i=i_0}^{\infty}$ of subcontinua of X such that $\phi \neq C_i - Bd(A_{x_i}) \subset \mathrm{Int}(A_{x_i})$, C_{i_0} is the component of $A_{x_{i_0}}$ containing $f^{i_0}(x)$, and for each $i \geq i_0 + 1$, C_i is the component of $f(C_{i-1}) \cap A_{x_i}$ containing $f^i(x)$. Note that $ls_{C_{i_0}}(f^{i_0}(x)) \leq ls_X(f^{i_0}(x)) \leq \max\{ls_X(a) \mid a \in B\}$ and $ls_{C_i}(f^i(x)) \geq ls_{C_{i+1}}(f^{i+1}(x))$ for $i \geq i_0$.

Suppose that for some $i_1 > i_0$ and $A \in \mathcal{A}$ with $A \neq A_{x_{i_1}}$,

$$x \in \overline{\bigcap_{j=0}^{i_1-1} f^{-j}(\mathrm{Int}(A_{x_j})) \cap f^{-i_1}(\mathrm{Int}(A))}.$$

Then $f^{i_1}(x) \in Bd(A_{x_{i_1}})$ and $f(C_{i_1-1})$ contains a point of $\mathrm{Int}(A)$. Then we see that $ls_{C_{i_1}}(f^{i_1}(x)) < ls_{C_{i_1-1}}(f^{i_1-1}(x))$. By continuing this procedure, we see that

$$|\pi_1^{-1}(x)| \leq \max\{ls_X(a) \mid a \in B\},$$

which implies that $h(\sigma_1, \pi_1^{-1}(x)) = 0$ for each $x \in X$. Then $h(f) = h(\sigma_1)$ (see Proposition 2.3).

Next, we show that $h(\sigma_2, \pi_2^{-1}((x_i)_{i=0}^{\infty})) = 0$ for each $(x_i)_{i=0}^{\infty} \in \sum(f, \mathcal{A})$. Note that

$$\pi_2^{-1}((x_i)_{i=0}^{\infty}) = \bigcap_{n=0}^{\infty} \overline{\bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i}))}.$$

We will show that

$$\left|\mathrm{Comp}\left(\overline{\bigcap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(A_{x_i}))}\right)\right| \le n \cdot |B|,$$

where $B = \bigcup\{Bd(A) | \ A \in \mathcal{A}\}$. Define a function

$$u : \mathrm{Comp}\left(\overline{\bigcap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(A_{x_i}))}\right) \to \{0, 1, 2, \ldots, n-1\} \times B$$

by $u(C) = (i, y)$, where i and y satisfy the following condition: $f^i(C) \cap B \neq \phi$ and $y \in f^i(C) \cap B$. Clearly, we see that u is injective. Hence,

$$\left|\mathrm{Comp}\left(\overline{\bigcap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(A_{x_i}))}\right)\right| \le n \cdot |B|.$$

Let $\epsilon > 0$ and $\mathcal{A}_{\epsilon/2}$ be a regular partition of X with mesh$\mathcal{A}_{\epsilon/2} < \epsilon/2$. By the proof of Lemma 2.4, we see that there is an $(n, \epsilon/2)$-spanning set F for f with respect to $\overline{\bigcap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(A_{x_i}))}$ such that $|F| \le (n \cdot L') \cdot (n \cdot |B|)$, where

$$L' = \left|\bigcup\{Bd(A') \mid A' \in \mathcal{A}_{\epsilon/2}\}\right|.$$

Hence, we see that if E is any (n, ϵ)-separating set for f with respect to $\bigcap_{n=0}^{\infty} \overline{\bigcap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(A_{x_i}))}$ such that $E \subset \bigcap_{n=0}^{\infty} \overline{\bigcap_{i=0}^{n-1} f^{-i}(\mathrm{Int}(A_{x_i}))}$, we can choose a function $T : E \to F$ such that

$$d(f^i(x), f^i(T(x))) < \left(\frac{1}{2}\right)\epsilon$$

for $x \in E$ and each $i = 0, 1, \ldots, n-1$. Then $T : E \to F$ is one-to-one and, hence,

$$|E| \le |F| \le n^2 \cdot L' \cdot |B|.$$

By using this fact, we see that $h(\sigma_1, \pi_2^{-1}((x_i)_{i=0}^{\infty})) = 0$. Then $h(\sigma_2) = h(\sigma_1)$. Since f is a piecewise embedding map with respect to $\mathcal{A}$ and B is a finite set, we see that

$$I(f, n; \mathcal{A}) = \left|\left\{(x_i)_{i=0}^{n-1} \ \middle| \ (x_i)_{i=0}^{\infty} \in \sum(f, \mathcal{A})\right\}\right|.$$

By [**10**, Theorem 1.9(a), p. 340], we know that

$$h(\sigma_2) = \limsup_{n\to\infty}(1/n) \log I(f, n; \mathcal{A}) = \lim_{n\to\infty}(1/n) \log I(f, n; \mathcal{A}).$$

Consequently, we conclude that

$$h(f) = \lim_{n\to\infty}(1/n) \log I(f, n; \mathcal{A}).$$

This completes the proof. □

By the proof of Theorem 2.2, we have the following.

THEOREM 2.7. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a regular partition $\mathcal{A}$ of X, then*

$$h(f) \leq \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

Let $f : X \to X$ be a piecewise embedding map of a regular curve X with respect to a regular partition $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ of X. Note that $m = |\mathcal{A}|$. Define a $m \times m$ matrix $M_f = (a_{ij})$ by the following: $a_{ij} = 1$ if $f(\mathrm{Int}(A_i)) \supset \mathrm{Int}(A_j)$, and $a_{ij} = 0$ otherwise. Also, define an $m \times m$ matrix $N_f = (b_{ij})$ by the following: $b_{ij} = 1$ if $f(\mathrm{Int}(A_i)) \cap \mathrm{Int}(A_j) \neq \phi$, and $b_{ij} = 0$ otherwise. Let $\lambda(M_f)$ be the real eigenvalue of M_f such that $\lambda(M_f) \geq |\lambda|$ for all the other eigenvalue λ of M_f. Then we have the following corollary (see [**10**, Theorem 1.9(b), p. 340]).

COROLLARY 2.8. *Let X be a regular curve. If a map $f : X \to X$ is a piecewise embedding map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$\lambda(M_f) \leq h(f) \leq \lambda(N_f).$$

Example 1. (1) The assertions of Theorems 2.2 and 2.7 are not true for piecewise embedding maps on Peano curves. Let $X = \mu^1$ be the Menger universal curve. We can choose a homeomorphism $f : X \to X$ such that $h(f) \neq 0$ (see [**5**, Examples (2)]). Then f is also a piecewise embedding map with respect to $\mathcal{A} = \{X\}$ and

$$h(f) > 0 = \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

(2) There is a piecewise embedding map $f : X \to X$ of a regular curve X with respect to a regular partition $\mathcal{A}$ of X such that

$$h(f) < \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

In the plane R^2, let $p = (0, 0) \in R^2$. Take two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of points of R^2 such that $\lim_{n\to\infty} a_n = p = \lim_{n\to\infty} b_n$, where p, a_n and $b_n (n = 1, 2, \ldots)$ are distinct points of R^2. Let $\overline{p\,q}$ be the segment from p to a point q in R^2. We assume that the sets $\overline{p\,a_n}$ and $\overline{p\,b_n}$ $(n = 1, 2, \ldots)$ have the only one common point $p \in R^2$. Put

$$X = \bigcup_{n=1}^{\infty} \overline{p\,a_n} \cup \bigcup_{n=1}^{\infty} \overline{p\,b_n}.$$

Note that $ls_X(p) = \chi_0$. Let $A_0 = \bigcup_{n=1}^{\infty} \overline{p\,a_n}$ and $A_1 = \bigcup_{n=1}^{\infty} \overline{p\,b_n}$. Then $\mathcal{A} = \{A_0, A_1\}$ is a regular partition of the dendrite X. Take bijections $g_0 : \{a_n \mid n = 1, 2, \ldots\} \to \{a_n, b_n \mid n = 1, 2, \ldots\}$ and $g_1 : \{b_n \mid n = 1, 2, \ldots\} \to \{a_n, b_n \mid n = 1, 2, \ldots\}$. Define a surjective function $g : \{a_n, b_n \mid n = 1, 2, \ldots\} \to \{a_n, b_n \mid n = 1, 2, \ldots\}$ by $g|\{a_n \mid n = 1, 2, \ldots\} = g_0$ and $g|\{b_n \mid n = 1, 2, \ldots\} = g_1$. Define a map $f : X \to X$ by

$$f((1-t)p + ty) = (1-t)p + t((1/2)p + (1/2)g(y)),$$

where $y \in \{a_n, b_n \mid n = 1, 2, \ldots\}$. Note that $I(f, n; \mathcal{A}) = 2^n$ and $\bigcap_{n=0}^{\infty} f^n(X) = \{p\}$. Then f is a piecewise embedding map with respect to the regular partition $\mathcal{A}$ such that

$$h(f) = 0 < \log 2 = \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

Note that $\mathcal{A}$ is not a strongly regular partition of X. Hence, the assertion of Theorem 2.2 is not true for piecewise embedding maps with respect to regular partitions of regular curves.

(3) Moreover, there is a homeomorphism $f : X \to X$ of a dendrite X such that

$$h(f) < \lim_{n\to\infty}(1/n)\log I(f, n; \mathcal{A})$$

for some regular partition $\mathcal{A}$ of X. Let p, a_n, b_n $(n = 1, 2, \dots)$, X and $\mathcal{A}$ be the same as in (2). Put $H_0 = \{a_n | \, n = 1, 2, \dots\}$ and $H_1 = \{b_n \mid n = 1, 2, \dots\}$. Let K be any subset of $H_0 \cup H_1 = \{a_n, b_n \mid n = 1, 2, \dots\}$ such that $K \cap H_0$ and $K \cap H_1$ are (countable) infinite sets. Let $(x_1, x_2, \dots, x_m) \in \{0, 1\}^m$ $(m \geq 2)$. Take infinite subsets G_i $(i = 1, \dots, m)$ of K such that G_i $(i = 1, \dots, m)$ are mutually disjoint, $G_i \subset H_{x_i}$ and $K - \bigcup_{i=1}^m G_i$ is an infinite set. Take a bijection $g_{(x_1,x_2,\dots,x_n)} : K \to K$ such that $g_{(x_1,x_2,\dots,x_n)}(G_i) = G_{i+1}$ for each $i = 1, 2, \dots, m-1$ and $g_{(x_1,x_2,\dots,x_n)}\big(K - \bigcup_{i=1}^{m-1} G_i\big) = K - \bigcup_{i=2}^m G_i$. Put $H_0 = \bigcup_{i=1}^\infty H_0^i$ and $H_1 = \bigcup_{i=1}^\infty H_1^i$, where H_0^i $(i = 1, 2, \dots)$ and H_1^i $(i = 1, 2, \dots)$ are mutually disjoint infinite sets. Put $K_j = H_0^j \cup H_1^j$. Since $F = \{(x_1, x_2, \dots, x_m) \in \{0, 1\}^m \mid m = 2, 3, \dots\}$ is a countable set, we can put $F = \{s_j \mid j = 1, 2, 3, \dots\}$. For each $s_j = (x_1, x_2, \dots, x_m)$, we obtain a bijection $g_{s_j} = g_{(x_1,x_2,\dots,x_m)} : K_j \to K_j$ as above. Define a bijection $g : H_0 \cup H_1 \to H_0 \cup H_1$ by $g|K_j = g_{sj}$. Finally, we define a homeomorphism $f : X \to X$ such that $f(p) = p$, $f(q) = g(q)$ for $q \in H_0 \cup H_1$ and $f|\overline{p\,q} : \overline{p\,q} \to \overline{p\,g(q)}$ $(q \in H_0 \cup H_1)$ is the natural homeomorphism. Note that $I(f, n; \mathcal{A}) = 2^n$. Then $f : X \to X$ is a homeomorphism of the dendrite X such that

$$h(f) = 0 < \log 2 = \lim_{n\to\infty}(1/n)\log I(f, n; \mathcal{A}).$$

3. *Topological entropy and piecewise monotone maps on dendrites*

In this section, we generalize the theorem of Misiurewicz–Szlenk and Young to the case of piecewise monotone maps with respect to strongly regular partitions of dendrites. Let $f : X \to X$ be a map and $x \in X$. Then x is *non-wandering* if for each neighborhood U of x in X, there exists $n \geq 1$ such that $f^n(U) \cap U \neq \phi$. $\Omega(f)$ is the set of all points which are non-wandering. It is well known that $\Omega(f)$ is a closed subset of X, $f(\Omega(f)) \subset \Omega(f)$ and $h(f) = h(f|\Omega(f))$ (see [**10**, Theorem 1.4, p. 336]). Recall the following notation. For a map $f : X \to X$ of a regular curve X and a regular partition $\mathcal{A} = \{A_i \mid i = 1, 2, \dots, m\}$ of X, we put

$$\sum(f, \mathcal{A}) = \left\{ (x_i)_{i=0}^\infty \;\middle|\; A_{x_i} \in \mathcal{A} \text{ and } \bigcap_{i=0}^n f^{-i}(\mathrm{Int}(A_{x_i})) \neq \phi \text{ for all } n = 0, 1, 2, \dots \right\}.$$

Also, let $\sigma_{(f,\mathcal{A})} = \sigma_2 : \sum(f, \mathcal{A}) \to \sum(f, \mathcal{A})$ be the shift map defined by $\sigma_{(f,\mathcal{A})}((x_i)_{i=0}^\infty) = (x_{i+1})_{i=0}^\infty$.

Then we have the following theorem.

THEOREM 3.1. *Let X be a dendrite. If a map $f : X \to X$ is a piecewise monotone map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$h(f) = h(\sigma_{(f,\mathcal{A})}).$$

We need the following simple lemma. For completeness, we give the proof.

LEMMA 3.2. *Let X, Y and Z be dendrites and let $f : X \to f(X) \subset Y$ and $g : Y \to g(Y) \subset Z$ be monotone maps. If C is a subcontinuum of X, then $g \cdot f|C : C \to g \cdot f(C)$ is also a monotone map and $\mathrm{ls}_C(p) \geq \mathrm{ls}_{f(C)}(f(p))$ for each $p \in C$.*

Proof. Note that if C' and C'' are subcontinua of the dendrite X and $C' \cap C'' \neq \phi$, then $C' \cap C''$ is also a continuum. Hence, we see that $f|C : C \to f(C)$ and $g|f(C) : f(C) \to g \cdot f(C)$ are onto monotone maps. Then we see that the composition $g \cdot f|C : C \to g \cdot f(C)$ is also a monotone map.

Suppose, on the contrary, that $ls_C(p) < ls_{f(C)}(f(p))$ for some $p \in C$. Then we can choose a component D of $C - \{p\}$ such that $f(D)$ intersects to two components E_1, E_2 of $f(C) - \{f(p)\}$. Then we see that $f^{-1}(f(p)) \cap C$ is not connected. This is a contradiction. □

Proof of Theorem 3.1. The notations are the same as in the proof of Theorem 2.2. Let $\mathcal{A} = \{A_i \mid i = 1, 2, \ldots, m\}$. Put

$$\sum(f, \mathcal{A}) = \left\{ (x_i)_{i=0}^{\infty} \;\middle|\; A_{x_i} \in \mathcal{A} \text{ and } \bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i})) \neq \phi \text{ for all } n = 0, 1, 2, \ldots \right\}$$

and

$$Y = \bigcup \left\{ \bigcap_{n=0}^{\infty} \left(\overline{\bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i}))} \right) \;\middle|\; (x_i)_{i=0}^{\infty} \in \sum(f, \mathcal{A}) \right\}.$$

We see that Y is a closed subset of X and $f(Y) \subset Y$. In fact, let $\{y_j\}_{j=1}^{\infty}$ be a sequence of points of Y such that $\lim_{j\to\infty} y_j = y \in X$. We can choose $(x_i^j)_{i=0}^{\infty} \in \sum(f, \mathcal{A})$ $(j = 1, 2, \ldots)$ such that $y_j \in \bigcap_{n=0}^{\infty}(\overline{\bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i^j}))})$. Since $\sum(f, \mathcal{A})$ is compact, we may assume that $\lim_{j\to\infty}(x_i^j)_{i=0}^{\infty} = (x_i)_{i=0}^{\infty}$. Then we see that $y \in \bigcap_{n=0}^{\infty}(\overline{\bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i}))})$, which implies that Y is a closed subset of X.

Consider the set

$$\sum_Y(f, \mathcal{A}) = \left\{ (y, (x_i)_{i=0}^{\infty}) \in Y \times \sum(f, \mathcal{A}) \;\middle|\; y \in \bigcap_{n=0}^{\infty} \overline{\bigcap_{i=0}^{n} f^{-i}(\mathrm{Int}(A_{x_i}))} \right\}.$$

Then $\sum_Y(f, \mathcal{A})$ is a closed subset of the compact set $Y \times \sum(f, \mathcal{A})$.

Define maps $\sigma_1 : \sum_Y(f, \mathcal{A}) \to \sum_Y(f, \mathcal{A})$, $\sigma_{(f,\mathcal{A})} : \sum(f, \mathcal{A}) \to \sum(f, \mathcal{A})$ by

$$\sigma_1(y, (x_i)_{i=0}^{\infty}) = (f(y), (x_{i+1})_{i=0}^{\infty}) \text{ and } \sigma_{(f,\mathcal{A})}((x_i)_{i=0}^{\infty}) = (x_{i+1})_{i=0}^{\infty}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc}
\sum(f, \mathcal{A}) & \xrightarrow{\sigma_{(f,\mathcal{A})}} & \sum(f, \mathcal{A}) \\
{\scriptstyle \pi_2}\uparrow & & \uparrow{\scriptstyle \pi_2} \\
\sum_Y(f, \mathcal{A}) & \xrightarrow{\sigma_1} & \sum_Y(f, \mathcal{A}) \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
Y & \xrightarrow{f|Y} & Y
\end{array}$$

By Lemma 3.2 and a similar argument to the proof of Theorem 2.2, we see that if $y \in Y$, then $|\pi_1^{-1}(y)| \leq \max\{ls_X(a) \mid a \in B\}$, which implies that $h(\sigma_1, \pi_1^{-1}(y)) = 0$ for each $y \in Y$. Then $h(f|Y) = h(\sigma_1)$ (see Proposition 2.3). Also, we see that $h(\sigma_{(f,\mathcal{A})}) = h(\sigma_1)$. Let $x \in X - Y$. Since B is a finite set and x is not contained in Y, we see that there is a neighborhood U of x in X such that for some $m \geq 1$, $f^m(U)$ is a point of B. If $x \in \Omega(f) - Y$, then x is a periodic point of f and x is an isolated point of $\Omega(f)$. Hence $h(f|Y) = h(f|\Omega(f)) = h(f)$. Consequently, we conclude that

$$h(f) = h(f|Y) = h(\sigma_{(f,\mathcal{A})}) \leq \lim_{n\to\infty}(1/n)\log I(f, n; \mathcal{A}).$$

This completes the proof. □

THEOREM 3.3. *Let X be a dendrite. If a map $f : X \to X$ is a piecewise monotone map with respect to a regular partition $\mathcal{A}$ of X, then*

$$h(f) \leq h(\sigma_{(f,\mathcal{A})}) \leq \lim_{n\to\infty}(1/n)\log I(f, n; \mathcal{A}).$$

Then we have the following corollary.

COROLLARY 3.4. *Let X be a dendrite. If a map $f : X \to X$ is a piecewise monotone map with respect to a strongly regular partition $\mathcal{A}$ of X, then*

$$\lambda(M_f) \leq h(f) \leq \lambda(N_f).$$

If X is a regular curve with $ls(X) < \infty$, then each regular partition is a strongly regular partition. Hence, we have the following.

COROLLARY 3.5. *Let X be a regular curve with $ls(X) < \infty$. If a map $f : X \to X$ is a piecewise embedding map with respect to a regular partition $\mathcal{A}$ of X, then*

$$h(f) = \lim_{n\to\infty}(1/n)\log I(f, n; \mathcal{A}).$$

COROLLARY 3.6. *Let X be a dendrite with $ls(X) < \infty$. If a map $f : X \to X$ is a piecewise monotone map with respect to a regular partition $\mathcal{A}$ of X, then*

$$h(f) = h(\sigma_{(f,\mathcal{A})}) \leq \lim_{n\to\infty}(1/n)\log I(f, n; \mathcal{A}).$$

Now, we have a slight generalization of the theorem of Misiurewicz–Szlenk [**8**] and Young [**13**].

COROLLARY 3.7. *If a map $f : I \to I$ of an interval I is a piecewise monotone map with respect to a regular partition $\mathcal{A}$ of I, then*

$$h(f) = h(\sigma_{(f,\mathcal{A})}) \leq \lim_{n\to\infty}(1/n)\log I(f, n; \mathcal{A}).$$

Example 2. Since the composition of onto monotone maps is also monotone, we see that for an onto monotone map $f : X \to X$ of a continuum X, $f^n : X \to X$ is also monotone for each $n \geq 1$. However, it is easy to see that there is a map $f : X \to X$ such that X is a regular curve, f is not onto, $f : X \to f(X)$ is monotone and $f^2 : X \to f^2(X)$ is not monotone. In fact, let $A = [0, 4]$ be the closed interval of the real line R and let P_i $(i = 1, 2, 3)$ be arcs such that $P_1 \cap A = \partial P_1 = \{0, 1\}$ and $P_i \cap A = \partial P_i = P_2 \cap P_3 = \{2, 3\}$ $(i = 2, 3)$. Also, let S_j $(j = 1, 2)$ be simple closed curves such that $S_1 \cap S_2 = \{4\}$. We assume that $P_1 \cap P_{j+1} = \phi = S_j \cap (P_1 \cup P_2 \cup P_3)$ $(j = 1, 2)$. Put $X = A \cup \bigcup_{i=1}^{3} P_i \cup \bigcup_{j=1}^{2} S_j$. Let $f : X \to X$ be a map satisfying the following conditions (1)–(3):

(1) $f|P_1 : P_1 \to f(P_1) = P_2$, $f|[0, 1] : [0, 1] \to f([0, 1]) = [2, 3]$, and $f|[1, 2] : [1, 2] \to f([1, 2]) = [3, 4]$ are the natural homeomorphisms;

(2) $f(P_3 \cup [3, 4] \cup \bigcup_{j=1}^{2} S_j) = \{4\}$, $f(P_2) = S_1$, $f([2, 3]) = S_2$; and

(3) $f|P_2 - \partial P_2 : P_2 - \partial P_2 \to S_1 - \{4\}$ and $f|(2, 3) : (2, 3) \to S_2 - \{4\}$ are homeomorphisms.

Then we see that $f : X \to f(X)$ is a desired monotone map on the graph X.

Finally, we have the following problems.

Problem 3.8. In the statement of Theorem 3.1, is the following equality true?

$$h(\sigma_{(f,\mathcal{A})}) = \lim_{n\to\infty} (1/n) \log I(f, n; \mathcal{A}).$$

Problem 3.9. Let X be a regular curve. Is it true that if a map $f : X \to X$ is a piecewise monotone map with respect to a strongly regular partition $\mathcal{A}$ of X, then $h(f) = \lim_{n\to\infty}(1/n) \log I(f, n; \mathcal{A})$?

REFERENCES

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.* **114** (1965), 309–319.

[2] M. Barnsley. *Fractal Everywhere*. Academic Press, Orlando, FL.

[3] L. S. Efremova and E. N. Makhrova. The dynamics of monotone maps of dendrites. *Sb. Math.* **192** (2001), 807–821.

[4] G. A. Edgar. *Measure, Topology, and Fractal Geometry (Undergraduate Texts in Mathematics)*. Springer, Berlin, 1990.

[5] H. Kato. Topological entropy of monotone maps and confluent maps on regular curves. *Topology Proc.* **28**(2) (2004), 587–593.

[6] H. Kato. Topological entropy of maps on regular curves. *Preprint*.

[7] W. de Melo and S. van Strien. *One Dimensional Dynamics*. Springer, Berlin, 1993.

[8] M. Misiurewicz and W. Szlenk. Entropy of piecewise monotone mappings. *Studia Math.* **67** (1980), 45–63.

[9] S. B. Nadler Jr. Continuum theory, an introduction. *Pure Appl. Math.* **158** (1992).

[10] C. Robinson. *Dynamical Systems: Stability, Symbolic Dynamics and Chaos*. CRC Press, Boca Raton, FL, 1995.

[11] G. T. Seidler. The topological entropy of homeomorphisms on one-dimensional continua. *Proc. Amer. Math. Soc.* **108** (1990), 1025–1030.

[12] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, Berlin, 1982.

[13] L. S. Young. On the prevalence of the horseshoes. *Trans. Amer. Math. Soc.* **263** (1981), 75–88.