# Convergence rates of approximate sums of Riemann integrals

Hiroyuki Tasaki
Graduate School of Pure and Applied Science, University of Tsukuba
Tsukuba Ibaraki 305-8571 Japan
tasaki@math.tsukuba.ac.jp

Keywords : convergence rate; Riemann sum; Riemann integral; trapezoidal sum

**Abstract**

We represent the convergence rates of the Riemann sums and the trapezoidal sums with respect to regular divisions and optimal divisions of a bounded closed interval to the Riemann integrals as some limits of their expanded error terms.

## 1 Introduction

The Riemann sums and the trapezoidal sums of functions defined on a bounded closed interval are well known as approximate sums of the Riemann integrals of the functions. In this paper the author represents the convergence rates of the Riemann sums and the trapezoidal sums as some limits of their expanded error terms.

Let $[a, b]$ be a bounded closed interval. We take an $n$-division $\Delta$ of $[a, b]$ defined by

$$\Delta : a = s_0 \leq s_1 \leq \cdots \leq s_{n-1} \leq s_n = b.$$

We denote by $D_n$ the division of $[a, b]$ defined by $s_i = a + i(b-a)/n$ and call it the regular $n$-division. For a function $f$ defined on $[a, b]$ and $s_{i-1} \leq \xi_i \leq s_i$ we define the Riemann sum $R(f; \Delta, \xi_i)$ by

$$R(f; \Delta, \xi_i) = \sum_{i=1}^{n} (s_i - s_{i-1}) f(\xi_i).$$

The width $d(\Delta)$ of $\Delta$ is defined as $d(\Delta) = \max\{s_i - s_{i-1} \mid 1 \leq i \leq n\}$. The Riemann integral of $f$ is defined as

$$\int_a^b f(x)dx = \lim_{d(\Delta)\to 0} R(f; \Delta, \xi_i)$$

and textbooks on calculus usually show that this limit exists for a continuous function $f$. In this paper we consider some limits of expanded error terms like

$$n\left|\int_a^b f(x)dx - R(f; \Delta, \xi_i)\right|, \qquad n^2\left|\int_a^b f(x)dx - R(f; \Delta, \xi_i)\right|$$

as $n \to \infty$. Chui [1] obtained such a limit of an expanded error term.

**Theorem 1.1 (Chui)** *If $f$ is twice differentiable and $f''$ is bound and almost everywhere continuous on $[a, b]$, then*

$$\begin{aligned} &\lim_{n\to\infty} n^2 \left\{\int_a^b f(x)dx - R\left(f; D_n, \frac{1}{2}(s_{i-1} + s_i)\right)\right\} \\ = \ &\frac{(b-a)^2}{24}\int_a^b f''(x)dx = \frac{(b-a)^2}{24}(f'(b) - f'(a)). \end{aligned}$$

In [1] the above theorem is formulated for the interval $[0, 1]$.

We consider not only regular divisions $D_n$ but also optimal divisions for lower Riemann sums and trapezoidal sums, so we explain about optimal divisions. We take a continuous function $f$ defined on $[a, b]$. For any division $\Delta$ of $[a, b]$ we take $s_{i-1} \leq \xi_i \leq s_i$ which satisfies $f(\xi_i) = \min_{[s_{i-1}, s_i]} f$ and define the lower Riemann sum

$$R(f; \Delta, \min) = R(f; \Delta, \xi_i).$$

The set of all $n$-divisions of $[a, b]$ is compact and

$$\Delta \mapsto R(f; \Delta, \min)$$

is continuous, so there exists an $n$-division $\Delta_n^\#$ at which the above function attains its maximum. This $n$-division $\Delta_n^\#$ is optimal for the lower Riemann sum $R(f; \Delta, \min)$. It may not be unique, but the sum $R(f; \Delta_n^\#, \min)$ is unique. Thus we can consider $R(f; \Delta_n^\#, \min)$. One of the main theorems of this paper is as follows:

**Theorem 1.2** *If $f$ is a function of class $C^1$ defined on $[a,b]$, then*

$$\lim_{n\to\infty} n\left\{\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right\} = \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2.$$

The trapezoidal sum $T(f;\Delta)$ of $f$ is defined as

$$T(f;\Delta) = \sum_{i=1}^{n}(s_i - s_{i-1})\frac{1}{2}(f(s_{i-1}) + f(s_i)).$$

We can obtain the limit of the expanded error term of the trapezoidal sum as follows:

**Theorem 1.3** *If $f$ is twice differentiable and $f''$ is bound and almost everywhere continuous on $[a,b]$, then*

$$\lim_{n\to\infty} n^2\left\{\int_a^b f(x)dx - T(f;D_n)\right\}$$
$$= -\frac{(b-a)^2}{12}\int_a^b f''(x)dx = -\frac{(b-a)^2}{12}(f'(b) - f'(a)).$$

We consider an optimal division for the trapezoidal sum.

$$\Delta \mapsto \left|\int_a^b f(x)dx - T(f;\Delta)\right|$$

is continuous, so there exists an $n$-division $\Delta_n^{t\#}$ at which the above function attains its minimum. This $n$-division $\Delta_n^{t\#}$ is optimal for the trapezoidal sum $T(f;\Delta)$. It may not be unique, but $\left|\int_a^b f(x)dx - T(f;\Delta_n^{t\#})\right|$ is unique. Thus we can consider it.

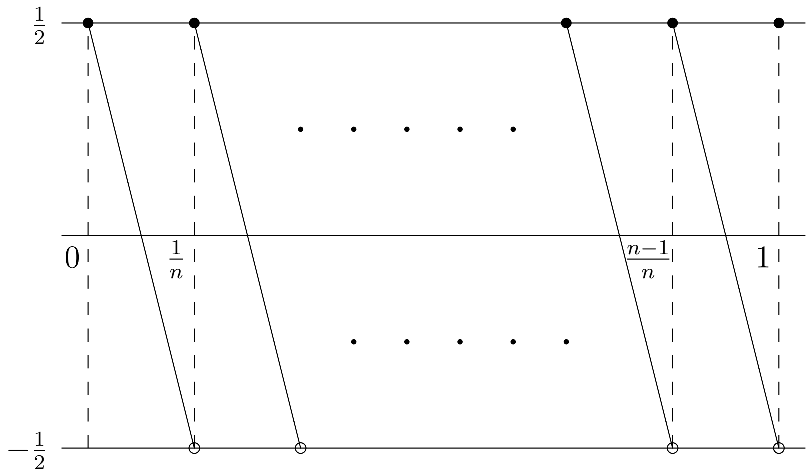
**Theorem 1.4** *If $f$ is a function of class $C^2$ defined on $[a,b]$ which satisfies $f'' \geq 0$ or $f'' \leq 0$, then*

$$\lim_{n\to\infty} n^2\left|\int_a^b f(x)dx - T(f;\Delta_n^{t\#})\right| = \frac{1}{12}\left(\int_a^b |f''(x)|^{1/3}dx\right)^3.$$

In the case where $f(x) = x^3$ on $[-1,1]$, if we take any division $\Delta$ of $[-1,1]$ all of whose points are symmetric at 0, then $\Delta$ satisfies $\int_{-1}^1 f(x)dx - T(f;\Delta) = 0$, however its trapezoidal sum on $[-1,0]$ may not be close to $\int_{-1}^0 f(x)dx$. So we only consider the case where $f'' \geq 0$ or $f'' \leq 0$.

## 2 Trapezoidal sums for regular divisions

In this section we treat regular divisions of intervals and prove Theorem 1.3. We can prove it in a way similar to that of Chui [1]. First we assume that $[a, b] = [0, 1]$. For each positive integer $n$ we define a function $v_n$ defined by the following graph.




This is a function of bounded variation which satisfies

$$-\frac{1}{2} < v_n(t) \leq \frac{1}{2}, \qquad v_n(0) = v_n(1) = \frac{1}{2}.$$

The Riemann-Stieltjes integral of $f$ with respect to $v_n$ is given by

$$(*) \qquad \int_0^1 f(t)dv_n(t) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_0^1 f(t)dt.$$

Thus we get

$$\begin{aligned}\frac{1}{n}\int_0^1 f(t)dv_n(t) &= \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(t)dt \\ &= T(f; D_n) + \frac{1}{2n}(f(1) - f(0)) - \int_0^1 f(t)dt.\end{aligned}$$

Since $f$ is Riemann integrable and $v_n$ is of bounded variation, we have

$$\begin{aligned}\int_0^1 f(t)dv_n(t) &= [f(t)v_n(t)]_0^1 - \int_0^1 v_n(t)df(t) \\ &= \frac{1}{2}(f(1) - f(0)) - \int_0^1 v_n(t)df(t)\end{aligned}$$

and

$$T(f;D_n)-\int_0^1 f(t)dt=-\frac{1}{n}\int_0^1 v_n(t)df(t)=-\frac{1}{n}\int_0^1 f'(t)v_n(t)dt.$$

We define

$$u_n(x)=\int_0^x v_n(t)dt,$$

which satisfies

$$u_n(0)=u_n\left(\frac{k}{n}\right)=0 \qquad (k=1,\ldots,n)$$

and is periodic with period $1/n$. From the above results we have

$$\begin{aligned}
& n^2\left\{T(f;D_n)-\int_0^1 f(t)dt\right\} \\
= \;& n\int_0^1 u_n(t)f''(t)dt=n\sum_{k=1}^{n}\int_{(k-1)/n}^{k/n} u_n(t)f''(t)dt \\
= \;& n\sum_{k=1}^{n}\int_0^{1/n} u_n\left(t+\frac{k-1}{n}\right)f''\left(t+\frac{k-1}{n}\right)dt \\
= \;& \sum_{k=1}^{n}\int_0^{1/n} nu_n(t)f''\left(t+\frac{k-1}{n}\right)dt \\
= \;& \sum_{k=1}^{n}\int_0^{1} u_n\left(\frac{x}{n}\right)f''\left(\frac{x+k-1}{n}\right)dx \\
= \;& \int_0^{1} u_n\left(\frac{x}{n}\right)\sum_{k=1}^{n}f''\left(\frac{x+k-1}{n}\right)dx.
\end{aligned}$$

For $0\le x\le 1$ we have

$$u_n\left(\frac{x}{n}\right)=\int_0^{x/n}\left(\frac{1}{2}-nt\right)dt=\frac{x-x^2}{2n}.$$

In particular $u_n(1/2n)=1/8n$. We define a function $w$ by

$$w(x)=\frac{1}{2}(x-x^2) \qquad (0\le x\le 1).$$

Then we get

$$u_n\left(\frac{x}{n}\right)=\frac{w(x)}{n} \qquad \text{and} \qquad \int_0^1 w(x)dx=\frac{1}{12}.$$

Using $w$ we obtain

$$\begin{aligned} n^2\left\{T(f;D_n) - \int_0^1 f(t)dt\right\} &= \int_0^1 u_n\left(\frac{x}{n}\right)\sum_{k=1}^n f''\left(\frac{x+k-1}{n}\right)dx \\ &= \int_0^1 w(x)\frac{1}{n}\sum_{k=1}^n f''\left(\frac{x+k-1}{n}\right)dx \\ &= \int_0^1 w(x)R\left(f''; D_n, \frac{x+k-1}{n}\right)dx. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \lim_{n\to\infty} n^2\left\{T(f;D_n) - \int_0^1 f(t)dt\right\} &= \lim_{n\to\infty}\int_0^1 w(x)R\left(f''; D_n, \frac{x+k-1}{n}\right)dx \\ &= \int_0^1 w(x)\lim_{n\to\infty} R\left(f''; D_n, \frac{x+k-1}{n}\right)dx \\ &= \int_0^1 w(x)\int_0^1 f''(t)dtdx \\ &= \frac{1}{12}\int_0^1 f''(t)dt = \frac{1}{12}(f'(1)-f'(0)), \end{aligned}$$

which completes the proof of Theorem 1.3 in the case where $[a,b] = [0,1]$. We can get the general statement of the theorem by the variable change $x = a + (b-a)t$.

# 3 Lower Riemann sums for optimal divisions

We prove Theorem 1.2 in this section. We need the following lemma obtained by Gleason [2] and Lemma 3.2 in order to consider the lower Riemann sums for optimal divisions.

**Lemma 3.1 (Gleason)** *Let $\varphi(t)$ be a nonnegative continuous function defined on $[a,b]$. For any positive integer $n$ there exists a division of $[a,b]$:*

$$a = s_0 < s_1 < \cdots < s_{n-1} < s_n = b$$

*such that all of*

$$(s_i - s_{i-1})\max_{[s_{i-1},s_i]}\varphi(t) \qquad (1 \le i \le n)$$

*are equal to each other. We denote by $J_n$ the equal value. Then we obtain*

$$\lim_{n\to\infty} nJ_n = \int_a^b \varphi(t)dt.$$

**Lemma 3.2** *For any function of class $C^1$ defined on $[a,b]$ we have*

$$\int_a^b f(x)dx - (b-a)\min_{[a,b]} f(x) \le \frac{1}{2}(b-a)^2 \max_{[a,b]} |f'(x)|.$$

The estimate in this lemma is well known, so we omit its proof.

**Proof of Theorem 1.2** We first prove the following inequality.

$$\limsup_{n\to\infty} n\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) \le \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2.$$

We apply Lemma 3.1 to the function $|f'(x)|^{1/2}$ and obtain a division $\Delta_n^e$ : $s_1, s_2, \ldots, s_{n-1}$ of $[a,b]$ such that all of

$$(s_i - s_{i-1}) \max_{[s_{i-1},s_i]} |f'(x)|^{1/2} \qquad (1 \le i \le n)$$

are equal to each other. We denote by $J_n$ the equal value. Then we obtain

$$\lim_{n\to\infty} nJ_n = \int_a^b |f'(x)|^{1/2}dx.$$

By the estimate of Lemma 3.2 we have

$$\begin{aligned}
& \int_a^b f(x)dx - R(f;\Delta_n^e,\min) \\
= & \sum_{i=1}^n \left(\int_{s_{i-1}}^{s_i} f(x)dx - (s_i - s_{i-1}) \min_{[s_{i-1},s_i]} f(x)\right) \\
\le & \frac{1}{2}\sum_{i=1}^n (s_i - s_{i-1})^2 \max_{[s_{i-1},s_i]} |f'(x)| = \frac{n}{2}J_n^2.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \limsup_{n\to\infty} n\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) \\
\le & \limsup_{n\to\infty} n\left(\int_a^b f(x)dx - R(f;\Delta_n^e,\min)\right) \\
\le & \limsup_{n\to\infty} \frac{n^2}{2}J_n^2 = \frac{1}{2}\lim_{n\to\infty}(nJ_n)^2 = \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2.
\end{aligned}$$

In order to complete the proof of Theorem 1.2 we have to estimate

$$n\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right)$$

from below. We prepare the following lemmas for this purpose.

**Lemma 3.3** *Under the assumption of Theorem 1.2, we define* $\omega_1$ *by*

$$\omega_1(r) = \sup\{\,||f'(x)| - |f'(y)||\mid x, y \in [a,b],\ |x-y| \le r\}.$$

*Then* $\omega_1$ *is a continuous function defined on* $[0, b-a]$ *which is monotone increasing and satisfies* $\lim_{r\to 0}\omega_1(r) = 0$*. If* $f'(x) \neq 0$ *in a subinterval* $[p,q]$ *of* $[a,b]$*, then for any* $\xi$ *in* $[p,q]$ *we have the following inequality.*

$$\begin{aligned}&\left|\int_p^q f(x)dx - (q-p)\min_{[p,q]} f(x) - \frac{1}{2}(q-p)^2|f'(\xi)|\right|\\ \le\ &\frac{1}{2}\omega_1(q-p)(q-p)^2.\end{aligned}$$

By the use of the mean value theorem we can prove this lemma.

**Lemma 3.4** *Under the assumption of Theorem 1.2, for any* $\epsilon > 0$ *there exists a positive integer* $N$ *such that for any* $n \ge N$ *and any* $n$*-division* $\Delta$ *of* $[a,b]$ *we have the following inequality.*

$$n\left(\int_a^b f(x)dx - R(f;\Delta,\min)\right) \ge \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2 - \epsilon.$$

**Proof** For the proof of the lemma, we show the following statement: For any $\delta > 0$ there exists a positive integer $r$ such that for any $n$-division $\Delta$ of $[a,b]$ we have the following inequality.

$$(n+r)^{1/2}\left(\int_a^b f(x)dx - R(f;\Delta,\min)\right)^{1/2} \ge \frac{1}{2^{1/2}}\int_a^b |f'(x)|^{1/2}dx - \delta(b-a).$$

Since the function $x \mapsto x^{1/2}$ is uniformly continuous on $[0,\infty)$, there exists $\delta_1 > 0$ such that for any $x$ and $y$ in $[0,\infty)$ if $|x-y| < \delta_1$ then $|x^{1/2} - y^{1/2}| < \delta$.

We take a subinterval $[p,q]$ of $[a,b]$ and suppose that $f'(x) \neq 0$ in $[p,q]$. We can use Lemma 3.3 and get

$$\left| \frac{\int_p^q f(x)dx - (q-p)\min_{[p,q]} f(x)}{(q-p)^2} - \frac{1}{2}|f'(\xi)| \right| \leq \frac{1}{2}\omega_1(q-p)$$

for any $\xi$ in $[p,q]$. Because of continuity of $\omega_1$ and $\omega_1(0)=0$, there exists $\eta > 0$ such that $0 \leq z \leq \eta$ implies $\omega_1(z)/2 \leq \delta_1$. Thus $q-p \leq \eta$ implies $\omega_1(q-p)/2 \leq \delta_1$. Therefore we have

$$\left| \frac{\left(\int_p^q f(x)dx - (q-p)\min_{[p,q]} f(x)\right)^{1/2}}{q-p} - \frac{1}{2^{1/2}}|f'(\xi)|^{1/2} \right| \leq \delta,$$

that is,

$$\left| \left(\int_p^q f(x)dx - (q-p)\min_{[p,q]} f(x)\right)^{1/2} - \frac{1}{2^{1/2}}|f'(\xi)|^{1/2}(q-p) \right| \leq \delta(q-p).$$

Since $f'$ is uniformly continuous on $[a,b]$, for the above $\delta > 0$ there exists $\beta > 0$ such that $|x-y| < \beta$ implies $|f'(x) - f'(y)| \leq \delta^2$. We denote by $Z(f')$ the zero set of $f'$:

$$Z(f') = \{x \in [a,b] \mid f'(x) = 0\}$$

and define the $\beta$-neighborhood $Z(f')_\beta$ of $Z(f')$ by

$$Z(f')_\beta = \{y \in [a,b] \mid \exists x \in Z(f')\ |x-y| < \beta\}.$$

Then for any $y$ in $Z(f')_\beta$ we have $|f'(y)| \leq \delta^2$ and $f'$ is not equal to 0 on the complement of $Z(f')_\beta$. By the definition of $Z(f')_\beta$ we can see that $Z(f')_\beta$ is a disjoint union of finitely many intervals. We denote by $r_1$ the number of all endpoints of the intervals constructing $Z(f')_\beta$. For $\eta > 0$ obtained above we take a positive integer $r_2$ satisfying $(b-a)/r_2 \leq \eta$ and set $r = r_1 + r_2$. For any $n$-division $\Delta$ of $[a,b]$ we can add at most $r_2$ points to $\Delta$ such that the width of each subinterval is less than or equal to $\eta$. Moreover we add all the endpoints of the intervals constructing $Z(f')_\beta$ and denote the new division by

$$\Delta' : s_0 = a, s_1, \ldots, s_t = b.$$

By the definition of $\Delta'$ we have $t \leq n + r$ and $s_i - s_{i-1} \leq \eta$. Each interval $[s_{i-1}, s_i]$ satisfies $[s_{i-1}, s_i] \subset \overline{Z(f')_\beta}$ or $[s_{i-1}, s_i] \subset [a, b] - Z(f')_\beta$. In both cases, according to the first mean value theorem for integration we can take $s_i'$ in $[s_{i-1}, s_i]$ satisfying

$$\int_{s_{i-1}}^{s_i} |f'(x)|^{1/2} dx = |f'(s_i')|^{1/2}(s_i - s_{i-1}).$$

In the case where $[s_{i-1}, s_i] \subset \overline{Z(f')_\beta}$

$$\begin{aligned} &\frac{1}{2^{1/2}}|f'(s_i')|^{1/2}(s_i - s_{i-1}) \leq \frac{1}{2^{1/2}}\delta(s_i - s_{i-1}) \\ \leq\ &\left(\int_{s_{i-1}}^{s_i} f(x)dx - (s_i - s_{i-1}) \min_{[s_{i-1}, s_i]} f(x)\right)^{1/2} + \delta(s_i - s_{i-1}) \end{aligned}$$

holds. In the case where $[s_{i-1}, s_i] \subset [a, b] - Z(f')_\beta$, $f'$ is not equal to 0 in $[s_{i-1}, s_i]$, thus

$$\begin{aligned} &\frac{1}{2^{1/2}}|f'(s_i')|^{1/2}(s_i - s_{i-1}) \\ \leq\ &\left(\int_{s_{i-1}}^{s_i} f(x)dx - (s_i - s_{i-1}) \min_{[s_{i-1}, s_i]} f(x)\right)^{1/2} + \delta(s_i - s_{i-1}). \end{aligned}$$

Finally the same inequality holds in both cases. We add the above inequalities for $i = 1, \ldots, t$ and get

$$\begin{aligned} &\frac{1}{2^{1/2}} \int_a^b |f'(x)|^{1/2} dx = \sum_{i=1}^t \frac{1}{2^{1/2}}|f'(s_i')|^{1/2}(s_i - s_{i-1}) \\ \leq\ &\sum_{i=1}^t \left(\int_{s_{i-1}}^{s_i} f(x)dx - (s_i - s_{i-1}) \min_{[s_{i-1}, s_i]} f(x)\right)^{1/2} + \delta(b - a). \quad (*) \end{aligned}$$

We apply the Cauchy-Schwarz inequality to the first term of $(*)$ and get

$$\begin{aligned} &\sum_{i=1}^t \left(\int_{s_{i-1}}^{s_i} f(x)dx - (s_i - s_{i-1}) \min_{[s_{i-1}, s_i]} f(x)\right)^{1/2} \\ \leq\ & t^{1/2} \left(\sum_{i=1}^t \left(\int_{s_{i-1}}^{s_i} f(x)dx - (s_i - s_{i-1}) \min_{[s_{i-1}, s_i]} f(x)\right)\right)^{1/2} \\ =\ & t^{1/2} \left(\int_a^b f(x)dx - R(f; \Delta', \min)\right)^{1/2}. \end{aligned}$$

From these we have

$$\frac{1}{2^{1/2}}\int_a^b |f'(x)|^{1/2}dx \le t^{1/2}\left(\int_a^b f(x)dx - R(f;\Delta',\min)\right)^{1/2} + \delta(b-a).$$

The inequality

$$\int_a^b f(x)dx - R(f;\Delta',\min) \le \int_a^b f(x)dx - R(f;\Delta,\min),$$

the estimate obtained above and $t \le n + r$ imply

$$\frac{1}{2^{1/2}}\int_a^b |f'(x)|^{1/2}dx - \delta(b-a) \le (n+r)^{1/2}\left(\int_a^b f(x)dx - R(f;\Delta,\min)\right)^{1/2}.$$

Using the result obtained above, we prove Lemma 3.4. Since the function $x \mapsto x^2$ is continuous, for any $\epsilon > 0$ there exists $\xi > 0$ such that if

$$\frac{1}{2^{1/2}}\int_a^b |f'(x)|^{1/2}dx - x \le \xi$$

then we have

$$\frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2 - x^2 \le \frac{\epsilon}{2}.$$

So we take $\delta > 0$ which satisfies $\delta(b-a) \le \xi$. We can apply the result obtained above and get a positive integer $r$ such that for any $n$-division $\Delta$ of $[a,b]$

$$\begin{aligned}\xi &\ge \delta(b-a) \\ &\ge \frac{1}{2^{1/2}}\int_a^b |f'(x)|^{1/2}dx - (n+r)^{1/2}\left(\int_a^b f(x)dx - R(f;\Delta,\min)\right)^{1/2},\end{aligned}$$

which implies

$$\frac{\epsilon}{2} \ge \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2 - (n+r)\left(\int_a^b f(x)dx - R(f;\Delta,\min)\right).$$

We can substitute the optimal division $\Delta_n^{\#}$ for $\Delta$ in the above inequality and get

$$(n+r)\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) \ge \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2 - \frac{\epsilon}{2}.$$

Since

$$\lim_{n\to\infty}\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) = 0,$$

we can choose a positive integer $N$ such that for $n \geq N$

$$0 \leq r\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) \leq \frac{\epsilon}{2}$$

holds. Thus for $n \geq N$ we have

$$\begin{aligned} & n\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) \\ \geq\ & (n+r)\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) - \frac{\epsilon}{2} \\ \geq\ & \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2 - \epsilon. \end{aligned}$$

Therefore for any $n$-division $\Delta$ of $[a,b]$ we have

$$\begin{aligned} n\left(\int_a^b f(x)dx - R(f;\Delta,\min)\right) &\geq n\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) \\ &\geq \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2 - \epsilon, \end{aligned}$$

which completes the proof of Lemma 3.4.

**Proof of Theorem 1.2** We have already proved

$$\limsup_{n\to\infty} n\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right) \leq \frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2$$

and by Lemma 3.4 we can see that

$$\frac{1}{2}\left(\int_a^b |f'(x)|^{1/2}dx\right)^2 \leq \liminf_{n\to\infty} n\left(\int_a^b f(x)dx - R(f;\Delta_n^{\#},\min)\right).$$

Therefore the limit of the left-hand side stated in the theorem exists and the equation holds.

# 4 Trapezoidal sums for optimal divisions

We prove Theorem 1.4 in this section. We need Lemmas 3.1 and 4.1 in order to consider the trapezoidal sums for optimal divisions.

**Lemma 4.1** *For any function $f$ of class $C^2$ defined on $[a,b]$ we have*

$$\left|\int_a^b f(x)dx - \frac{1}{2}(f(a)+f(b))(b-a)\right| \leq \frac{1}{12}(b-a)^3 \max_{[a,b]} |f''(x)|.$$

By the use of the mean value theorem we can prove this lemma.

**Proof of Theorem 1.4** We first prove the following inequality for any function $f$ of class $C^2$ defined on $[a,b]$.

$$\limsup_{n\to\infty} n^2 \left|\int_a^b f(x)dx - T(f;\Delta_n^{t\#})\right| \leq \frac{1}{12}\left(\int_a^b |f''(x)|^{1/3}dx\right)^3.$$

We apply Lemma 3.1 to the function $|f''(x)|^{1/3}$ and obtain a division $\Delta_n^{te}$ : $s_1, s_2, \ldots, s_{n-1}$ of $[a,b]$ such that all of

$$(s_i - s_{i-1}) \max_{[s_{i-1},s_i]} |f''(x)|^{1/3} \qquad (1 \leq i \leq n)$$

are equal to each other. We denote by $J_n$ the equal value. Then we obtain

$$\lim_{n\to\infty} nJ_n = \int_a^b |f''(x)|^{1/3}dx.$$

By the estimate of Lemma 4.1 we have

$$\begin{aligned}
&\left|\int_a^b f(x)dx - T(f;\Delta_n^{te})\right| \\
\leq\ & \sum_{i=1}^n \left|\int_{s_{i-1}}^{s_i} f(x)dx - \frac{1}{2}(f(s_{i-1})+f(s_i))(s_i - s_{i-1})\right| \\
\leq\ & \frac{1}{12}\sum_{i=1}^n (s_i - s_{i-1})^3 \max\{|f''(x)| \mid s_{i-1} \leq x \leq s_i\} = \frac{n}{12}J_n^3.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
& \limsup_{n\to\infty} n^2 \left| \int_a^b f(x)dx - T(f;\Delta_n^{t\#}) \right| \\
\le\ & \limsup_{n\to\infty} n^2 \left| \int_a^b f(x)dx - T(f;\Delta_n^{te}) \right| \\
\le\ & \limsup_{n\to\infty} \frac{n^3}{12} J_n^3 = \frac{1}{12} \lim_{n\to\infty} (nJ_n)^3 = \frac{1}{12} \left( \int_a^b |f''(x)|^{1/3} dx \right)^3 .
\end{aligned}$$

In order to complete the proof of Theorem 1.4 we have to estimate

$$n^2 \left| \int_a^b f(x)dx - T(f;\Delta_n^{t\#}) \right|$$

from below. We prepare the following lemmas for this purpose.

**Lemma 4.2** *Let $f$ be a function of class $C^2$ defined on $[a,b]$. We define $\omega_2$ by*

$$\omega_2(r) = \sup\{ |\,|f''(x)| - |f''(y)|\,| \mid x, y \in [a,b],\ |x-y| \le r \}.$$

*Then $\omega_2$ is a continuous function defined on $[0, b-a]$ which is monotone increasing and satisfies $\lim_{r\to 0} \omega_2(r) = 0$. If $f''(x) \ge 0$ or $f''(x) \le 0$ in a subinterval $[p,q]$ of $[a,b]$, then for any $\xi$ in $[p,q]$ we have the following inequality.*

$$\begin{aligned}
& \left| \left| \int_p^q f(x)dx - \frac{1}{2}(f(p)+f(q))(q-p) \right| - \frac{1}{12}(q-p)^3 |f''(\xi)| \right| \\
\le\ & \frac{1}{12} \omega_2(q-p)(q-p)^3.
\end{aligned}$$

By the use of the mean value theorem we can prove this lemma.

**Lemma 4.3** *Under the assumption of Theorem 1.4, for any $\epsilon > 0$ there exists a positive integer $N$ such that for any $n \ge N$ and any $n$-division $\Delta$ of $[a,b]$ we have the following inequality.*

$$n^2 \left| \int_a^b f(x)dx - T(f;\Delta) \right| \ge \frac{1}{12} \left( \int_a^b |f''(x)|^{1/3} dx \right)^3 - \epsilon.$$

**Proof** For the proof of the lemma, we show the following statement: For any $\delta > 0$ there exists a positive integer $r$ such that for any $n$-division $\Delta$ of $[a,b]$ we have the following inequality.

$$(n+r)^{2/3}\left|\int_a^b f(x)dx - T(f;\Delta)\right|^{1/3} \geq \frac{1}{12^{1/3}}\int_a^b |f''(x)|^{1/3}dx - \delta(b-a).$$

Since the function $x \mapsto x^{1/3}$ is uniformly continuous on $[0,\infty)$, there exists $\delta_1 > 0$ such that for any $x$ and $y$ in $[0,\infty)$ if $|x-y| < \delta_1$ then $|x^{1/3}-y^{1/3}| \leq \delta$. We take a subinterval $[p,q]$ of $[a,b]$. We can use Lemma 4.2 and get

$$\left|\frac{\left|\int_p^q f(x)dx - \frac{1}{2}(f(p)+f(q))(q-p)\right|}{(q-p)^3} - \frac{1}{12}|f''(\xi)|\right| \leq \frac{1}{12}\omega_2(q-p)$$

for any $\xi$ in $[p,q]$. Because of continuity of $\omega_2$ and $\omega_2(0) = 0$, there exists $\eta > 0$ such that $0 \leq z \leq \eta$ implies $\omega_2(z)/12 \leq \delta_1$. Thus $q-p \leq \eta$ implies $\omega_2(q-p)/12 \leq \delta_1$. Therefore we have

$$\left|\frac{\left|\int_p^q f(x)dx - \frac{1}{2}(f(p)+f(q))(q-p)\right|^{1/3}}{q-p} - \frac{1}{12^{1/3}}|f''(\xi)|^{1/3}\right| \leq \delta,$$

that is,

$$\left|\left|\int_p^q f(x)dx - \frac{1}{2}(f(p)+f(q))(q-p)\right|^{1/3} - \frac{1}{12^{1/3}}|f''(\xi)|^{1/3}(q-p)\right| \leq \delta(q-p).$$

For $\eta > 0$ obtained above we take a positive integer $r$ satisfying $(b-a)/r \leq \eta$. For any $n$-division $\Delta$ of $[a,b]$ we can add at most $r$ points to $\Delta$ such that the width of each subinterval is less than or equal to $\eta$. We denote the new division by

$$\Delta' : s_0 = a, s_1, \ldots, s_t = b.$$

we have $t \leq n+r$. According to the first mean value theorem for integration we can take $s_i'$ in $[s_{i-1}, s_i]$ satisfying

$$\int_{s_{i-1}}^{s_i} |f''(x)|^{1/3}dx = |f''(s_i')|^{1/3}(s_i - s_{i-1}).$$

By the estimate obtained above we get

$$\begin{aligned}
&\frac{1}{12^{1/3}}|f''(s_i')|^{1/3}(s_i - s_{i-1}) \\
\le\ &\left|\int_{s_{i-1}}^{s_i} f(x)dx - \frac{1}{2}(f(s_{i-1}) + f(s_i))(s_i - s_{i-1})\right|^{1/3} + \delta(s_i - s_{i-1}).
\end{aligned}$$

We add the above inequalities for $i = 1, \ldots, t$ and get

$$\begin{aligned}
&\frac{1}{12^{1/3}}\int_a^b |f''(x)|^{1/3}dx = \sum_{i=1}^{t} \frac{1}{12^{1/3}}|f'(s_i')|^{1/3}(s_i - s_{i-1}) \\
\le\ &\sum_{i=1}^{t}\left|\int_{s_{i-1}}^{s_i} f(x)dx - \frac{1}{2}(f(s_{i-1}) + f(s_i))(s_i - s_{i-1})\right|^{1/3} + \delta(b-a). \quad (**)
\end{aligned}$$

We apply the Hölder inequality to the first term of $(**)$ and get

$$\begin{aligned}
&\sum_{i=1}^{t}\left|\int_{s_{i-1}}^{s_i} f(x)dx - \frac{1}{2}(f(s_{i-1}) + f(s_i))(s_i - s_{i-1})\right|^{1/3} \\
\le\ & t^{2/3}\left(\sum_{i=1}^{t}\left|\int_{s_{i-1}}^{s_i} f(x)dx - \frac{1}{2}(f(s_{i-1}) + f(s_i))(s_i - s_{i-1})\right|\right)^{1/3} \\
=\ & t^{2/3}\left|\int_a^b f(x)dx - T(f;\Delta')\right|^{1/3}.
\end{aligned}$$

From these we have

$$\frac{1}{12^{1/3}}\int_a^b |f''(x)|^{1/3}dx \le t^{2/3}\left|\int_a^b f(x)dx - T(f;\Delta')\right|^{1/3} + \delta(b-a).$$

The inequality

$$\left|\int_a^b f(x)dx - T(f;\Delta')\right| \le \left|\int_a^b f(x)dx - T(f;\Delta)\right|,$$

the estimate obtained above and $t \le n + r$ imply

$$\frac{1}{12^{1/3}}\int_a^b |f''(x)|^{1/3}dx - \delta(b-a) \le (n+r)^{2/3}\left|\int_a^b f(x)dx - T(f;\Delta)\right|^{1/3}.$$

Using the result obtained above, we prove Lemma 4.3. Since the function $x \mapsto x^3$ is continuous, for any $\epsilon > 0$ there exists $\xi > 0$ such that if

$$\xi \geq \frac{1}{12^{1/3}} \int_a^b |f''(x)|^{1/3} dx - x$$

then we have

$$\frac{\epsilon}{2} \geq \frac{1}{12} \left( \int_a^b |f''(x)|^{1/3} dx \right)^3 - x^3.$$

So we take $\delta > 0$ which satisfies $\delta(b-a) \leq \xi$. We can apply the result obtained above and get a positive integer $r$ such that for any $n$-division $\Delta$ of $[a, b]$

$$\xi \geq \delta(b-a) \geq \frac{1}{12^{1/3}} \int_a^b |f''(x)|^{1/3} dx - (n+r)^{2/3} \left| \int_a^b f(x)dx - T(f; \Delta) \right|^{1/3},$$

which implies

$$\frac{\epsilon}{2} \geq \frac{1}{12} \left( \int_a^b |f''(x)|^{1/3} dx \right)^3 - (n+r)^2 \left| \int_a^b f(x)dx - T(f; \Delta) \right|.$$

We can substitute the optimal division $\Delta_n^{t\#}$ for $\Delta$ in the above inequality and get

$$(n+r)^2 \left| \int_a^b f(x)dx - T(f; \Delta^{t\#}) \right| \geq \frac{1}{12} \left( \int_a^b |f''(x)|^{1/3} dx \right)^3 - \frac{\epsilon}{2}.$$

Since

$$\limsup_{n \to \infty} (2nr + r^2) \left| \int_a^b f(x)dx - T(f; \Delta_n^{t\#}) \right| = 0,$$

we can choose a positive integer $N$ such that for $n \geq N$

$$0 \leq (2nr + r^2) \left| \int_a^b f(x)dx - T(f; \Delta_n^{t\#}) \right| \leq \frac{\epsilon}{2}$$

holds. Thus for $n \geq N$ we have

$$\begin{aligned} n^2 \left| \int_a^b f(x)dx - T(f; \Delta_n^{t\#}) \right| &\geq (n+r)^2 \left| \int_a^b f(x)dx - T(f; \Delta_n^{t\#}) \right| - \frac{\epsilon}{2} \\ &\geq \frac{1}{12} \left( \int_a^b |f''(x)|^{1/3} dx \right)^3 - \epsilon \end{aligned}$$

Therefore for any $n$-division $\Delta$ of $[a,b]$ we have

$$\begin{aligned} n^2 \left| \int_a^b f(x)dx - T(f;\Delta) \right| &\geq n^2 \left| \int_a^b f(x)dx - T(f;\Delta_n^{t\#}) \right| \\ &\geq \frac{1}{12} \left( \int_a^b |f''(x)|^{1/3} dx \right)^3 - \epsilon, \end{aligned}$$

which completes the proof of Lemma 4.3.

**Proof of Theorem 1.4** We can combine the inequality

$$\limsup_{n\to\infty} n^2 \left| \int_a^b f(x)dx - T(f;\Delta_n^{t\#}) \right| \leq \frac{1}{12} \left( \int_a^b |f''(x)|^{1/3} dx \right)^3.$$

and Lemma 4.3 and see the assertion of the theorem.

# References


[1] Charles K. Chui, Concerning rates of convergence of Riemann sums, Journal of Approximation Theory 4 (1971) 279–287.

[2] Andrew M. Gleason, A curvature formula, Amer. J. Math. 101 (1979) 86–93.