RESTRICTED ENERGY INEQUALITIES AND NUMERICAL APPROXIMATIONS

By
Reiko SAKAMOTO

Introduction

Let \{A, B_j\} be linear partial differential operators. Let \( \Omega(\subset \mathbb{R}^n) \) be a bounded domain with smooth boundary \( \Gamma \). Our boundary value problem is to find \( u \in L^2(\Omega) \) satisfying

\[
\begin{align*}
Au &= f \quad \text{in} \ \Omega, \\
B_ju &= f_j \quad \text{on} \ \Gamma \ (j \in J)
\end{align*}
\]

for given data \( \{f, f_j\} \). We are particularly interested in a method of numerical approximation of solutions of \( (P) \).

The problem \( (P) \) is closely connected with its adjoint problem \( (P^*) \). The adjoint problem is to find \( v \in L^2(\Omega) \) satisfying

\[
\begin{align*}
A^*v &= g \quad \text{in} \ \Omega \\
\partial_{\nu}^*v &= g_j \quad \text{on} \ \Gamma \ (j \in J^*)
\end{align*}
\]

for given data \( \{g, g_j\} \).

Recently, it has become clear that a solution \( u \in L^2(\Omega) \) of \( (P) \) can be constructed numerically, assuming an energy inequality

\[
(E^*) \quad \|v\| \leq C \left( \|A^*v\| + \sum_{j \in J^*} \langle \partial_{\nu}^*v, j \rangle_{\mu_j} \right) \quad (v \in H^q(\Omega))
\]

([1]).

Here we have two questions:

1. In case when \( L^2 \)-solutions of \( (P) \) are not unique, how can we characterize the solution, obtained in [1]?

2. In case when \( L^2 \)-solutions of \( (P^*) \) are not unique, \( (E^*) \) can not be satisfied. Is there any numerical method to approach to one of solutions of \( (P) \)?
In this paper, instead of \((E^*)\), we assume a restricted energy inequality:

\[
\|(v)\| \leq C\|A^*v\| \quad (v \in M^*, \mathcal{B}_j^*v|\Gamma = 0 \quad (j \in J^*)),
\]

where \(M^*\) is a subspace in \(H^q(\Omega)\) defined in §1. Then we will see that the method in [1] is applicable. Moreover, we will see that the solution obtained by our method is unique in a subspace \(\tau\) in \(L^2(\Omega)\).

\section{Restricted Energy Inequalities}

Let

\[
A = \sum_{|\nu| \leq m} a_{\nu}(x) \partial_x^\nu
\]

be a differential operator with smooth coefficients defined in a neighborhood of \(\Omega\). Let

\[
B_j = \sum_{|\nu| \leq J} b_{\nu j}(x) \partial_x^\nu \quad (j \in J, J \subset \{0, 1, \ldots, m-1\})
\]

be differential operators with smooth coefficients defined in a neighborhood of \(\Gamma\). We assume that \(\Gamma\) is non-characteristic for \(\{A, B_j \ (j \in J)\}\). Namely,

\[
\sum_{|\nu| = m} a_{\nu}(x)n(x)^\nu \neq 0 \quad \text{on } \Gamma,
\]

\[
\sum_{|\nu| = J} b_{\nu j}(x)n(x)^\nu \neq 0 \quad \text{on } \Gamma,
\]

where \(n(x)\) is a unit inner normal at \(x \in \Gamma\).

Set

\[
J^c \cup J = \{0, 1, \ldots, m-1\}, \quad J^c \cap J = \phi, \quad J^* = \{j \mid m-1 - j \in J^c\},
\]

\[
B_j = (d/dn)^j \quad (j \in J^c).
\]

Then we can define

\[
\mathcal{B}_j = \sum_{|\nu| \leq J} \beta_{\nu j}(x) \partial_x^\nu \quad (j \in \{0, 1, \ldots, m-1\}),
\]

for which \(\Gamma\) is non-characteristic, such that the following Green’s Theorem holds.
**Lemma 1.1 (Green's Theorem).** Suppose that $u, Au \in L^2(\Omega)$, then it holds that

$$
\langle (d/dn)^k u \rangle_{-k-m+1/2} \leq C(\|u\| + \|Au\|) \quad (k = 0, 1, \ldots, m - 1)
$$

and

$$
(Au, v) - (u, A^* v) = - \sum_{j \in J} \langle B_j u \rangle_{\Gamma}, \mathcal{B}^*_{m-1-j} v \rangle_{\Gamma} - \sum_{j \in J} \langle B^*_{m-1-j} u \rangle_{\Gamma}, \mathcal{B}^*_{m-1-j} v \rangle_{\Gamma} \quad (v \in H^{2m-1}(\Omega)),
$$

where

$$
A^* = \sum_{|\alpha| \leq m} (-\partial_x)^\alpha a_{\alpha}(x), \quad \mathcal{B}^* = \sum_{|\alpha| \leq j} (-\partial_x)^\alpha b_{\alpha}(x).
$$

**Notations.**

1. $(u, v) = (u, v)_{L^2(\Omega)}$, $\|u\| = \|u\|_{L^2(\Omega)}$,
2. $\Lambda = (1 - \Delta)^{1/2}$, where $\Delta$ is the Laplace-Beltrami operator on $\Gamma$,
3. $\langle u, v \rangle_{\sigma} = (u, v)_{H^\sigma(\Gamma)} = (\Lambda^\sigma u, \Lambda^\sigma v)_{L^2(\Gamma)}$, $\langle u \rangle_{\sigma} = \|u\|_{H^\sigma(\Gamma)}$ for $u, v \in H^\sigma(\Gamma)$ ($\sigma$: real),
4. $u \in H^{-\sigma}(\Gamma)$: $H^\sigma(\Gamma) \ni v \mapsto \langle u, v \rangle \in \mathbb{C}$ ($\sigma > 0$).

Lemma 1.1 is well known for $u \in H^m(\Omega)$. See Appendix of [3] in case when $u, Au \in L^2(\Omega)$.

**Remark.** Set

$$
Pu = \{Au, B_j u \} \quad (j \in J), \quad Qu = \{u, -B_{m-1-j} u \} \quad (j \in J^*),
$$

$$
P^\ast v = \{A^* v, \mathcal{B}^*_{m-1-j} v \} \quad (j \in J^*), \quad Q^\ast v = \{v, \mathcal{B}^*_{m-1-j} v \} \quad (j \in J),
$$

then the problem $(P)$ denotes $Pu = \{f, f_j \}$ ($j \in J$) and the problem $(P^\ast)$ denotes $P^\ast v = \{g, g_j \}$ ($j \in J^*$), and Green's Theorem is stated as follows.

**Green's Theorem.** Suppose that $u, Au \in L^2(\Omega)$, then it holds that

$$
\langle (d/dn)^k u \rangle_{-k-m+1/2} \leq C(\|u\| + \|Au\|) \quad (k = 0, 1, \ldots, m - 1)
$$

and

$$
[Pu, Q^* v] = [Qu, P^* v] \quad (v \in H^{2m-1}(\Omega)),
$$
where .

\[ [F, G] = (f, g) + \sum_{j \in J} \langle f_j, g_j \rangle \text{ for } F = \{f, f_j \ (j \in J)\} \text{ and } G = \{g, g_j \ (j \in J)\}, \]

\[ [F, G]_* = (f, g) + \sum_{j \in J^*} \langle f_j, g_j \rangle \text{ for } F = \{f, f_j \ (j \in J^*)\} \text{ and } G = \{g, g_j \ (j \in J^*)\}. \]

**NULL SPACES.** Set

\[ K = \{\phi \in L^2(\Omega) \mid P\phi = 0\}, \quad K^* = \{\phi \in L^2(\Omega) \mid P^*\phi = 0\}. \]

Owing to Green's Theorem, we have

\[ K = \{\phi \in L^2(\Omega) \mid [Q\phi, P^*v]_* = 0 \ (\forall v \in H^{2m-1}(\Omega))\}, \]

\[ K^* = \{\phi \in L^2(\Omega) \mid [Pu, Q^*\phi] = 0 \ (\forall u \in H^{2m-1}(\Omega))\}. \]

Therefore, \( K \) and \( K^* \) are closed subspaces in \( L^2(\Omega) \). Set

\[ K^\perp = \{f \in L^2(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K)\}, \]

\[ K^{\ast \perp} = \{f \in L^2(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K^*)\}. \]

We assume

(A-1) there exists an integer \( p(\geq 2m - 1) \) such that

\[ K, K^* \subset H^p(\Omega) \]

throughout this paper.

We define

\[ M^* = K^{\ast \perp} \cap H^q(\Omega) = \{f \in H^q(\Omega) \mid (f, \phi) = 0 \ (\forall \phi \in K^*)\} \]

for an integer \( q \ (m \leq q \leq p) \). Then \( M^* \) is a closed subspace in \( H^q(\Omega) \). Let \( u \in H^q(\Omega) \), then there exist \( \phi \in K^* \) and \( \xi \in M^* \) such that

\[ u = \phi + \xi \quad \text{and} \quad \|u\|^2 = \|\phi\|^2 + \|\xi\|^2. \]

We say that restricted energy inequality \((\mathcal{E}^*)\) holds, if it holds

\[ (\mathcal{E}^*) \quad \|v\| \leq C \left( \|A^*v\| + \sum_{j \in J^*} \langle \mathcal{A}_j^*v, v \rangle_{\mu_j} \right) \quad (v \in M^*), \]

where \( \mu_j = q - 1/2 - j \).
We say that restricted energy inequality \((\mathcal{E}_0^*)\) holds, if it holds
\[
\|v\| \leq C\|A^*v\| \quad (v \in M^*, \mathcal{B}_j^*v|_{\Gamma} = 0 \quad (j \in J^*)).
\]
Since \(\{B_j \ (j = 0, 1, \ldots, m - 1)\}\) and \(\{\mathcal{B}_j^* \ (j = 0, 1, \ldots, m - 1)\}\) are Dirichlet sets, we have \((|4|)\)

**Lemma 1.2.** Let \(s \geq m\).

i) Let \(f_j \in H^{s-1/2-j}(\Gamma) \ (j = 0, 1, \ldots, m - 1)\), then there exists \(U \in H^s(\Omega)\) such that
\[
B_j U|_{\Gamma} = f_j \quad (j = 0, 1, \ldots, m - 1), \quad \|U\|_s \leq C \sum_{j \in \{0, 1, \ldots, m-1\}} \langle f_j \rangle_{s-1/2-j}.
\]

ii) Let \(g_j \in H^{s-1/2-j}(\Gamma) \ (j = 0, 1, \ldots, m - 1)\), then there exists \(V \in H^s(\Omega)\) such that
\[
\mathcal{B}_j^* V|_{\Gamma} = g_j \quad (j = 0, 1, \ldots, m - 1), \quad \|V\|_s \leq C \sum_{j \in \{0, 1, \ldots, m-1\}} \langle g_j \rangle_{s-1/2-j}.
\]

**Lemma 1.3.** \((\mathcal{E}^*)\) holds iff \((\mathcal{E}_0^*)\) holds.

**Proof.** Suppose that \((\mathcal{E}_0^*)\) holds. Let \(v \in M^*(\subset H^q(\Omega))\).

1) Set
\[
g = A^*v \in H^{q-m}(\Omega), \quad g_j = \mathcal{B}_j^*v|_{\Gamma} \in H^{q-1/2-j}(\Gamma) \quad (j \in J^*).
\]
Then there exists \(V \in H^q(\Omega)\) such that
\[
\mathcal{B}_j^* V|_{\Gamma} = g_j \quad (j \in J^*), \quad \|V\|_q \leq C \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j}
\]
from (ii) of Lemma 1.2 \((s = q)\).

2) Set \(w = v - V\), then \(w \in H^q(\Omega)\) satisfies
\[
\begin{cases}
A^*w = g - A^*V, \\
\mathcal{B}_j^*w|_{\Gamma} = 0 \quad (j \in J^*).
\end{cases}
\]
Since \(w \in H^q(\Omega)\), there exist \(\phi \in K^*\) and \(\xi \in M^*\) such that
\[
w = \phi + \xi, \quad \|w\|^2 = \|\phi\|^2 + \|\xi\|^2.
\]
Therefore \(\xi \in M^*\) satisfies
\[
\begin{cases}
A^*\xi = g - A^*V, \\
\mathcal{B}_j^*\xi|_\Gamma = 0 \quad (j \in J^*).
\end{cases}
\]

Since \((\mathcal{E}_0^*)\) holds, we have
\[
\|\xi\| \leq C\|A^*\xi\| = C\|g - A^*V\| \leq C'\left(\|g\| + \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j}\right).
\]

(3) In the same way, since \(V \in H^q(\Omega)\), there exist \(\psi \in K^*\) and \(\eta \in M^*\) such that
\[
V = \psi + \eta, \quad \|V\|^2 = \|\psi\|^2 + \|\eta\|^2.
\]
Hence we have
\[
v = w + V = (\phi + \psi) + (\xi + \eta), \quad \phi + \psi \in K^*, \quad \xi + \eta \in M^*.
\]
Since \(v \in M^*\), we have
\[
v = \xi + \eta.
\]
Hence we have
\[
\|v\| \leq \|\xi\| + \|\eta\| \leq \|\xi\| + \|V\| \leq C\left(\|g\| + \sum_{j \in J^*} \langle g_j \rangle_{q-1/2-j}\right). \quad \square
\]

We assume
\((A-II)\) \((\mathcal{E}_0^*)\)
throughout this paper. Then we can define a Hilbert space \(\mathcal{H}\) as the closure of \(M^*\) by the norm \([\cdot]\):
\[
[v]^2 = \|A^*v\|^2 + \sum_{j \in J^*} \langle \mathcal{B}_j^*v|_\Gamma \rangle_{\mu_j}^2.
\]
Inner product of \(\mathcal{H}\) is defined by
\[
[w, v] = (A^*w, A^*v) + \sum_{j \in J^*} \langle \mathcal{B}_j^*w|_\Gamma, \mathcal{B}_j^*v|_\Gamma \rangle_{\mu_j}.
\]
For a fixed \(f \in L^2(\Omega)\), define
\[
f : \mathcal{H} \ni v \mapsto (v, f) \in \mathbb{C}
\]
then \(f\) is a continuous linear functional on \(\mathcal{H}\). In fact, it holds
\[
|(v, f)| \leq \|v\| \|f\| \leq C[v]\|f\| \quad (v \in \mathcal{H})
\]
from Lemma 1.3. Therefore, owing to Riesz' Theorem in $\mathcal{H}$, there exists $w \in \mathcal{H}$ such that

$$(f, v) = [w, v] \quad (v \in \mathcal{H}),$$

where we say that $w \in \mathcal{H}$ is a Riesz function of $f \in L^2(\Omega)$.

§ 2. Existence and Uniqueness

THEOREM 2.1. Assume (A-I) and (A-II). Suppose that $f \in K^*$. Let $w \in \mathcal{H}$ be a Riesz function of $f$. Set $u = A^* w \in L^2(\Omega)$, then $u$ satisfies

$$(P_0) \quad \begin{cases} Au = f & \text{in } \Omega, \\ B_j u |_\Gamma = 0 & (j \in J), \end{cases}$$

and

$$B_{m-1-j} u |_\Gamma = -A^{2\mu} \mathcal{B}^*_j w |_\Gamma \quad (j \in J^*).$$

PROOF. (1) Since $w \in \mathcal{H}$ satisfy

$$(f, v) = (A^* w, A^* v) + \sum_{j \in J^*} \langle \mathcal{B}^*_j w |_\Gamma, \mathcal{B}^*_j v |_\Gamma \rangle \quad (v \in \mathcal{H}),$$

$u = A^* w$ satisfies

$$(f, v) - (u, A^* v) = \sum_{j \in J^*} \langle \mathcal{B}^*_j w |_\Gamma, \mathcal{B}^*_j v |_\Gamma \rangle \quad (v \in \mathcal{H}) \ldots \ldots \Omega.$$ (1)

(2) Moreover, we have

$$(f, v) - (u, A^* v) = \sum_{j \in J^*} \langle \mathcal{B}^*_j w |_\Gamma, \mathcal{B}^*_j v |_\Gamma \rangle \quad (v \in H^q(\Omega)) \ldots \ldots \Omega'.$$

In fact, let $v \in H^q(\Omega)$. Then there exist $\phi \in K^*$ and $\xi \in M^*$ such that $v = \phi + \xi$. Since $\xi \in M^* \subset \mathcal{H}$, we have from (1)

$$(f, \xi) - (u, A^* \xi) = \sum_{j \in J^*} \langle \mathcal{B}^*_j w |_\Gamma, \mathcal{B}^*_j \xi |_\Gamma \rangle \mu_j.$$ (1')

We remark that $\phi$ satisfies

$$A^* \phi = 0, \quad \mathcal{B}^*_j \phi |_\Gamma = 0 \quad (j \in J^*),$$
and \((f, \phi) = 0\). Hence we have

\[
(f, v) - (u, A^* v) = \sum_{j \in J^*} \langle B^*_j w|_\Gamma, B^*_j v|_\Gamma \rangle_{\mu_j}.
\]

(3) From \(\mathcal{I}'\), we have

\[
(f, v) - (u, A^* v) = 0 \quad (v \in \mathcal{D}'(\Omega)).
\]

which means

\[
Au = f \quad \text{in} \ \mathcal{D}'(\Omega).
\]

Therefore we have

\[
(Au, v) - (u, A^* v) = \sum_{j \in J^*} \langle B^*_j w|_\Gamma, B^*_j v|_\Gamma \rangle_{\mu_j} \quad (v \in H^q(\Omega)) \ldots \ldots \mathcal{I}''.
\]

(4) Owing to Green's Theorem, we have

\[
(Au, v) - (u, A^* v) = -\sum_{j \in J} \langle B_j u|_\Gamma, B^*_j v|_\Gamma \rangle - \sum_{j \in J} \langle B_{m-1-j} u|_\Gamma, B^*_j v|_\Gamma \rangle \quad (v \in H^{q'}(\Omega), \ q' = \max(q, 2m-1)),
\]

Hence, from \(\mathcal{I}''\) and \(\mathcal{I}\), we have

\[
\sum_{j \in J^*} \langle B^*_j w|_\Gamma, B^*_j v|_\Gamma \rangle_{\mu_j} = -\sum_{j \in J} \langle B_j u|_\Gamma, B^*_j v|_\Gamma \rangle - \sum_{j \in J} \langle B_{m-1-j} u|_\Gamma, B^*_j v|_\Gamma \rangle \quad (v \in H^q(\Omega)),
\]

which means

\[
B_{m-1-j} u|_\Gamma = 0 \quad (j \in J), \quad B_{m-1-j} u|_\Gamma = -\lambda_{2j} B^*_j w|_\Gamma \quad (j \in J^*). \quad \Box
\]

**Corollary 2.1.** Assume \((\text{A-I})\) and \((\text{A-II})\). Suppose that \(\{f \in L^2(\Omega), \ f_j \in H^{m-1/2-j}(\Gamma) \ (j \in J)\}\) satisfy

\[
\mathcal{R} \quad (f, \phi) + \sum_{j \in J} \langle f_j, B^*_j \phi|_\Gamma \rangle = 0 \quad (\phi \in K^*),
\]

that is,

\[
\mathcal{R} \quad [[f, f_j], Q^* \phi] = 0 \quad (\phi \in K^*).
\]
Restricted energy inequalities and numerical approximations

Let $U \in H^m(\Omega)$ satisfy $\{B_j U|_\Gamma = f_j \ (j \in J)\}$. Set $u = A^*w + U$, where $w$ is a Riesz function of $f - AU$. Then $u \in L^2(\Omega)$ satisfies $(P)$.

**Proof.** Since $K^* \subset H^{2m-1}(\Omega)$, we have

$$[P U, Q^* \phi] = [Q U, P^* \phi], = 0 \quad (\phi \in K^*),$$

owing to Green’s Theorem. Namely, we have

$$\{\{A U, B_j U|_\Gamma \ (j \in J)\}, \{\phi, R_{m-1-j}^* \phi|_\Gamma \ (j \in J)\}\} = 0 \quad (\phi \in K^*),$$

which means

$$(A U, \phi) + \sum_{j \in J} \langle f_j, R_{m-1-j}^* \phi|_\Gamma \rangle = 0 \quad (\phi \in K^*).$$

Set $F = f - AU$, then we have

$$(F, \phi) = (f - AU, \phi) = (f, \phi) + \sum_{j \in J} \langle f_j, R_{m-1-j}^* \phi|_\Gamma \rangle \quad (\phi \in K^*).$$

Therefore, we have $F \in K^* \perp$, iff $\{f, f_j\}$ satisfies $(R)$.

Now we apply Theorem 2.1, then there exists $v = A^*w \in L^2(\Omega)$ satisfying

$$Av = F, \quad B_j v|_\Gamma = 0 \quad (j \in J),$$

where $w$ is a Riesz function of $F$. Hence $u = v + U \in L^2(\Omega)$ is a solution of $(P)$. $\square$

Now we define a subspace $\tau$ in $L^2(\Omega)$:

$$\tau = \left\{ u \in L^2(\Omega) \mid Au \in L^2(\Omega), \langle u, \phi \rangle + \sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, B_{m-1-j} \phi|_\Gamma \rangle_{-\mu_j} = 0 \ (\forall \phi \in K) \right\}.$$

We remark that $K \cap \tau = \{0\}$, because $u \in K \cap \tau$ satisfies

$$(u, u) + \sum_{j \in J^*} \langle B_{m-1-j} u|_\Gamma, B_{m-1-j} u|_\Gamma \rangle_{-\mu_j} = 0.$$

**Theorem 2.2.** Assume (A-I) and (A-II). Suppose that $f \in K^* \perp$. Let $w$ be a Riesz function of $f$. Then $u = A^*w \in \tau$ and $u$ is a solution of $(P_0)$. Moreover, a solution $u$ of $(P_0)$ is unique in $\tau$.

**Proof.** (1) From Theorem 2.1, we have
\[ Qu = \{ u, -B_{m-1-j}u|_\Gamma \ (j \in J^*) \} \]
\[ = \{ A^*w, \Lambda^{2\mu} B_j^*w|_\Gamma \ (j \in J^*) \}, \]
that is,
\[ P^*w = \{ A^*w, B_j^*w|_\Gamma \ (j \in J^*) \} \]
\[ = \{ u, -\Lambda^{-2\mu} B_{m-1-j}u|_\Gamma \ (j \in J^*) \}. \]

(2) Since \( w, A^*w \in L^2(\Omega) \), we have, owing to Green's Theorem,
\[ [P\phi, Q^*w] = [Q\phi, P^*w], \quad (\phi \in H^{2m-1}(\Omega)). \]
Since \( K \subset H^{2m-1}(\Omega) \), we have
\[ [Q\phi, P^*w] = 0 \quad (\phi \in K), \]
that is,
\[ (\phi, u) + \sum_{j \in J^*} \langle B_{m-1-j} \phi|_\Gamma, \Lambda^{-2\mu} B_{m-1-j}u|_\Gamma \rangle = 0 \quad (\phi \in K), \]
which means \( u \in \tau \).

(3) (Uniqueness) Let \( u_1 \) and \( u_2 \) be solutions of \( (P_0) \), belonging to \( \tau \). Then
\[ u = u_1 - u_2 \in K \cap \tau \]. Since \( K \cap \tau = \{0\} \), we have \( u = 0 \).

Finally, we consider a method to construct a function belonging to \( K - \{0\} \).

**Lemma 2.1.** Let \( U \in H^m(\Omega) \) satisfy \( \{ B_jU|_\Gamma = 0 \ (j \in J) \} \) and \( U \notin \tau \). Set \( \phi = A^*w + U \), where \( w \) is a Riesz function of \( -AU \). Then \( \phi \in K - \{0\} \).

**Proof.** From Theorem 2.1, we have \( \phi = A^*w + U \in K \). On the other hand, from Theorem 2.2, we have \( A^*w \in \tau \). Then we have \( \phi \notin \tau \). In fact, if we suppose \( \phi \in \tau \), then \( U = \phi - A^*w \in \tau \), which contradicts to \( U \notin \tau \). The fact \( \phi \neq 0 \) follows from \( \phi \notin \tau \).

**Lemma 2.2.** Assume that there exists \( \phi_0 \in K - \{0\} \) such that \( \phi_0(x) > 0 \) \( (x \in \Omega) \). Let \( U \) be a non-negative function satisfying
\[ U \in H^m(\Omega) - \{0\} \quad \text{and} \quad \text{supp}[U] \subseteq \Omega, \]
then \( U \notin \tau \).
§ 3. Numerical Approximation

Let us say that \( \{v_k \ (k = 1, 2, \ldots)\} \) is a basis of \( \mathcal{H} \), if any finite subset of \( \{v_k \ (k = 1, 2, \ldots)\} \) is linearly independent and the space spanned by \( \{v_k \ (k = 1, 2, \ldots)\} \) is dense in \( \mathcal{H} \).

The solution \( u \), obtained in §2, can be approximated by the method proposed in [1], that is,

**Theorem 3.1.** Assume (A-I) and (A-II). Let \( u = A^*w \), where \( w \) is a Riesz function of given \( f \in K^* \). Let \( \{v_k \ (k = 1, 2, \ldots)\} \) be a basis of \( \mathcal{H} \). Set

\[
\begin{align*}
  u_N &= ((f, v_1), \ldots, (f, v_N))\Gamma_N^{-1} \begin{pmatrix} A^*v_1 \\ \vdots \\ A^*v_N \end{pmatrix},
\end{align*}
\]

where

\[
\Gamma_N = \begin{pmatrix} [v_1, v_1] & \cdots & [v_1, v_N] \\ \vdots & \ddots & \vdots \\ [v_N, v_1] & \cdots & [v_N, v_N] \end{pmatrix}.
\]

Then

\[
u_N \to u \quad (N \to \infty) \quad \text{in} \quad L^2(\Omega).
\]

**Proof.** (1) (Theory of Fourier Series in \( \mathcal{H} \)) Let \( \{v_1, v_2, \ldots\} \) be Schmidt's orthonormalization of \( \{v_1, v_2, \ldots\} \) in \( \mathcal{H} \). For \( w \in \mathcal{H} \), we have

\[
w_N = \sum_{1 \leq k \leq N} [w, v_k]v_k ^\wedge \to w \quad \text{in} \quad \mathcal{H},
\]

that is,

\[
w_N = ([w, v_1], \ldots, [w, v_N])\Gamma_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \to w \quad \text{in} \quad \mathcal{H},
\]
where
\[ \Gamma_N = \{(v_j, v_k)\}_{j,k=1,\ldots,N}. \]

Moreover, since \( w_N \to w \) in \( H \), we have
\[ u_N = A^*w_N \to A^*w = u \quad \text{in } L^2(\Omega). \]

(2) Especially, since \( w \in H \) is a Riesz function of \( f \), that is,
\[ [w, v] = (f, v) \quad (v \in H), \]
we have
\[ [w, v_k] = (f, v_k) \quad (k = 1, 2, \ldots). \]
Hence we have
\[ w_N = ((f, v_1), \ldots, (f, v_N))\Gamma_N^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}, \]
\[ u_N = A^*w_N = ((f, v_1), \ldots, (f, v_N))\Gamma_N^{-1} \begin{pmatrix} A^*v_1 \\ \vdots \\ A^*v_N \end{pmatrix}. \]

Since the boundary of \( \Omega \) is smooth, we have

**Lemma 3.1.** Let \( \text{diam}(\Omega) < a\pi \) \((a > 0)\). Then
\[ \{ \exp(ia^{-1}x \cdot x) | x \in \mathbb{Z}^n \} \]
is a basis of \( H^q(\Omega) \).

As is shown easily, we have

**Lemma 3.2.** Let \( \{v_k \ (k = 1, 2, \ldots)\} \) be a basis of \( H^q(\Omega) \). Set
\[ v_k = \phi_k + \xi_k \quad (\phi_k \in K^*, \xi_k \in M^*), \]
then the space spanned by \( \{\xi_k \ (k = 1, 2, \ldots)\} \) is dense in \( H \). Therefore, we can obtain a subset \( \{\xi'_{j} \ (j = 1, 2, \ldots)\} = \{\xi_{kj} \ (j = 1, 2, \ldots)\} \) such that \( \{\xi'_{j} \ (j = 1, 2, \ldots)\} \) is a basis of \( H \).

Let \( \{v'_{j} \ (j = 1, 2, \ldots)\} = \{v_{kj} \ (j = 1, 2, \ldots)\} \) be a subset of \( \{v_k \ (k = 1, 2, \ldots)\} \),
corresponding to \( \{ \xi_j^i \ (j = 1, 2, \ldots) \} = \{ \xi_k \ (j = 1, 2, \ldots) \} \) in Lemma 3.2. Remark that it holds

\[
( [\xi_k, \xi_j^i] )_{k,s=1,2,\ldots,N} = ( [v_k, v_j^i] )_{k,s=1,2,\ldots,N},
\]

and

\[
( A^* \xi_k )_{k=1,2,\ldots,N} = ( A^* v_k )_{k=1,2,\ldots,N},
\]

and

\[
((f, \xi_k))_{k=1,2,\ldots,N} = ((f, v_k))_{k=1,2,\ldots,N} \quad (f \in K^{\perp}).
\]

Hence we have

**Corollary 3.1.** Assume (A-I) and (A-II). Let \( u = A^* w \), where \( w \) is a Riesz function of \( f \in K^{\perp} \). Let \( \{v_k \ (k = 1, 2, \ldots) \} \) be a basis of \( H^q(\Omega) \). Let \( \{v_k^i \ (k = 1, 2, \ldots) \} \) be a subset of \( \{v_k \ (k = 1, 2, \ldots) \} \) chosen in the above way. Set

\[
u_N = \left( (f, v_1^i), \ldots, (f, v_N^i) \right) \Gamma_N^{-1} \begin{pmatrix} A^* v_1^i \\ \vdots \\ A^* v_N^i \end{pmatrix},
\]

where

\[
\Gamma_N = \left( [v_k^i, v_j^i] \right)_{k,s=1,2,\ldots,N}.
\]

Then

\[
u_N \to u \quad (N \to \infty) \quad \text{in} \quad L^2(\Omega).
\]

Finally, we consider the approximation of \( \phi \in K - \{0\} \) in Theorem 2.3.

**Theorem 3.2.** Assume (A-I) and (A-II). Assume that there exists \( \phi_0 \in K - \{0\} \) such that \( \phi_0(x) > 0 \) (\( x \in \Omega \)). Let \( U \) be a non-negative function satisfying

\[
U \in H^m(\Omega) - \{0\} \quad \text{and} \quad \text{supp}[U] \subseteq \Omega.
\]

Let \( \{v_k \ (k = 1, 2, \ldots) \} \) be a basis of \( \mathcal{H} \). Set

\[
\phi_N = U - \left( (AU,v_1), \ldots, (AU,v_N) \right) \Gamma_N^{-1} \begin{pmatrix} A^* v_1 \\ \vdots \\ A^* v_N \end{pmatrix},
\]

where

\[
\Gamma_N = \left( [v_k, v_s] \right)_{k,s=1,2,\ldots,N}.
\]
Then
\[ \phi_N \to \phi \quad (N \to \infty) \quad \text{in} \quad L^2(\Omega), \]
and \( \phi \in K - \{0\} \).

§ 4. Examples

**Example 1.** Consider Neumann problem:

\[
\begin{cases}
-\Delta u = f & \text{in } \Omega, \\
(d/dn)u = f_1 & \text{on } \Gamma,
\end{cases}
\]

where \( \Omega \subseteq (-\pi, \pi)^n \). Then \( K(= K^*) \) is a space spanned by 1.

**Lemma 4.1.** It holds

\[
(\delta_0) \quad \|u\| \leq C\|\Delta u\| \quad (u \in M, (d/dn)u|_\Gamma = 0),
\]

where

\[
M = \{ u \in H^2(\Omega) \mid (u, 1) = 0 \}.
\]

**Proof.** Let \( \{ \phi_k \mid k = 0, 1, \ldots \} \) be a complete set of eigen-functions, corresponding to eigen-values \( \{ \lambda_k \mid k = 0, 1, \ldots \} \) such that

\[
-\Delta \phi_k = \lambda_k \phi_k \quad \text{in } \Omega, \quad (d/dn)\phi_k = 0 \quad \text{on } \Gamma,
\]

and \( (\phi_j, \phi_k) = \delta_{jk} \), where \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \).

Let \( u \in M \) satisfy

\[
\begin{cases}
-\Delta u = f \in L^2(\Omega) & \text{in } \Omega, \\
(d/dn)u = 0 & \text{on } \Gamma.
\end{cases}
\]

Then, owing to Green's Theorem, we have

\[
(f, \phi_k) = \lambda_k (u, \phi_k) \quad (k = 0, 1, \ldots).
\]

Therefore, we have \( f \in K^\perp \),

\[
u = \sum_{k \neq 0} (1/\lambda_k) (f, \phi_k) \phi_k,
\]

and
Restricted energy inequalities and numerical approximations

\[ \|u\|^2 = \sum_{k \neq 0} |\lambda_k|^{-2} |(f, \phi_k)|^2 \leq c^{-2} \|f\|^2, \]

where \( c = |\lambda_1| \).

Let \( \mathcal{H} \) be a Hilbert space, defined by the completion of \( M \) by the norm \([ \cdot ]\):

\[ [v]^2 = \|\Delta v\|^2 + \langle (d/dn)v \rangle_{1/2}^2. \]

Since \( \{e^{ik \cdot x} \ (k \in \mathbb{Z}^n)\} \) is a basis of \( H^2(\Omega) \),

\[ \{e^{ik \cdot x} - |\Omega|^{-1}(e^{ik \cdot x}, 1) \ (k \in \mathbb{Z}^n - \{0\})\} \]

is a basis of \( \mathcal{H} \). From Theorem 2.1, Theorem 2.2 and Corollary 3.1, we have

**Proposition 4.1.** Suppose that \( f \in L^2(\Omega) \) and \( (f, 1) = 0 \). Let \( w \in \mathcal{H} \) be a Riesz function of \( f \) in \( \mathcal{H} \). Set \( u = -\Delta w \), then \( u \in L^2(\Omega) \) satisfies

\[
\begin{aligned}
& \begin{cases} 
-\Delta u = f & \text{in } \Omega, \\
(d/dn)u = 0 & \text{on } \Gamma
\end{cases}
\end{aligned}
\]

and \( (u, 1) + \langle u|_{\Gamma}, 1 \rangle_{-1/2} = 0 \). Moreover, set

\[
\begin{aligned}
f_N &= ((f, e^{ik \cdot x}))_{0 < |k| \leq N}, \\
\Gamma_N &= ((e^{ik \cdot x}, e^{is \cdot x}))_{0 < |k| \leq N, 0 < |s| \leq N} \\
&= ([|k|^2 |s|^2 (e^{ik \cdot x}, e^{is \cdot x}) + (k \cdot n)(s \cdot n) (e^{ik \cdot x}|_{\Gamma}, e^{is \cdot x}|_{\Gamma})_{1/2})_{0 < |k| \leq N, 0 < |s| \leq N}, \\
V_N &= (-\Delta e^{ik \cdot x})_{0 < |k| \leq N} = ([|k|^2 e^{ik \cdot x})_{0 < |k| \leq N}, \\
\end{aligned}
\]

and

\[
u_N = f_N \Gamma_N^{-1} V_N,
\]

then it holds

\[
u_N \to u \ (N \to \infty) \ in \ L^2(\Omega).
\]

**Example 2.** Consider Dirichlet problem:

\[
\begin{aligned}
& \begin{cases} 
(-\Delta - \lambda_0)u = f & \text{in } \Omega, \\
u = f_0 & \text{on } \Gamma,
\end{cases}
\end{aligned}
\]

where \( \Omega \subseteq (-\pi, \pi)^n \) and \( \lambda_0 \) is the least eigen-value for the eigen-value problem:
Then, $K(=K^*)$ is a space spanned by $\phi_0$, where $\phi_0$ is an eigen-function corresponding to the eigen-value $\lambda_0$.

**Lemma 4.2.** It holds
\[
\|u\| \leq C\|(-\Delta - \lambda_0)u\| \quad (u \in M, u|_{\Gamma} = 0),
\]
where $M = \{u \in H^2(\Omega) | (u, \phi_0) = 0\}$.

**Proof.** Let $\{\phi_k \ (k = 0, 1, \ldots)\}$ be a complete set of eigen-functions, corresponding to eigen-values $\{\lambda_k \ (k = 0, 1, \ldots)\}$ such that
\[
-\Delta \phi_k = \lambda_k \phi_k \quad \text{in } \Omega, \quad \phi_k = 0 \quad \text{on } \Gamma,
\]
and $(\phi_j, \phi_k) = \delta_{jk}$, where $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$.

Let $u \in M$ satisfy
\[
\begin{cases}
-\Delta u = f \in L^2(\Omega) & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma.
\end{cases}
\]
Then, owing to Green's Theorem, we have
\[
(f, \phi_k) = (\lambda_k - \lambda_0)(u, \phi_k) \quad (k = 0, 1, \ldots).
\]
Therefore, we have $f \in K^\perp$,
\[
u = \sum_{k \neq 0} (\lambda_k - \lambda_0)^{-1} (f, \phi_k) \phi_k,
\]
and
\[
\|u\|^2 = \sum_{k \neq 0} (\lambda_k - \lambda_0)^{-2} |(f, \phi_k)|^2 \leq c^{-2} \|f\|^2,
\]
where $c = |\lambda_1 - \lambda_0|$.

Let $\mathcal{H}$ be a Hilbert space, defined by the completion of $M$ by the norm $[\cdot]$:
\[
[v]^2 = \|(-\Delta - \lambda_0)v\|^2 + \langle v|_{\Gamma} \rangle^2_{L^{1+1/2}}.
\]
Since $\{e^{ik \cdot x} \ (k \in \mathbb{Z}^n)\}$ is a basis of $H^2(\Omega)$,
\[
\{\xi_k(x) = e^{ik \cdot x} - (e^{ik \cdot x}, \phi_0) \phi_0 \ (k \in \mathbb{Z}^n)\}
\]
is a basis of $\mathcal{H}$, we have from Theorem 2.1, Theorem 2.2 and Corollary 3.1, we have

**Proposition 4.2.** Suppose that $f \in L^2(\Omega)$ and $(f, \phi_0) = 0$. Let $w \in \mathcal{H}$ be a Riesz function of $f$ in $\mathcal{H}$. Set $u = (-\Delta - \lambda_0)w$, then $u \in L^2(\Omega)$ satisfies

$$
\begin{cases}
(-\Delta - \lambda_0)u = f & \text{in } \Omega, \\
u = 0 & \text{on } \Gamma
\end{cases}
$$

and $(u, \phi_0) + \langle (d/dn)u|_{\Gamma}, (d/dn)\phi_0|_{\Gamma}\rangle_{-1-1/2} = 0$. Moreover, set

$$
f_N = ((f, e^{ik \cdot x}), |k| \leq N),
$$

$$
\Gamma_N = \langle (e^{ik \cdot x}, e^{is \cdot x}), |k| \leq N, |s| \leq N \rangle
$$

$$
= \langle (|k|^2 - \lambda_0)(|s|^2 - \lambda_0)(e^{ik \cdot x}, e^{is \cdot x}) + \langle e^{ik \cdot x}, e^{is \cdot x} \rangle_{1+1/2}, |k| \leq N, |s| \leq N \rangle,
$$

$$
V_N = ((-\Delta - \lambda_0)e^{ik \cdot x}, |k| \leq N) = \langle (|k|^2 - \lambda_0)e^{ik \cdot x}, |k| \leq N \rangle,
$$

and

$$
u_N = f_N \Gamma_N^{-1} V_N,
$$

then it holds

$$
u_N \to u \quad (N \to \infty) \quad \text{in } L^2(\Omega).
$$

**References**


28-2-507 Fukakusa-Sekiyashiki-cho
Fushimi-ku, Kyoto, Japan, 612-0037