INVERSE SPECTRAL PROBLEMS ON HYPERBOLIC MANIFOLDS AND THEIR APPLICATIONS TO INVERSE BOUNDARY VALUE PROBLEMS IN EUCLIDEAN SPACE

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Abstract. We propose a new approach to solve the inverse boundary value problems in the Euclidean space. The idea consists in embedding the problem into hyperbolic manifolds and using their spectral properties. As a by-product, one can discuss the reconstruction of local conformal deformation of the metric of hyperbolic manifold from the spectral data at infinity. We also propose a new spectral data observed from the cusp neighborhood at infinity.

1. Introduction

There are two fundamental problems in the inverse spectral theory. One is the inverse boundary value problem (*IBVP*). Let Ω be a bounded domain in $\mathbf{R}^n$ and consider the Dirichlet problem

$$(-\Delta + V)u = 0 \quad \text{in} \quad \Omega, \quad u = f \quad \text{on} \quad \partial\Omega.$$

The Dirichlet-Neumann map is defined by

$$\Lambda_V f = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega},$$

ν being the outer unit normal to the boundary. In IBVP, we try to reconstruct V from Λ_V.

Another is the inverse scattering problem (*ISP*). For Schrödinger operastors $H_0 = -\Delta$, $H = H_0 + V$, define the wave operators $W_{\pm} = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$, and the scattering operator $S = (W_+)^* W_-$. The S-matrix $S(\lambda;\theta,\theta')$ is defined to be the integral kernel of $\mathcal{F}S\mathcal{F}^*$, where $\mathcal{F}$ is the Fourier tarnsformation and λ is the energy parameter. The S-matrix $S(\lambda;\theta,\theta')$ is observed in the behavior at infinity of solutions to the Schrödinger equation $(H-\lambda)\psi = 0$. In fact if we put $A(\lambda;\theta,\omega) = S(\lambda;\theta,\omega) - \delta(\theta-\omega)$, we have

$$\psi(x,\lambda,\omega) \sim e^{i\sqrt{\lambda}\cdot\omega x} - C(\lambda)\frac{e^{i\sqrt{\lambda}r}}{r^{(n-1)/2}}A(\lambda;\theta,\omega),$$

as $r = |x| \to \infty$, $\theta = x/r$. In ISP, we try to reconstruct V from the scattering matrix of H. We restrict ourselves here to the fixed energy problem, namely, the reconstruction of V from the S-matrix of arbitrarily given fixed energy.

There is an extensive literature dealing with these subjects. For $n \geq 2$, these two problems are known to be equivalent and for $n \geq 3$, they are affirmatively solved by Sylvester-Uhlmann [36], Nachman [25], Khenkin-Novikov [19].

There is also an abundance of articles devoted to the spectral theory of Schrödinger operators on Riemannian manifolds (see e.g. Lax-Phillips [21], Agmon [1]). The uniqueness (up to diffeomorphism) of the metric with given D-N map was proved by Lee and Uhlmann [23], Sylvester [34] and Nachman [26]. Belishev and Kurylev constructed the Riemannian manifold itself from the boundary spectral data [2], [18]. However, the above mentioned problems in the fixed metric have not been considered yet.

For the moment, there is essentially only one method for solving IBVP and ISP. In IBVP it is called the method of complex geometrical optics, or exponentially growing solutions, and in ISP it is called Faddeev's Green function. These are two sides of one thing.

In this paper, we propose a new approach based on the spectral properties of hyperbolic manifolds, which we explain breifly here.

The fisrt fact we have to notice is:

The IBVP in the Euclidean space and that in the hyperbolic manifold are equivalent.

This can be easily observed in the 2-dimensional case. Multiplying the Schrödinger equation

$$-\Delta u + qu = 0 \tag{1.1}$$

in $\mathbf{R}^2$ by x_2^2, we have

$$-x_2^2\Delta u + x_2^2 qu = 0,$$

which is just the Schrödinger equation in $\mathbf{H}^2$. Therefore the D-N maps $\widetilde{\Lambda}_V$ in $\mathbf{R}^2$ and $\Lambda_{x_2^2 V}$ in $\mathbf{H}^2$ are related as follows

$$\widetilde{\Lambda}_V = x_2\Lambda_{x_2^2 V}.$$

In the general case, by putting $u = x_n^{(2-n)/2}v$, we are led to the equation

$$(-x_n^2\partial_n^2 + (n-2)x_n\partial_n - x_n^2\Delta_x + V)v = 0 \quad \text{in} \quad \Omega \subset \mathbf{H}^n, \tag{1.2}$$

where $V = x_n^2 q - \frac{n(n-2)}{4}$, and the points in $\mathbf{H}^n$ are denoted as (x, x_n), $x_n > 0$.

The next step is to use the gauge transformation $v = e^{i\theta\cdot x}u$ to introduce a parameter θ in the above equation :

$$(-x_n^2\partial_n^2 + (n-2)x_n\partial_n - x_n^2(\partial_x + i\theta)^2 + V)u = 0 \quad \text{in} \quad \Omega \subset \mathbf{H}^n, \quad \theta \in \mathbf{R}^{n-1}. \tag{1.3}$$

In the 3rd step, we consider the action of simple discrete groups. We take a sufficiently large lattice Γ of rank $n-1$ in $\mathbf{R}^{n-1}$ so that Ω is contained in one coordinate patch of the quotient space $\Gamma\backslash\mathbf{H}^n$. Then the above equation (1.3) can be regarded as the one in a domain in $\Gamma\backslash\mathbf{H}^n$. Here one should note that the operator $-x_n^2\partial_n^2 + (n-2)x_n\partial_n + x_n^2(\partial_x + i\theta)^2$ is just the Floquet operator in the theory of periodic Schrödinger equation.

In the 4th step, we use the equivalence of IBVP and ISP in $\Gamma\backslash\mathbf{H}^n$ to construct the scattering matrix from the D-N map.

The final step is the complex Born approximation. By passing to the Fourier series, the Green's function of the Floquet operators are written by modified Bessel functions, $K_{i\sigma}(\zeta x_n)$, $I_{i\sigma}(\zeta x_n)$, $\zeta = \sqrt{(\gamma^* + \theta)^2}$. They are analytic with respect to θ. Hence by varying θ along the imaginary axis, one can recover the potential q from the scattering matrix. This procedure works for $n \geq 3$.

In summary, one can show the following theorem.

Let $\mathcal{M}$ be an n-dimensional hyperbolic manifold with $n \geq 3$, and Ω a contractible bounded open set in $\mathcal{M}$ with smooth boundary. Suppose 0 is not a Dirichlet eigenvalue of $-\Delta_g + V$. Then V is uniquely reconstructed from the D-N map.

In fact $\mathcal{M} = \Gamma\backslash\mathbf{H}^n$ for some discrete group Γ. Hence by passing to the universal covering space, we see that the Dirichlet problem on a bounded contractible open set in $\mathcal{M}$ is equivalent to that on a similar set in $\mathbf{H}^n$.

By examining the above procedures, one can introduce a new notion of spectral data, the scattering amplitude associated with the cusp neighborhood at infinity. One can show that

Let $n \geq 2$. Suppose that $\Gamma\backslash\mathbf{H}^n$ has a cusp and that $-\Delta_g$ is perturbed in a contractible compact set in $\Gamma\backslash\mathbf{H}^n$. Then to know the associated D-N map is equivalent to know the scattering amplitude at the cusp.

Using this fact, one can reconstruct the conformal perturbation of the metric for $n \geq 3$ and the general perturbation of the metric modulo conformal deformation for $n = 2$ from the scattering amplitude at the cusp of $\Gamma\backslash\mathbf{H}^n$. This fact also holds for the arithmetic surface $SL(2, \mathbf{Z})\backslash\mathbf{H}^2$.

Although our method is apparently very different from the usual ones, there is a close connection between our approach and the methods already established for the multi-dimensional inverse problems. The crucial trick in this paper is to consider the analytic continuation with respect to the Floquet parameter θ of the resolvent $R_0(\lambda + i0; \theta)$ (see §5). This idea is inspired by the direction dependent Green operator of Faddeev ([6], [25], [19]). Employing the family of scattering amplitudes for the Floquet operators as the spectral data avoids the difficulty of exceptional points, the main barrier of the inverse scattering problem. This is analogus to the situation in the approach of Eskin-Ralston ([5], [12]). Our reconstruction procedure is the complex Born approximation, the method frequently used in the inverse scattering theory.

To study the continuous spectrum of the Schrödinger operator, the first important step is the limiting absorption principle (LAP). Basically there are three methods for the resolvent estimates ; the framework of scattering metric (Melrose [24]), the commutator method based on the Mourre inequality (Froese-Hislop [7], Hislop [8]), and the methods of a-priori estimates (Perry [28]). In this paper we adopt a more classical approach based on integration by parts developed by Ikebe-Saito [9]. This will make the argument self-contained except for §9, in which we need word by word translation of a part of the work of Saito [30]. Let us remark that if the potential is compactly supported, the contents of §9 can be replaced by an elementary argument using the properties of Bessel functions.

The plan of this paper is as follows. In §2, we prepare basic properties of Bessel functions. We study spectral properties of Schrödinger operators on $\mathbf{H}^n$ and $\Gamma\backslash\mathbf{H}^n$ in §3 and §4, leaving the proof in later sections. In §5, we reconstruct the Fourier coefficients of the potential from the scattering amplitudes of Floquet operators on $\Gamma\backslash\mathbf{H}^n$. In §6, we study the relation between ISP and IBVP in $\Gamma\backslash\mathbf{H}^n$, and derive the main theorem on IBVP in $\mathbf{R}^n$. We shall prove in §7 LAP for the 1-dimensional operator $-\partial_y^2 + e^{2y}\lambda$, which is utilized in §8 to prove LAP for Schrödinger operators on $\mathbf{H}^n$ and $\Gamma\backslash\mathbf{H}^n$. In §9 we state an abstract version of the theorem due to Saito [30] on the growth properties of solutions to the ordinary differential equation with operator-valued coefficients. In §10, we shall discuss the scattering amplitude at the cusp for general hyperbolic manifold and reconstruction of the metric. More detailed arguments of this section are given in [13]. We also note that in [14] an application of this hyperbolic space approach is given to derive local properties of Dirichlet-to-Neumann map for the inverse boundary value problem in the Euclidean space.

The notation in this paper is almost standard. C's denote various constants. For two Banach spaces X and Y, $\mathbf{B}(X;Y)$ denotes the totality of bounded operators from X to Y, and $\mathbf{B}(X) = \mathbf{B}(X;X)$. For $x \in \mathbf{R}^n$, $\langle x\rangle = (1+|x|^2)^{1/2}$. For $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbf{C}^n$, $\zeta^2 = \sum_{j=1}^n \zeta_j^2$. We denote by $L^2(\Omega; d\mu)$ the L^2-space over Ω with respect to the measure $d\mu$. The spectrum of a closed operator A is denoted by $\sigma(A)$. In sections 7 and 8, we use a non-standard notation. For a set $\{\cdots\}$, $F(\cdots)$ denotes its characteristic function. For example $F(y<0)$ means the characteristic function of the set $\{y \in \mathbf{R}; y < 0\}$ in §7, while in §8 it will mean that of $\{(x,y); x \in \mathbf{E}, y < 0\}$ or that of $\{(x,y) \in \mathbf{R}^n; y < 0\}$. There will be no fear of confusion.

The author is indebted to James Ralston for pointing out errors in the earlier version of this paper. He also thanks to Gunther Uhlmann for useful discussions and encouragements.

2. Bessel functions

2.1 *Bessel functions.* Let $J_\nu(x)$ be the Bessel function of order $\nu \notin \mathbf{Z}$, and for $x > 0$ let

$$I_\nu(x) = e^{-\nu\pi i/2} J_\nu(ix), \quad K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)} \tag{2.1}$$

be modified Bessel functions. They are analytic in the complex plane with cut along the negative real axis and

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{(z^2/4)^k}{k!\Gamma(k+\nu+1)}. \tag{2.2}$$

By (2.2), for $x > 0, \sigma \in \mathbf{R}$,

$$\overline{I_{i\sigma}(x)} = I_{-i\sigma}(x), \quad \overline{K_{i\sigma}(x)} = K_{-i\sigma}(x) = K_{i\sigma}(x). \tag{2.3}$$

One also has for $0 < x < 1$, $\sigma \neq 0$,

$$|I_{i\sigma}(x)| \leq Cx^{-\mathrm{Im}\,\sigma}, \quad |K_{i\sigma}(x)| \leq Cx^{-|\mathrm{Im}\,\sigma|}. \tag{2.4}$$

The following asymptotic expansion is well-known :

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} + \frac{e^{-z+(\nu+1/2)\pi i}}{\sqrt{2\pi z}}, \quad |z| \to \infty, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \tag{2.5}$$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} + \frac{e^{-z-(\nu+1/2)\pi i}}{\sqrt{2\pi z}}, \quad |z| \to \infty, \quad -\frac{3\pi}{2} < \arg z < \frac{\pi}{2}, \tag{2.6}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad |z| \to \infty, \quad -\pi < \arg z < \pi. \tag{2.7}$$

In particular

$$K_\nu(-ix) \sim \sqrt{\frac{\pi}{2x}} e^{i(x+\pi/4)}, \quad x \to \infty. \tag{2.8}$$

(See e.g. [39] p. 202.) The following formulas are known as Kontorovich-Lebedev inversion formulas

$$\frac{1}{\pi^2}\int_0^\infty \frac{K_{i\sigma}(x)K_{i\sigma}(y)}{\sqrt{xy}} 2\sigma\sinh(\pi\sigma)d\sigma = \delta(x-y), \tag{2.9}$$

$$\frac{2\sigma\sinh(\pi\sigma)}{\pi^2}\int_0^\infty \frac{K_{i\sigma}(x)K_{i\tau}(x)}{x}dx = \delta(\sigma-\tau). \tag{2.10}$$

(See e.g. [37] pp.138, 145.)

2.2 *Auxiliary operator.* We reduce the analysis of our Schrödinger operators on hyperbolic manifolds to that of the operator

$$L_0(\zeta) = -\partial_y^2 + e^{2y}\zeta^2 \tag{2.11}$$

in $L^2(\mathbf{R}; dy)$ with parameter $\zeta \in \mathbf{C}$. Here $\partial_y = d/dy$. If $\zeta \geq 0$, $L_0(\zeta)|_{C_0^\infty(\mathbf{R})}$ is essentially self-adjoint, whose self-adjoint realization is also denoted by $L_0(\zeta)$. For $s \in \mathbf{R}$, we introduce the weighted L^2-space $L^{2,s} = L^{2,s}(\mathbf{R}^1)$ by

$$u \in L^{2,s} \Longleftrightarrow \|u\|_s^2 = \int_{-\infty}^\infty (1+|y|)^{2s}|u(y)|^2 dy < \infty. \tag{2.12}$$

We shall prove in §7 that for $\zeta \geq 0, E > 0$ and $s > 1/2$, there exist the following strong limits

$$\mathrm{s}-\lim_{\epsilon\downarrow 0}(L_0(\zeta) - (E \pm i\epsilon))^{-1} =: (L_0(\zeta) - E \mp i0)^{-1} \in \mathbf{B}(L^{2,s}; L^{2,-s}). \tag{2.13}$$

2.3 *Green function.* For $\zeta, \sigma \in \mathbf{C}$ with $\zeta \neq 0$, the equation

$$(L_0(\zeta) - \sigma^2)u = 0 \tag{2.14}$$

has two linearly independent solutions

$$K_{i\sigma}(\zeta e^y), \quad I_{i\sigma}(\zeta e^y). \tag{2.15}$$

In the following, we always assume that Re $\zeta \geq 0$. We put

$$G(y, y'; \sigma, \zeta) = \begin{cases} K_{i\sigma}(\zeta e^y) I_{i\sigma}(\zeta e^{y'}), & y > y' \\ \\ I_{i\sigma}(\zeta e^y) K_{i\sigma}(\zeta e^{y'}), & y' > y. \end{cases} \tag{2.16}$$

By (2.4) $\sim$ (2.7), it satisfies for $\sigma \in \mathbf{R} \setminus \{0\}$

$$|G(y, y'; \sigma, \zeta)| \leq \frac{C\exp(-\mathrm{Re}\,\zeta|e^y - e^{y'}|)}{(1+|\zeta e^y|^{1/2})(1+|\zeta e^{y'}|^{1/2})}, \tag{2.17}$$

where C is a constant independent of y, y', ζ. We shall use this estimate in the following two ways :

$$|G(y, y'; \sigma, \zeta)| \leq C, \tag{2.18}$$

$$|G(y, y'; \sigma, \zeta)| \leq C|\zeta|^{-1}e^{-(y+y')/2}. \tag{2.19}$$

For $f \in C_0^\infty(\mathbf{R})$, we define $G(\sigma, \zeta)f$ by

$$G(\sigma, \zeta)f = \int_{-\infty}^\infty G(y, y'; \sigma, \zeta)f(y')dy'. \tag{2.20}$$

Then $u = G(\sigma, \zeta)f$ satisfies

$$(L_0(\zeta) - \sigma^2)u = f. \tag{2.21}$$

If $\zeta > 0$ and Im $\sigma < 0$, $G(\sigma, \zeta)f \in L^2(\mathbf{R})$ by (2.4). Therefore

$$G(\mp\sqrt{E \pm i\epsilon}, \zeta) = (L_0(\zeta) - (E \pm i\epsilon))^{-1}. \tag{2.22}$$

In particular

$$G(\mp\sqrt{E},\zeta) = (L_0(\zeta) - E \mp i0)^{-1}, \quad E > 0, \quad \zeta > 0. \tag{2.23}$$

2.4 *Spectral representation.* Let $f \in L^{2,s}, s > 1/2$. Since

$$I_\nu(z) \sim \frac{1}{\Gamma(1+\nu)}\left(\frac{z}{2}\right)^\nu \quad (z \to 0), \tag{2.24}$$

we have for $\sigma \in \mathbf{R}\setminus\{0\}, \zeta > 0$ and $y \to -\infty$

$$G(\sigma,\zeta)f \sim \frac{1}{\Gamma(1+i\sigma)}\left(\frac{\zeta}{2}\right)^{i\sigma} e^{i\sigma y}\int_{-\infty}^{\infty} K_{i\sigma}(\zeta e^t)f(t)dt. \tag{2.25}$$

Let $K(\zeta)$ be the operator defined by

$$(K(\zeta)f)(\sigma) = \frac{1}{\pi}\int_{-\infty}^{\infty} K_{i\sigma}(\zeta e^y)f(y)dy. \tag{2.26}$$

It follows from (2.1) and (2.3) that the integral kernel of $\frac{1}{2\pi i}\Big[G(-\sqrt{E},\zeta) - G(\sqrt{E},\zeta)\Big]$ is

$$\frac{\sinh(\sqrt{E}\pi)}{\pi^2}K_{i\sqrt{E}}(\zeta e^y)K_{i\sqrt{E}}(\zeta e^{y'}). \tag{2.27}$$

This implies that for $f \in C_0^\infty(\mathbf{R}), \zeta > 0, E > 0$

$$\frac{1}{2\pi i}([(L_0(\zeta) - E - i0)^{-1} - (L_0(\zeta) - E + i0)^{-1}]f, f)$$
$$= |(K(\zeta)f)(\sqrt{E})|^2 \sinh(\sqrt{E}\pi). \tag{2.28}$$

Therefore by the standard argument from spectral theory and the formulas (2.9), (2.10), one can show that

$$K(\zeta) : L^2(\mathbf{R}^1; dy) \to L^2((0,\infty); 2\sigma\sinh(\pi\sigma)d\sigma) \tag{2.29}$$

is unitary and

$$(K(\zeta)L_0(\zeta)f)(\sigma) = \sigma^2(K(\zeta)f)(\sigma). \tag{2.30}$$

In contrast to (2.25), if we let $y \to \infty$ in $G(\sigma,\zeta)f(y)$ for $\sigma \in \mathbf{R}\setminus\{0\}, \zeta > 0$ and compactly supported f, we have

$$G(\sigma,\zeta)f \sim \sqrt{\frac{\pi}{2\zeta}}\exp(-\frac{y}{2} - \zeta e^y)\int_{-\infty}^{\infty} I_{i\sigma}(\zeta e^t)f(t)dt. \tag{2.31}$$

3. Schrödinger operators on $\mathbf{H}^n$

3.1 *Basic spectral properties.* Let $\mathbf{H}^n = \mathbf{R}^n_+ = \{(x,x_n); x \in \mathbf{R}^{n-1}, x_n > 0\}$ be the hyperbolic space equipped with the metric $g = ds^2 = x_n^{-2}((dx)^2 + (dx_n)^2)$. The Laplace-Beltrami operator on $\mathbf{H}^n$ is

$$\Delta_g = x_n^2\partial_n^2 - (n-2)x_n\partial_n + x_n^2\Delta_x, \tag{3.1}$$

where $\partial_n = \partial/\partial x_n$, $\Delta_x = \sum_{i=1}^{n-1}(\partial/\partial x_i)^2$. By the change of variable $y = \log x_n$ and the unitary operator

$$L^2(\mathbf{H}^n; x_n^{-n}dxdx_n) \ni u \to v = e^{-(n-1)y/2}u \in L^2(\mathbf{R}^n; dxdy), \tag{3.2}$$

the equation

$$(-\Delta_g - \frac{(n-1)^2}{4} - z)u = f \tag{3.3}$$

is transformed into

$$(H_0 - z)v = e^{-(n-1)y/2}f, \tag{3.4}$$

$$H_0 = -\partial_y^2 - e^{2y}\Delta_x. \tag{3.5}$$

Let $H = H_0 + V$, where $V = V_1(x, y) + V_2(x, y)$ is a real function on $\mathbf{R}^n$. We shall assume that V_1 is compactly supported and

$$V_1(x, y) \in L^p(\mathbf{R}^n), \tag{3.6}$$

where $p = 2$ for $n \leq 3$, $p > n/2$ for $n \geq 4$,

$$|V_2(x, y)| \leq C(1 + |x|)^{-\rho}(1 + |y|)^{-1-\rho} \tag{3.7}$$

for some constants $C, \rho > 0$. $H|_{C_0^\infty(\mathbf{R}^n)}$ is essentially self-adjoint, whose self-adjoint extension is denoted by H also. Let $R(z) = (H - z)^{-1}$, and $R_0(z) = (H_0 - z)^{-1}$. For $t, s \in \mathbf{R}$, let $\mathcal{H}^{t,s}$ be the weighted Hilbert space endowed with the norm

$$\|u\|_{t,s} = \|(1 + |x|)^t(1 + |y|)^s u(x, y)\|_{L^2(\mathbf{R}^n; dxdy)} < \infty. \tag{3.8}$$

The following theorem will be proved in §8.

Theorem 3.1. *(1)* $\sigma_e(H) = [0, \infty)$.
(2) $\sigma_p(H) \cap (0, \infty) = \emptyset$.
(3) For $0 < \delta < \rho, s > 1/2$ *and* $\lambda > 0$, *there exist the strong limits*

$$\mathrm{s} - \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon) =: R(\lambda \pm i0) \in \mathbf{B}(\mathcal{H}^{0,s}; \mathcal{H}^{-\delta,-s}).$$

(4) For $s > 1/2$ *and* $\lambda > 0$, *there exist the weak limits*

$$\mathrm{w} - \lim_{\epsilon \downarrow 0} R_0(\lambda \pm i\epsilon) =: R_0(\lambda \pm i0) \in \mathbf{B}(\mathcal{H}^{0,s}; \mathcal{H}^{0,-s}).$$

3.2 *Spectral representation for* H_0. Let F_0 be the Fourier transformation on $\mathbf{R}^{n-1}$:

$$F_0 u(\xi) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{-ix\cdot\xi} u(x) dx. \tag{3.9}$$

Let $R_0(z) = (H_0 - z)^{-1}$. Then by (2.11)

$$R_0(z) = F_0^{-1}(L_0(|\xi|) - z)^{-1} F_0. \tag{3.10}$$

The right-hand side should be written as $(F_0 \otimes 1)^{-1} 1 \otimes (L_0(|\xi|) - z)^{-1}(F_0 \otimes 1)$. However, we employ the abbreviation of writing AB instead of $(A \otimes 1)(1 \otimes B)$ throughout the paper. By (2.28) we have

$$\begin{aligned} &\frac{1}{2\pi i}([R_0(\lambda + i0) - R_0(\lambda - i0)]f, f) \\ &= \sinh(\sqrt{\lambda}\pi) \int_{\mathbf{R}^{n-1}} |(K(|\xi|) F_0 f(\xi, \cdot))(\sqrt{\lambda})|^2 d\xi. \end{aligned} \tag{3.11}$$

With these preparations, we introduce the spectral representation for H_0.

Definition 3.2. We define

$$(\mathcal{F}_0^{(\pm)} f)(w, \lambda) = \left(F_0^{-1} |\xi|^{\mp i\sqrt{\lambda}} (K(|\xi|)(F_0 f)(\xi, \cdot))(\sqrt{\lambda})\right)(w).$$

Then by (3.11), we have

$$\frac{1}{2\pi i}([R_0(\lambda + i0) - R_0(\lambda - i0)]f, f) = \sinh(\sqrt{\lambda}\pi) \int_{\mathbf{R}^{n-1}} |(\mathcal{F}_0^{(\pm)} f)(w, \lambda)|^2 dw. \tag{3.12}$$

Using (2.29) and (2.30), one can show the following theorem.

Theorem 3.3.

$$\mathcal{F}_0^{(\pm)} : L^2(\mathbf{R}^n; dxdy) \to L^2(\mathbf{R}^{n-1} \times (0,\infty); \sinh(\pi\sqrt{\lambda})dwd\lambda)$$

is unitary and diagonalizes H_0 :

$$(\mathcal{F}_0^{(\pm)} H_0 f)(w,\lambda) = \lambda(\mathcal{F}_0^{(\pm)} f)(w,\lambda).$$

This spectral representation is related to the asymptotic behavior of the resolvent in the following way.

Theorem 3.4. *For $f \in \mathcal{H}^{0,s}, s > 1/2$, we have*

$$\lim_{y\to-\infty} e^{\pm i\sqrt{\lambda}y}(R_0(\lambda \pm i0)f)(\cdot,y) = C_\pm(\lambda)(\mathcal{F}_0^{(\pm)} f)(\cdot,\lambda) \quad \text{in} \quad L^2(\mathbf{R}^{n-1}),$$

$$C_\pm(\lambda) = \frac{2^{\pm i\sqrt{\lambda}}\pi}{\Gamma(1 \mp i\sqrt{\lambda})}.$$

Proof. This follows from (2.18), (2.25) and Lebesgue's convergence theorem. $\diamondsuit$

3.3 *Spectral representation for H.* One can construct the spectral representation for $H = H_0 + V$ by the method of perturbation. We define for $\lambda > 0$ and $f \in C_0^\infty(\mathbf{R}^n)$

$$(\mathcal{F}_0^{(\pm)}(\lambda)f)(w) = (\mathcal{F}_0^{(\pm)} f)(w,\lambda). \tag{3.13}$$

By Theorem 3.1 (4) and (3.12)

$$\mathcal{F}_0^{(\pm)}(\lambda) \in \mathbf{B}(\mathcal{H}^{0,s}; L^2(\mathbf{R}^{n-1})), \quad s > 1/2. \tag{3.14}$$

We define

$$\mathcal{F}^{(\pm)}(\lambda) = \mathcal{F}_0^{(\pm)}(\lambda)\left(1 - VR(\lambda \pm i0)\right). \tag{3.15}$$

Then by Theorem 3.1 (3) and (3.14)

$$\mathcal{F}^{(\pm)}(\lambda) \in \mathbf{B}(\mathcal{H}^{0,s}; L^2(\mathbf{R}^{n-1})), \quad s > 1/2. \tag{3.16}$$

For $\varphi \in L^2(\mathbf{R}^{n-1})$, $\mathcal{F}^{(\pm)}(\lambda)^*\varphi \in \mathcal{H}^{0,-s}$ is a generalized eigenfunction in the following sense

$$(H - \lambda)\mathcal{F}^{(\pm)}(\lambda)^*\varphi = 0. \tag{3.17}$$

The following theorem is proved by the well-known method of perturbation. See e.g. Kuroda [20].

Theorem 3.5. *Let $(\mathcal{F}^{(\pm)}f)(w,\lambda) = (\mathcal{F}^{(\pm)}(\lambda)f)(w)$. Then $\mathcal{F}^{(\pm)}$ are uniquely extended to partial isometries with initial set $\mathcal{H}_{ac}(H)$ = the absolutely continuous subspace for H and final set $L^2(\mathbf{R}^{n-1} \times (0,\infty); \sinh(\pi\sqrt{\lambda})dwd\lambda)$. They diagonalize H :*

$$(\mathcal{F}^{(\pm)} Hf)(w,\lambda) = \lambda(\mathcal{F}^{(\pm)} f)(w,\lambda).$$

By the resolvent equation

$$R(\lambda \pm i0) = R_0(\lambda \pm i0)\left(1 - VR(\lambda \pm i0)\right)$$

and Theorem 3.4, we have

Theorem 3.6. *For $f \in \mathcal{H}^{0,s}, s > 1/2$, we have*

$$\lim_{y\to-\infty} e^{\pm i\sqrt{\lambda}y} R(\lambda \pm i0)f(\cdot,y) = C_\pm(\lambda)\mathcal{F}^{(\pm)}(\lambda)f \quad \text{in} \quad L^2(\mathbf{R}^{n-1}),$$

where $C_\pm(\lambda)$ are given in Theorem 3.4.

3.4 *Scattering amplitude.* The wave operators $W_\pm$ are defined by

$$W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}. \tag{3.18}$$

They are asymptotically complete in the sense that Ran $W_\pm = \mathcal{H}_{ac}(H)$. Let S be the scattering operator

$$S = W_+^* W_-. \tag{3.19}$$

It is well-known that $\hat{S} = \mathcal{F}_0^{(+)} S \mathcal{F}_0^{(+)*}$ has the direct integral representation :

$$(\hat{S}f)(w,\lambda) = (\hat{S}(\lambda)f(\cdot,\lambda))(w), \quad \forall f \in L^2(\mathbf{R}^{n-1}\times(0,\infty); \sinh(\pi\sqrt{\lambda})dwd\lambda), \tag{3.20}$$

where $\hat{S}(\lambda)$ is untary on $L^2(\mathbf{R}^{n-1})$ and is written as

$$\hat{S}(\lambda) = 1 - 2\pi i \sinh(\pi\sqrt{\lambda})A(\lambda), \tag{3.21}$$

$$A(\lambda) = \mathcal{F}_0^{(+)}(\lambda)(V - VR(\lambda+i0)V)\mathcal{F}_0^{(+)}(\lambda)^*. \tag{3.22}$$

The last formula is called the *scattering amplitude*.

We next relate $A(\lambda)$ to the asymptotic behavior at infinity of the generalized eigenfunction of H. For $\lambda > 0$, let $Q(\lambda)$ be defined by

$$Q(\lambda) = F_0^{-1}|\xi|^{2i\sqrt{\lambda}}F_0. \tag{3.23}$$

Then we have

$$\mathcal{F}_0^{(-)}(\lambda) = Q(\lambda)\mathcal{F}_0^{(+)}(\lambda). \tag{3.24}$$

With this in mind, we define the *geometric scattering amplitude* $\widetilde{A}(\lambda)$ by

$$\widetilde{A}(\lambda) = A(\lambda)Q(\lambda)^*. \tag{3.25}$$

Theorem 3.7. *The geometric scattering amplitude $\widetilde{A}(\lambda)$ has the following expression*

$$\widetilde{A}(\lambda) = \mathcal{F}_0^{(+)}(\lambda)(V - VR(\lambda+i0)V)\mathcal{F}_0^{(-)}(\lambda)^*. \tag{3.26}$$

It is related to the asymptotic behavior of a generalized eigenfunction in the following way : For $\varphi \in L^2(\mathbf{R}^{n-1})$, we have as $y \to -\infty$

$$\mathcal{F}^{(-)}(\lambda)^*\varphi - \mathcal{F}_0^{(-)}(\lambda)^*\varphi \sim -C_+(\lambda)e^{-i\sqrt{\lambda}y}\widetilde{A}(\lambda)\varphi \quad \text{in} \quad L^2(\mathbf{R}^{n-1}). \tag{3.27}$$

Proof. The formula (3.26) follows from (3.22) and (3.24). The formula (3.27) follows from Theorem 3.6. $\diamondsuit$

4. Schrödinger operators on $\Gamma\backslash\mathbf{H}^n$

4.1 *Floquet operators on the quotient manifold.* Let $\Gamma \subset \mathbf{R}^{n-1}$ be a lattice of rank $n-1$, and $\Gamma^* = \{\gamma^* \in \mathbf{R}^{n-1}; \gamma\cdot\gamma^* \in 2\pi\mathbf{Z}, \forall\gamma \in \Gamma\}$ be its dual lattice. The fundamental domains of Γ and Γ^* are

$$\mathbf{E} = \mathbf{R}^{n-1}/\Gamma, \quad \mathbf{E}^* = \mathbf{R}^{n-1}/\Gamma^*. \tag{4.1}$$

By the natural identification of Γ with the discrete translation group acting on $\mathbf{H}^n$, we introduce the hyperbolic manifold $\mathcal{M} = \Gamma\backslash\mathbf{H}^n$, whose Laplace-Beltrami operator is

$$\Delta_{\mathcal{M}} = x_n^2\partial_n^2 - (n-2)x_n\partial_n - x_n^2 P_0, \tag{4.2}$$

where $P_0 = -\Delta_x$ is the Laplacian on $\mathbf{E}$ with periodic boundary condition. As in §3, the change of variable $y = \log x_n$ transforms $-\Delta_{\mathcal{M}} - (n-1)^2/4$ to

$$H_0 = -\partial_y^2 + e^{2y}P_0 \quad \text{in} \quad L^2(\mathbf{E}\times\mathbf{R}; dxdy). \tag{4.3}$$

For $\theta \in \mathbf{E}^*$, we introduce the following operator on $\mathbf{E} \times \mathbf{R}$

$$H_0(\theta) = -\partial_y^2 + e^{2y} P_0(\theta), \quad P_0(\theta) = (-i\partial_x + \theta)^2, \tag{4.4}$$

where $\partial_x = \nabla_x$ and the periodic boundary condition is imposed on $P_0(\theta)$.

Let $H(\theta) = H_0(\theta) + V$. We assume that $V(x,y)$ is a real function and is split into two parts $V = V_1 + V_2$, where $V_1(x,y)$ is compactly supported and

$$V_1(x,y) \in L^p(\mathbf{E} \times \mathbf{R}), \tag{4.5}$$

with $p = 2$ $(n \leq 3)$, $p > n/2$ $(n \geq 4)$, and where V_2 verifies

$$|V(x,y)| \leq C(1 + |y|)^{-1-\epsilon} \tag{4.6}$$

for some constants $C, \epsilon > 0$.

$H(\theta)|_{C_0^\infty(\Omega)}$ is essentially self-adjoint, whose self-adjoint extension is denoted by $H(\theta)$ again. By the reasoning to be given in 4.4, we call $H(\theta)$ *Floquet operator*. Let $R(z;\theta) = (H(\theta) - z)^{-1}$, and $R_0(z;\theta) = (H_0(\theta) - z)^{-1}$. For $s \in \mathbf{R}$, let $L^{2,s}$ be the weighted Hilbert space equipped with the norm

$$\|u\|_s^2 = \int_\Omega (1 + |y|)^{2s} |u(x,y)|^2 dxdy < \infty. \tag{4.7}$$

We shall prove the following theorem in §8.

Theorem 4.1. *(1)* $\sigma_e(H(\theta)) = [0, \infty)$.
(2) $\sigma_p(H(\theta)) \cap (0, \infty) = \emptyset$.
(3) For $s > 1/2$ *and* $\lambda > 0$*, there exist the strong limits*

$$\text{s} - \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon; \theta) =: R(\lambda \pm i0; \theta) \in \mathbf{B}(L^{2,s}; L^{2,-s}).$$

4.2 *Spectral representation for* $H_0(\theta)$. We put

$$\hat{f}(\gamma^*, y) = \int_{\mathbf{E}} e^{-i\gamma^* \cdot x} f(x,y) dx. \tag{4.8}$$

Then as in (3.10), we have for $f \in C_0^\infty(\Omega)$

$$R_0(\lambda \pm i0; \theta) f = \frac{1}{|\mathbf{E}|} \sum_{\gamma^* \in \Gamma^*} e^{i\gamma^* \cdot x} (L_0(|\gamma^* + \theta|) - \lambda \mp i0)^{-1} \hat{f}(\gamma^*, \cdot). \tag{4.9}$$

Definition 4.2. We define

$$(\mathcal{F}_0^{(\pm)}(\theta) f)(w, \lambda) = \sum_{\gamma^* \in \Gamma^*} e^{i\gamma^* \cdot w} |\gamma^* + \theta|^{\mp i\sqrt{\lambda}} (K(|\gamma^* + \theta|) \hat{f}(\gamma^*, \cdot))(\sqrt{\lambda}).$$

Then by (2.28) and (4.9), we have

$$\frac{1}{2\pi i}([R_0(\lambda + i0; \theta) - R_0(\lambda - i0; \theta)] f, f) = \sinh(\sqrt{\lambda}\pi) \int_{\mathbf{E}} |\mathcal{F}_0^{(\pm)}(\theta) f(w, \lambda)|^2 dw. \tag{4.10}$$

One can show the following theorem as in §3.

Theorem 4.3. *Assume* $\theta \notin \Gamma^*$.
(1) $\mathcal{F}_0^{(\pm)}(\theta)$ *is unitary*

$$\mathcal{F}_0^{(\pm)}(\theta) : L^2(\Omega; dxdy) \to L^2(\mathbf{E} \times (0, \infty); \sinh(\sqrt{\lambda}\pi) dw d\lambda)$$

and diagonalizes $H_0(\theta)$ *:*

$$(\mathcal{F}_0^{(\pm)}(\theta) H_0(\theta) f)(w, \lambda) = \lambda (\mathcal{F}_0^{(\pm)}(\theta) f)(w, \lambda).$$

(2) Let $f \in L^{2,s}, s > 1/2$. Then we have

$$\lim_{y\to-\infty} e^{\pm i\sqrt{\lambda}y}(R_0(\lambda \pm i0;\theta)f)(\cdot,y) = C_\pm(\lambda)(\mathcal{F}_0^{(\pm)}(\theta)f)(\cdot,\lambda) \quad \text{in} \quad L^2(\mathbf{E}),$$

$$C_\pm(\lambda) = \frac{2^{\pm i\sqrt{\lambda}}\pi}{\Gamma(1\mp i\sqrt{\lambda})|\mathbf{E}|}.$$

Proof. We give a sketch of the proof of (2). Using (2.18) we easily see

$$\sup_y \|(R_0(\lambda + i0;\theta)f(\cdot,y)\|_{L^2(\mathbf{E})} \leq C\|f\|_s. \tag{4.11}$$

To prove (2), therefore, we have only to prove it for $f \in C_0^\infty(\mathbf{E}\times\mathbf{R})$. Approximate f by a finite sum of the Fourier series and apply (2.25). ♢

4.3 *Scattering amplitude.* We define for $\lambda > 0$ and $f \in C_0^\infty(\mathbf{E}\times\mathbf{R})$

$$(\mathcal{F}_0^{(\pm)}(\lambda;\theta)f)(w) = (\mathcal{F}_0^{(\pm)}(\theta)f)(w,\lambda). \tag{4.12}$$

Then by Theorem 4.1 and (4.10)

$$\mathcal{F}_0^{(\pm)}(\lambda;\theta) \in \mathbf{B}(L^{2,s};L^2(\mathbf{E})), \quad s > 1/2. \tag{4.13}$$

We put

$$\mathcal{F}^{(\pm)}(\lambda;\theta) = \mathcal{F}_0^{(\pm)}(\lambda;\theta)(1 - VR(\lambda \pm i0;\theta)). \tag{4.14}$$

By Theorem 4.3 (2), we have the following theorem.

Theorem 4.4. *Let $f \in L^{2,s}, s > 1/2$. Then we have*

$$\lim_{y\to-\infty} e^{\pm i\sqrt{\lambda}y}(R(\lambda \pm i0;\theta)f)(\cdot,y) = C_\pm(\lambda)\mathcal{F}^{(\pm)}(\lambda;\theta)f \quad \text{in} \quad L^2(\mathbf{E}),$$

$$C_\pm(\lambda) = \frac{2^{\pm i\sqrt{\lambda}}\pi}{\Gamma(1\mp i\sqrt{\lambda})|\mathbf{E}|}.$$

As in §3, we define (geometric) scattering amplitudes as follows :

$$A(\lambda;\theta) = \mathcal{F}_0^{(+)}(\lambda;\theta)\,(V - VR(\lambda + i0;\theta)V)\,\mathcal{F}_0^{(+)}(\lambda;\theta)^*, \tag{4.15}$$

$$\widetilde{A}(\lambda;\theta) = A(\lambda;\theta)Q(\lambda;\theta)^*, \tag{4.16}$$

$$Q(\lambda;\theta)\varphi = \frac{1}{|\mathbf{E}|}\sum_{\gamma^*\in\Gamma^*} e^{i\gamma^*\cdot x}|\gamma^* + \theta|^{2i\sqrt{\lambda}}\hat{\varphi}(\gamma^*). \tag{4.17}$$

Since $\mathcal{F}_0^{(-)}(\lambda;\theta) = Q(\lambda;\theta)\mathcal{F}_0^{(+)}(\lambda;\theta)$, the following theorem is proved in the same way as Theorem 3.7.

Theorem 4.5. *The geometric scattering amplitude $\widetilde{A}(\lambda;\theta)$ has the following expression*

$$\widetilde{A}(\lambda;\theta) = \mathcal{F}_0^{(+)}(\lambda;\theta)(V - VR(\lambda + i0;\theta)V)\mathcal{F}_0^{(-)}(\lambda;\theta)^*. \tag{4.18}$$

For $\varphi \in L^2(\mathbf{E})$, we have as $y \to -\infty$

$$\mathcal{F}^{(-)}(\lambda;\theta)^*\varphi - \mathcal{F}_0^{(-)}(\lambda;\theta)^*\varphi \sim -C_+(\lambda)e^{-i\sqrt{\lambda}y}\widetilde{A}(\lambda;\theta)\varphi \quad \text{in} \quad L^2(\mathbf{E}) \tag{4.19}$$

$$C_+(\lambda) = \frac{2^{i\sqrt{\lambda}}\pi}{\Gamma(1 - i\sqrt{\lambda})|\mathbf{E}|}.$$

4.4 *Periodic Schrödinger operator.* Suppose we are given a Γ-periodic Schrödinger operator $H_0 + V$ on $\mathbf{H}^n$. A natural way to study its spectral structure is to investigate $H(\theta), \theta \in \mathbf{E}^*$, on $\Gamma\backslash\mathbf{H}^n$. In fact by the Floquet (or Bloch) theory it can be shown that (see e.g. [29] Vol 4, p. 279) there exists a unitary operator $U : L^2(\mathbf{H}^n) \to L^2(\mathbf{E}\times\mathbf{R}\times\mathbf{E}^*)$ such that

$$\Big(U(H_0+V)U^{-1}f\Big)(x,y,\theta) = \Big((H_0(\theta)+V)f(\cdot,\theta)\Big)(x,y) \tag{4.20}$$

We do not use this property in this paper, however.

4.5 *The case* $\theta = 0$. When $\theta = 0$, the spectral representation of $H_0(0)$ should be modified at the mode $\gamma^* = 0$. We put

$$F_{0,+\infty}(\lambda)f = (4\pi\sqrt{\lambda})^{-1/2}\int_{-\infty}^{\infty} e^{-i\sqrt{\lambda}y}\hat{f}(0,y)dy, \tag{4.21}$$

$$F_{0,-\infty}(\lambda)f = (4\pi\sqrt{\lambda})^{-1/2}\int_{-\infty}^{\infty} e^{i\sqrt{\lambda}y}\hat{f}(0,y)dy. \tag{4.22}$$

Recall that $\hat{f}(\gamma^*, y)$ is defined by (4.8). We also put

$$\mathcal{F}_0^{(\pm)}(\lambda;0)f(w) = \frac{1}{|\mathbf{E}|}\sum_{0\neq\gamma^*\in\Gamma^*} e^{i\gamma^*\cdot w}|\gamma^*|^{\mp i\sqrt{\lambda}}(K(|\gamma^*|)\hat{f}(\gamma^*,\cdot))(\sqrt{\lambda}). \tag{4.23}$$

Then we have

$$\begin{aligned}&\frac{1}{2\pi i}([R_0(\lambda+i0;0) - R_0(\lambda-i0;0)]f,f)\\ =\ &|F_{0,+\infty}(\lambda)f|^2 + |F_{0,-\infty}(\lambda)f|^2 + \sinh(\sqrt{\lambda}\pi)\|\mathcal{F}_0^{(+)}(\lambda;0)f\|^2_{L^2(\mathbf{E})}\end{aligned} \tag{4.24}$$

As in Theorem 4.3, the following theorem holds.

Theorem 4.6. *(1) The mapping*

$$\begin{aligned}L^2(\mathbf{E}\times\mathbf{R}, dxdy) \ni f \to (F_{0,+\infty}(\lambda)f, F_{0,-\infty}(\lambda)f, \mathcal{F}_0^{(\pm)}(\lambda;0)f)\\ \in \mathbf{C}\times\mathbf{C}\times L^2(\mathbf{E}\times(0,\infty); \sinh(\sqrt{\lambda}\pi)dwd\lambda)\end{aligned}$$

is unitary and diagonalizes H_0.
(1) Let $f \in L^{2,s}, s > 1/2$. *Then we have*

$$\lim_{y\to-\infty} e^{\pm i\sqrt{\lambda}y}(R_0(\lambda\pm i0;0)f)(\cdot,y) = C_0(\lambda)F_{0,-\infty}(\lambda)f + C_\pm(\lambda)\mathcal{F}_0^{(\pm)}(\lambda;0)f,$$

$$\lim_{y\to\infty} e^{\mp i\sqrt{\lambda}y}(R_0(\lambda\pm i0;0)f)(\cdot,y) = C_0(\lambda)F_{0,+\infty}(\lambda)f,$$

where $C_0(\lambda) = \pi^{1/2}\lambda^{-1/4}$ *and* $C_\pm(\lambda)$ *is from Theorem 4.3 (2).*

As in (4.14), the spectral representation for H is defined by

$$F_{+\infty}^{(\pm)}(\lambda) = F_{0,+\infty}(\lambda)(1 - VR(\lambda\pm i0;0)), \tag{4.25}$$

$$F_{-\infty}^{(\pm)}(\lambda) = F_{0,-\infty}(\lambda)(1 - VR(\lambda\pm i0;0)), \tag{4.26}$$

$$\mathcal{F}^{(\pm)}(\lambda;0) = \mathcal{F}_0^{(\pm)}(\lambda;0)(1 - VR(\lambda\pm i0;0)). \tag{4.27}$$

Then the following theorem holds.

Theorem 4.7. *For $f \in L^{2,s}$, $s > 1/2$, we have*

$$\lim_{y\to-\infty} e^{\pm i\sqrt{\lambda}y}(R(\lambda \pm i0;0)f)(\cdot,y) = C_0(\lambda)F^{(\pm)}_{-\infty}(\lambda)f + C_\pm(\lambda)\mathcal{F}^{(\pm)}(\lambda;0)f,$$

$$\lim_{y\to\infty} e^{\mp i\sqrt{\lambda}y}(R(\lambda \pm i0;0)f)(\cdot,y) = C_0(\lambda)F^{(\pm)}_{+\infty}(\lambda)f,$$

$C_0(\lambda)$ and $C_\pm(\lambda)$ being given in Theorem 4.6.

We finally define

$$\mathbf{F}^{(\pm)}_0(\lambda;0) = (F_{0,+\infty}(\lambda), F_{0,-\infty}(\lambda), \mathcal{F}^{(\pm)}_0(\lambda;0)), \tag{4.28}$$

$$\mathbf{F}^{(\pm)}(\lambda;0) = (F^{(\pm)}_{+\infty}(\lambda), F^{(\pm)}_{-\infty}(\lambda), \mathcal{F}^{(\pm)}(\lambda;0)), \tag{4.29}$$

and define the (geometric) scattering amplitude as follows :

$$\mathbf{A}(\lambda;0) = \mathbf{F}^{(+)}_0(\lambda;0)(V - VR(\lambda + i0;0)V)\mathbf{F}^{(+)}_0(\lambda;0)^*, \tag{4.30}$$

$$\widetilde{\mathbf{A}}(\lambda;0) = \mathbf{F}^{(+)}_0(\lambda;0)(V - VR(\lambda + i0;0)V)\mathbf{F}^{(-)}_0(\lambda;0)^*. \tag{4.31}$$

As in (4.16), $\widetilde{\mathbf{A}}(\lambda;0)$ is related with $\mathbf{A}(\lambda;0)$ in the following way :

$$\widetilde{\mathbf{A}}(\lambda;0) = \mathbf{A}(\lambda;0)\mathbf{Q}(\lambda;0)^*, \tag{4.32}$$

$$\mathbf{Q}(\lambda;0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Q(\lambda;0) \end{pmatrix}, \tag{4.33}$$

$$Q(\lambda;0)\varphi = \frac{1}{|\mathbf{E}|}\sum_{0\neq\gamma^*\in\Gamma^*} e^{i\gamma^*\cdot x}|\gamma^*|^{2i\sqrt{\lambda}}\hat{\varphi}(\gamma^*). \tag{4.34}$$

We also have the follwing theorem.

Theorem 4.8. *For $\Phi \in \mathbf{C} \times \mathbf{C} \times L^2(\mathbf{E})$, we have as $y \to \infty$,*

$$(\mathbf{F}^{(-)}(\lambda;0)^*\Phi)_1 - (\mathbf{F}^{(-)}_0(\lambda;0)^*\Phi)_1 \sim C_1(\lambda)e^{i\sqrt{\lambda}y}(\widetilde{\mathbf{A}}(\lambda;0)\Phi)_1,$$

and as $y \to -\infty$

$$(\mathbf{F}^{(-)}(\lambda;0)^*\Phi)_{2,3} - (\mathbf{F}^{(-)}_0(\lambda;0)^*\Phi)_{2,3} \sim C_{2,3}(\lambda)e^{-i\sqrt{\lambda}y}(\widetilde{\mathbf{A}}(\lambda;0)\Phi)_{2,3}.$$

Here for $\Psi = (a_+, a_-, \psi) \in \mathbf{C} \times \mathbf{C} \times L^2(\mathbf{E})$, $(\Psi)_1 = a_+$ and $(\Psi)_{2,3} = (a_-, \psi)$.

4.6 *Scattering amplitude at the cusp.* We shall observe the behavior of the resolvent as $y \to \infty$ and introduce the geometric scattering amplitude in the cusp neighborhood at infinity. In the sequel, the subscript c means the cusp. We treat the case $\theta = 0$.

We put for $\zeta > 0$

$$(I(\zeta)f)(\sigma) = \int_{-\infty}^{\infty} I_{i\sigma}(\zeta e^y)f(y)dy. \tag{4.35}$$

Lemma 4.9. *Let $s > 1/2$ and $\sigma \in \mathbf{R}$. Then there exists a constant $C > 0$ depending only on s and σ such that*

$$|(I(\zeta)f)(\sigma)| \leq Ce^{\zeta^2/2}\left(\int_{-\infty}^{\infty}\langle y\rangle^{2s}\exp(e^{2y})|f(y)|^2dy\right)^{1/2}, \quad \forall\zeta > 0.$$

Proof. By (2.5), we have $|I_{i\sigma}(z)| \leq Ce^z, z > 0$. Therefore using the inequality $\zeta e^y \leq (\zeta^2 + e^{2y})/2$, we get

$$\begin{aligned} |(I(\zeta)f)(\sigma)| &\leq Ce^{\zeta^2/2}\int_{-\infty}^{\infty} \exp(\frac{e^{2y}}{2})|f(y)|dy \\ &\leq Ce^{\zeta^2/2}\left(\int_{-\infty}^{\infty} \langle y\rangle^{2s}\exp(e^{2y})|f(y)|^2 dy\right)^{1/2}. \qquad \diamondsuit \end{aligned}$$

We use the notation

$$\langle e^{i\gamma^*\cdot x}, f\rangle = \int_{\mathbf{E}} e^{i\gamma^*\cdot x} f(x,y)dx = \hat{f}(-\gamma^*, y). \tag{4.36}$$

Then by virtue of (2.31), we have as $y \to \infty$

$$\langle e^{-i\gamma^*\cdot x}, R_0(\lambda \pm i0; 0)f\rangle \sim \sqrt{\frac{\pi}{2|\gamma^*|}}\exp(-\frac{y}{2} - |\gamma^*|e^y)\big(I(|\gamma^*|)\hat{f}(\gamma^*,\cdot)\big)(\mp\sqrt{\lambda}), \tag{4.37}$$

for $\gamma^* \neq 0$, and for $\gamma^* = 0$

$$\langle 1, R_0(\lambda \pm i0; 0)f\rangle \sim \mp i\sqrt{\frac{\pi}{2\lambda}}e^{\pm i\sqrt{\lambda}y}F_{0,\pm\infty}(\lambda)\hat{f}(0,\cdot). \tag{4.38}$$

For $s \in \mathbf{R}$, let $L^{2,s}_{exp}$ be the function space defined by

$$L^{2,s}_{exp} \ni f \Longleftrightarrow \|f\|_{s,exp} = \left(\int_{\mathbf{E}\times\mathbf{R}} \langle y\rangle^{2s}\exp(e^{2y})|f(x,y)|^2 dxdy\right)^{1/2} < \infty.$$

Lemma 4.9 implies that for $s > 1/2$

$$|(I(|\gamma^*|)\hat{f}(\gamma^*,\cdot))(\mp\sqrt{\lambda})| \leq Ce^{|\gamma^*|^2/2}\|f\|_{s,exp} \tag{4.39}$$

with C independent of $0 \neq \gamma^* \in \Gamma^*$.

In the following we shall assume that $V(x,y)$ is compactly supported. We define for $f \in L^{2,s}_{exp}, s > 1/2$,

$$\mathcal{F}^{(\pm)}_{0c\gamma^*}(\lambda)f = \begin{cases} (I(|\gamma^*|)\hat{f}(\gamma^*,\cdot))(\mp\sqrt{\lambda}), & \gamma^* \neq 0 \\ F_{0,\pm\infty}(\lambda)\hat{f}(0,\cdot), & \gamma^* = 0 \end{cases} \tag{4.40}$$

$$\mathcal{F}^{(\pm)}_{c\gamma^*}(\lambda) = \mathcal{F}^{(\pm)}_{0c\gamma^*}(\lambda)(1 - VR(\lambda \pm i0; 0)). \tag{4.41}$$

We then have the following theorem.

Theorem 4.10. *If $f \in L^{2,s}_{exp}, s > 1/2$, we have as $y \to \infty$*

$$\langle e^{-i\gamma^*\cdot x}, R(\lambda \pm i0; 0)f\rangle \sim \sqrt{\frac{\pi}{2|\gamma^*|}}\exp(-\frac{y}{2} - |\gamma^*|e^y)\mathcal{F}^{(\pm)}_{c\gamma^*}(\lambda)f, \quad \gamma^* \neq 0$$

$$\langle 1, R(\lambda \pm i0; 0)f\rangle \sim \mp i\sqrt{\frac{\pi}{2\lambda}}e^{\pm i\sqrt{\lambda}y}\mathcal{F}^{(\pm)}_{c\gamma^*}(\lambda)f, \quad \gamma^* = 0.$$

We put

$$\mu_{\gamma^*} = (1 + |\gamma^*|)^{-n}e^{-|\gamma^*|^2/2} \tag{4.42}$$

and define on $L^{2,s}_{exp}, s > 1/2$,

$$\mathcal{F}^{(\pm)}_{0c}(\lambda) = \sum_{\gamma^*\in\Gamma^*}\mu_{\gamma^*}e^{i\gamma^*\cdot x}\mathcal{F}^{(\pm)}_{0c\gamma^*}(\lambda), \tag{4.43}$$

$$\mathcal{F}^{(\pm)}_{c}(\lambda) = \sum_{\gamma^*\in\Gamma^*}\mu_{\gamma^*}e^{i\gamma^*\cdot x}\mathcal{F}^{(\pm)}_{c\gamma^*}(\lambda). \tag{4.44}$$

By (4.39), we have

$$\mathcal{F}_{0c}^{(\pm)}(\lambda), \mathcal{F}_{c}^{(\pm)}(\lambda) \in \mathbf{B}(L^{2,s}_{exp}; L^2(\mathbf{E})), \quad s > 1/2. \tag{4.45}$$

The *geometric scattering amplitude at the cusp* is then defined by

$$\widetilde{A}_c(\lambda) = \mathcal{F}_{0c}^{(+)}(\lambda)(V - VR(\lambda + i0; 0)V)\mathcal{F}_{0c}^{(-)}(\lambda)^*. \tag{4.46}$$

We also put

$$\begin{aligned} \widetilde{A}_{c\gamma^*}(\lambda) &= \langle e^{-i\gamma^*\cdot x}, \widetilde{A}_c(\lambda)\cdot\rangle \\ &= \mu_{\gamma^*}\mathcal{F}_{0c\gamma^*}^{(+)}(\lambda)^*(V - VR(\lambda + i0; 0)V)\mathcal{F}_{0c}^{(-)}(\lambda)^* \end{aligned} \tag{4.47}$$

Note that for $\varphi \in L^2(\mathbf{E})$, $\mathcal{F}_c^{(\pm)}(\lambda)^*\varphi$ satisfies the Schrödinger equation $(H - \lambda)u = 0$, and it grows up at the cusp. In fact for $\varphi = e^{i\gamma^*\cdot x}/\gamma^*|\mathbf{E}|$, $\mathcal{F}_{0c\gamma^*}^{(\pm)}(\lambda)^*\varphi = I_{\pm i\sqrt{\lambda}}(|\gamma^*|e^y)$, which behaves like $e^z/\sqrt{z}, z = |\gamma^*|e^y$, as $y \to \infty$. The following theorem, which follows easily from Theorem 4.10, shows that $\widetilde{A}_c(\lambda)$ is obtained by observing this growing solution at the cusp.

Theorem 4.11. *Let $\varphi \in L^2(\mathbf{E})$. For $\gamma^* \neq 0$, we have as $y \to \infty$*

$$\langle e^{-i\gamma^*\cdot x}, \mathcal{F}_c^{(-)}(\lambda)^*\varphi - \mathcal{F}_{0c}^{(-)}(\lambda)^*\varphi\rangle \sim -\sqrt{\frac{\pi}{2|\gamma^*|}}\exp(-\frac{y}{2} - |\gamma^*|e^y)\widetilde{A}_{c\gamma^*}(\lambda)\varphi,$$

For $\gamma^* = 0$ we have as $y \to \infty$

$$\langle 1, \mathcal{F}_c^{(-)}(\lambda)^*\varphi - \mathcal{F}_{0c}^{(-)}(\lambda)^*\varphi\rangle \sim -i\sqrt{\frac{\pi}{2\lambda}}e^{-i\sqrt{\lambda}y}\widetilde{A}_{c\gamma^*}(\lambda)\varphi.$$

5. Reconstruction from scattering amplitudes

We consider the inverse scattering problem on $\Gamma\backslash\mathbf{H}^n$. We use the same notation as in §4.

5.1 *Analytic continuation of the resolvent.* We take the fundamental domain $\mathbf{E}$ in such a way that it contains the origin in its interior. Let

$$\Gamma_{unit} = \left\{\frac{\gamma}{|\gamma|}; 0 \neq \gamma \in \Gamma\right\}. \tag{5.1}$$

For $\alpha = \gamma/|\gamma| \in \Gamma_{unit}$ (given α, we choose the smallest γ), we take a, b such that

$$-\frac{2\pi}{|\gamma|} < -a < -b < 0 \tag{5.2}$$

and put

$$D_\alpha = \{z \in \mathbf{C}; -a < \mathrm{Re}\, z < -b,\ \mathrm{Im}\, z > 0\}. \tag{5.3}$$

For $\gamma^* \in \Gamma^*$, we define

$$\zeta(\gamma^*, z) = \sqrt{(\gamma^* + z\alpha)^2}. \tag{5.4}$$

Here we take the branch of $\sqrt{\cdot}$ such that $\mathrm{Re}\sqrt{\cdot} \geq 0$.

Lemma 5.1. *$\zeta(\gamma^*, z)$ is analytic with respect to $z \in D_\alpha$ and there exists a constant $C > 0$ such that*

$$|\zeta(\gamma^*, z)| \geq C\sqrt{\mathrm{Im}\, z}, \quad \forall\gamma^* \in \Gamma^*, \quad z \in D_\alpha.$$

Proof. For $\gamma^* \in \Gamma^*$ such that $\gamma^* \cdot \alpha \neq 0$, $|\gamma^* \cdot \alpha| \geq 2\pi/|\gamma|$. Therefore there exists $C > 0$ such that

$$|\mathrm{Im}\,(\gamma^* + z\alpha)^2| \geq C\mathrm{Im}\, z, \quad z \in D_\alpha.$$

This also holds for $\gamma^* \in \Gamma^*$ such that $\gamma^* \cdot \alpha = 0$. This proves the lemma. ♢

We put

$$R_0(\lambda + i0; z\alpha)f = \frac{1}{|\mathbf{E}|}\sum_{\gamma^*\in\Gamma^*} e^{i\gamma^*\cdot x} G(-\sqrt{\lambda}, \zeta(\gamma^*, z))\hat{f}(\gamma^*, \cdot), \tag{5.5}$$

where $G(\sigma, \lambda)$ is defined by (2.20).

Fix $s > 1/2$. We define the function spaces $X_\pm$ by

$$u(x, y) \in X_\pm \Longleftrightarrow \|u\|^2_{X_\pm} = \int_{\mathbf{E}\times\mathbf{R}} (1 + |y|)^{\pm 2s} e^{\mp y}|u(x, y)|^2 dxdy < \infty. \tag{5.6}$$

Lemma 5.2. *(1) Let $s > 1/2$. Then for $f \in L^{2,s}$, $R_0(\lambda + i0; z\alpha)f$ is an $L^{2,-s}$-valued analytic finction of $z \in D_\alpha$ and as $z \to t \in (-a, -b)$, $R_0(\lambda + i0; z\alpha)f \to R_0(\lambda + i0; t\alpha)f$ in $L^{2,-s}$.*
(2) There exists a constant $C > 0$ such that

$$\|R_0(\lambda + i0; z\alpha)f\|_{-s} \leq C\|f\|_s, \quad \forall z \in D_\alpha.$$

(3) For a constant $C > 0$

$$\|R_0(\lambda + i0; z\alpha)f\|_{X_-} \leq \frac{C}{\sqrt{\mathrm{Im}\, z}}\|f\|_{X_+}, \quad \forall z \in D_\alpha.$$

Proof. For $f, g \in C_0^\infty(\Omega)$, we have

$$(R_0(\lambda+i0; z\alpha)f, g) = \frac{1}{|\mathbf{E}|}\sum_{\gamma^*\in\Gamma^*}\int\int_{\mathbf{R}^2} G(y, y'; -\sqrt{\lambda}, \zeta(\gamma^*, z))\hat{f}(\gamma^*, y')\overline{\hat{g}(\gamma^*, y)}dy'dy. \tag{5.7}$$

Using (2.18), we have

$$\begin{aligned} |(R_0(\lambda + i0; \theta)f, g)| &\leq C\int\int\sum_{\gamma^*\in\Gamma^*}|\hat{f}(\gamma^*, y')| \cdot |\hat{g}(\gamma^*, y)|dy'dy \\ &\leq C\int\|f(\cdot, y')\|_{L^2(\mathbf{E})}dy'\int\|g(\cdot, y)\|_{L^2(\mathbf{E})}dy \\ &\leq C\|f\|_s \cdot \|g\|_s, \end{aligned}$$

which proves (2). The assertion (1) follows from Lemma 5.1.

We also have by using (2.19), (5.7) and Lemma 5.1

$$\begin{aligned} |(R_0(\lambda + i0; z\alpha)f, g)| &\leq \frac{C}{\sqrt{\mathrm{Im}\, z}}\int\|f(\cdot, y')\|_{L^2(\mathbf{E})}e^{-y'/2}dy'\int\|g(\cdot, y)\|_{L^2(\mathbf{E})}e^{-y/2}dy \\ &\leq \frac{C}{\sqrt{\mathrm{Im}\, z}}\|f\|_{X_+} \cdot \|g\|_{X_+}, \end{aligned}$$

which implies (3). ♢

We turn to the analytic continuation of $R(\lambda + i0; t\alpha)$ with respect to t. We first consider the case $n \geq 3$. We assume that

$$|V(x, y)| \leq Ce^{-|y|}(1 + |y|)^{-\rho} \tag{5.8}$$

for some $\rho > 1$. We take $1/2 < s < \rho/2$ and define the spaces $Y_\pm$ by

$$u(x,y) \in Y_\pm \Longleftrightarrow \|u\|^2_{Y_\pm} = \int_{\mathbf{E}\times\mathbf{R}} (1+|y|)^{\pm 2s} e^{\pm|y|} |u(x,y)|^2 dxdy < \infty. \tag{5.9}$$

Lemma 5.3. *Let $n \geq 3$ and assume (5.8).*
(1) $VR_0(\lambda + i0; z\alpha)$ is a $\mathbf{B}(Y_+;Y_+)$-valued analytic function of $z \in D_\alpha$.
(2) For each $z \in D_\alpha$, $VR_0(\lambda + i0; z\alpha)$ is compact on Y_+.
(3) There exists a constant $C_0 > 0$ such that

$$-1 \notin \sigma(VR_0(\lambda + i0; z\alpha)) \quad \text{if} \quad \text{Im}\, z > C_0, \ z \in D_\alpha.$$

(4) $R(\lambda + i0; z\alpha)$ defined by

$$R(\lambda + i0; z\alpha) = R_0(\lambda + i0; z\alpha)(1 + VR_0(\lambda + i0; z\alpha))^{-1}$$

is a $\mathbf{B}(Y_+;Y_-)$-valued meromorphic function of $z \in D_\alpha$. There exists a set of measure 0, $\mathcal{E}$, in $(-a,-b)$ such that as $z \to t \in (-a,-b) \setminus \mathcal{E}$, $R(\lambda + i0; z\alpha)$ has a boundary value, which coincides with $R(\lambda + i0; t\alpha) = (H(t\alpha) - \lambda - i0)^{-1}$.
(5) There exists a constant $C > 0$ such that

$$\|R(\lambda + i0; z\alpha)f\|_{Y_-} \leq \frac{C}{\sqrt{\text{Im}\, z}} \|f\|_{Y_+}, \quad z \in D_\alpha, \quad \text{Im}\, z > C.$$

Proof. The first three assertions follow from Lemma 5.2. To prove (2), note that $R_0(\lambda - i0; \overline{z}\alpha)V$ is compact on Y_-. By the following Lemma 5.4, $R(\lambda + i0; z\alpha)$ is meromorphic in D_α and $R(\lambda + i0; z\alpha)$ tends to some $S(t)$ as $z \to t \in (-a,-b) \setminus \mathcal{E}$, $\mathcal{E}$ being a null set. Then

$$S(t)^* = R_0(\lambda - i0; t\alpha) - R_0(\lambda - i0; t\alpha)VS(t)^*. \tag{5.10}$$

As will be explained in §8, for $g \in L^{2,s}$, $u = R_0(\lambda - i0; \theta)g$ is an incoming solution of $(H_0(\theta) - \lambda)u = g$. Namely

$$u \in L^{2,-s}, \quad F(\pm y > 0)(i\partial_y \mp \sqrt{\lambda})u \in L^{2,-\alpha}$$

for some $0 < \alpha < 1/2 < s$. Therefore, for $f \in Y_+$, $S(t)^* f$ is an incoming solution of the equation $(H_0(t\alpha) + V - \lambda)u = f$. Moreover, $R(\lambda - i0; t\alpha)f$ is also an incoming solution and the incoming solution is unique. Therefore, $S(t)^* f = R(\lambda - i0; t\alpha)f$, hence $S(t) = R(\lambda + i0; t\alpha)$.

The assertion (5) follows from Lemma 5.2 (3). ♢

Lemma 5.4. *Let $\mathcal{H}$ be a Hilbert space. Let D be a unit disc in $\mathbf{C}$ and $f(z)$ a $\mathbf{B}(\mathcal{H})$-valued analytic function of $z \in D$, which is continuous on $\overline{D}$. Assume that $f(z)$ is compact for each $z \in \overline{D}$. Let $\mathcal{E} = \{z \in \overline{D}; 1 \in \sigma(f(z))\}$, and suppose that $\mathcal{E} \neq \overline{D}$. Then $\mathcal{E} \cap D$ is discrete, $\mathcal{E} \cap \partial D$ is a closed 1-dimensional null set and for $z_0 \in \partial D \setminus \mathcal{E}$, $(1 - f(z))^{-1} \to (1 - f(z_0))^{-1}$ as $D \setminus \mathcal{E} \ni z \to z_0$.*

This lemma follows from the analytic Fredholm theorem (see e.g. [29] Vol. 1, p. 201) and the well-known Fatou-Riesz theorem on the boundary value of analytic functions (see e.g. [38] p. 135).

5.2 *Estimates in 2-dimension.* When $n = 2$, one can allow L^2-local singularities for the potential.

Lemma 5.5. *Let $n = 2$ and assume that*

$$e^{-y/2}V(x,y) \in L^2(\mathbf{E}\times\mathbf{R}). \tag{5.11}$$

Then we have for $z \in D_\alpha$ and $s > 1/2$

$$\|VR_0(\lambda+i0;z\alpha)f\|_{L^2(\mathbf{E}\times\mathbf{R})} \leq \frac{C}{\sqrt{\mathrm{Im}\, z}}\|e^{-y/2}V\|_{L^2(\mathbf{E}\times\mathbf{R})}\|\langle y\rangle^s e^{-y/2}f\|_{L^2(\mathbf{E}\times\mathbf{R})}.$$

Proof. One can assume without loss of generality that $\Gamma^* = \mathbf{Z}$. Letting $\mathrm{Im}\, z = \tau$, we have

$$|\zeta(n,z)| \geq C(|n|+\tau)$$

for a constant $C > 0$. Then $u(x,y) = R_0(\lambda+i0;z\alpha)f$ is written as

$$u(x,y) = \sum_n a(n,y)e^{inx},$$

$$|a(n,y)| \leq \frac{C}{|n|+\tau}e^{-y/2}\int_{-\infty}^{\infty} e^{-y'/2}|\hat{f}(n,y')|dy'.$$

By the Schwarz and the Parseval (in)equalities

$$|u(x,y)|^2 \leq C\left(\sum_{n\in\mathbf{Z}}(|n|+\tau)^{-2}\right)e^{-y}\left(\int_{\mathbf{E}\times\mathbf{R}}\langle y'\rangle^{2s}e^{-y'}|f(x,y')|^2dxdy'\right),$$

which implies the lemma. $\diamondsuit$

Lemma 5.6. *When $n = 2$, Lemma 5.3 holds under the assumption that $V = V_1 + V_2$, where $V_1 \in L^2(\mathbf{E}\times\mathbf{R})$ and is compactly supported, V_2 satisfies (5.8).*

Proof. Pick $\chi \in C_0^\infty(\mathbf{E}\times\mathbf{R})$ such that $\chi = 1$ on supp V_1. Making use of the formula

$$V_1R_0(\lambda+i0;z\alpha) = V_1(-\Delta+1)^{-1}(-\Delta+1)\chi R_0(\lambda+i0;z\alpha),$$

one can see that $\|V_1R_0(\lambda+i0;z\alpha)\|_{\mathbf{B}(Y_+;Y_+)}$ is locally bounded on $\overline{D_\alpha}$. For $f,g \in C_0^\infty(\mathbf{E}\times\mathbf{R})$, $(R_0(\lambda+i0;z\alpha)f, Vg)$ is analytic with respect to $z \in D_\alpha$. This proves the analyticity of $VR_0(\lambda+i0;z\alpha)$ in $\mathbf{B}(Y_+;Y_+)$.

We split $V_1 = V_{1,\epsilon} + V_{\infty,\epsilon}$ in such a way that $\|V_{1,\epsilon}\|_{L^2} < \epsilon$, $V_{\infty,\epsilon} \in L^\infty$. Then $\|R_0(\lambda-i0;\overline{z}\alpha)V_{1,\epsilon}\|_{\mathbf{B}(Y_-;Y_-)} \leq C\epsilon$ and $R_0(\lambda-i0;\overline{z}\alpha)V_{\infty,\epsilon}$ is compact. Therefore $VR_0(\lambda+i0;z\alpha)$ is compact on Y_+.

The assertion (4) is proven without any change, and the assertions (3) and (5) follow from Lemma 5.5. $\diamondsuit$

5.3 *Reconstruction of the potential.* We are now in a position of extracting Fourier coefficients of $V(x,y)$ from scattering amplitudes $A(\lambda;\theta), \theta \in \mathbf{E}^*$. The following assumption is imposed on V :

(A) *If $n \geq 3$, there exist $\rho > 1$ and $\epsilon > 0$ such that*

$$|V(x,y)| \leq \begin{cases} C\exp(-\epsilon e^y) & (y > 0) \\ C\langle y\rangle^{-\rho}e^y & (y < 0). \end{cases} \tag{5.12}$$

If $n = 2$, $V = V_1 + V_2$, where $V_1 \in L^2(\mathbf{E} \times \mathbf{R})$ and is compactly supported, and V_2 satisfies

$$|V_2(x,y)| \leq \begin{cases} C_N \exp(-Ne^y) & (y > 0) \\ \\ C\langle y\rangle^{-\rho} e^y & (y < 0). \end{cases} \tag{5.13}$$

for a fixed $\rho > 1$ and any $N > 1$.

By Definition 4.2 and (2.3)

$$\mathcal{F}_0^{(+)}(\lambda;\theta)^*\varphi = \frac{1}{\pi}\sum_{\gamma^*\in\Gamma^*} e^{i\gamma^*\cdot x}|\gamma^*+\theta|^{i\sqrt{\lambda}} K_{i\sqrt{\lambda}}(|\gamma^*+\theta|e^y)\int_{\mathbf{E}} e^{-i\gamma^*\cdot w}\varphi(w)dw. \tag{5.14}$$

Therefore for $\xi \in \Gamma^*$

$$\mathcal{F}_0^{(+)}(\lambda;\theta)^* e^{i\xi\cdot w} = \frac{|\mathbf{E}|}{\pi} e^{i\xi\cdot x}|\xi+\theta|^{i\sqrt{\lambda}} K_{i\sqrt{\lambda}}(|\xi+\theta|e^y). \tag{5.15}$$

By making use of (2.3) and (5.15), we have

$$\begin{aligned} (A(\lambda;\theta)e^{i\xi\cdot w}, e^{i\eta\cdot w}) &= \left(\frac{|\mathbf{E}|}{\pi}\right)^2 \left(\frac{|\xi+\theta|}{|\eta+\theta|}\right)^{i\sqrt{\lambda}} \\ &\times \iint (V - VR(\lambda+i0;\theta)V)[e^{i\xi\cdot x}K_{i\sqrt{\lambda}}(|\xi+\theta|e^y)] \cdot \\ & e^{-i\eta\cdot x}K_{i\sqrt{\lambda}}(|\eta+\theta|e^y)dxdy. \end{aligned}$$

Let $k \in \Gamma^* \setminus \{0\}$ be arbitrarily given. If $n \geq 3$, a simple trigonometry shows that one can pick $\xi, \eta \in \Gamma^*$ and $\alpha \in \Gamma_{unit}$ such that

$$k = \eta - \xi, \quad \alpha\cdot\eta < \delta < \alpha\cdot\xi < \epsilon \tag{5.16}$$

for small $\delta > 0$, where ϵ is the constant specified in the assumption (A). If $n = 2$, by taking $\eta = 0$, $\xi = -k$ and $\alpha = -k/|k| = -\mathrm{sgn}\, k$, one has

$$k = \eta - \xi, \quad \alpha\cdot\eta < \delta < \alpha\cdot\xi \tag{5.17}$$

for small $\delta > 0$. We put $z = -\delta + i\tau$. Then since $\alpha\cdot\xi - \delta > 0$, $\alpha\cdot\eta - \delta < 0$, we have as $\tau \to \infty$,

$$\sqrt{(\xi + z\alpha)^2} \sim \tau i + (\alpha\cdot\xi - \delta), \tag{5.18}$$

$$\sqrt{(\eta + z\alpha)^2} \sim -\tau i - (\alpha\cdot\eta - \delta). \tag{5.19}$$

Therefore we have by (2.7)

$$K_{i\sqrt{\lambda}}(\sqrt{(\xi+z\alpha)^2}e^y) \sim \frac{C}{\sqrt{\tau e^y}}\exp(-i\tau e^y - (\alpha\cdot\xi - \delta)e^y), \tag{5.20}$$

$$K_{i\sqrt{\lambda}}(\sqrt{(\eta+z\alpha)^2}e^y) \sim \frac{C}{\sqrt{\tau e^y}}\exp(i\tau e^y + (\alpha\cdot\eta - \delta)e^y). \tag{5.21}$$

Now for ϵ small enough, we consider $(A(\lambda;t\alpha)e^{i\xi\cdot w}, e^{i\eta\cdot w})$ for $t \in (-2\epsilon, -\epsilon)$. By virtue of (4.15), Lemma 5.3 and the well-known Riesz's theorem on boundary values of analytic functions, it has a unique meromorphic extension to D_α. Let $z = -\delta + i\tau$, where $\epsilon < \delta < 2\epsilon$, and put

$$B(\tau,\alpha,\xi,\eta) = \tau(A(\lambda;z\alpha)e^{i\xi\cdot w}, e^{i\eta\cdot w}). \tag{5.22}$$

We let $\tau \to \infty$. By the assumption (A), the term containing $R(\lambda+i0; z\alpha)$ disappears by virtue of Lemma 5.3 (5). Hence we have by (5.20) and (5.21)

$$\begin{aligned}\lim_{\tau\to\infty} B(\tau,\alpha,\xi,\eta) &= \lim_{\tau\to\infty} \tau(V\mathcal{F}_0^{(+)}(\lambda;\theta)^* e^{i\xi\cdot w}, \mathcal{F}_0^{(+)}(\lambda;\theta)^* e^{i\eta\cdot w}) \\ &= Const. \int \hat{V}(k,y) e^{-y} \exp(\alpha\cdot k e^y) dy.\end{aligned}$$

If $n \geq 3$, one can vary α as long as it satisfies (5.16). Let us take notice of the fact that the set Γ_{unit} is dense in S^{n-2}. Then by the analytic continuation, one can reconstruct

$$\int_{\mathbf{E}\times(0,\infty)} e^{-ik\cdot x} e^{-itz} V(x,\log z) \frac{dxdz}{z^2}, \quad k \neq 0$$

from the scattering amplitudes $\{A(\lambda;\theta)\}_{\theta\in\mathbf{E}^*}$. If $n = 2$, $\alpha\cdot k = -|k|$. Therefore we can reconstruct

$$\int_{\mathbf{E}\times(0,\infty)} e^{-ik\cdot x} e^{-|k|z} V(x,\log z) \frac{dxdz}{z^2}, \quad k \neq 0$$

from the scattering amplitudes $\{A(\lambda;\theta)\}_{\theta\in\mathbf{E}^*}$. We have thus proven

Theorem 5.7. *(ISP in $\mathbf{H}^n/\Gamma$). Assume (A).*
(1) If $n \geq 3$, one can uniquely reconstruct

$$\int_{\mathbf{E}} e^{-ik\cdot x} V(x,y) dx, \quad \forall k \in \Gamma^* \setminus \{0\}$$

from the scattering amplitudes $\{A(\lambda;\theta)\}_{\theta\in\mathbf{E}^*}$ of arbitrarily fixed energy $\lambda > 0$.
(2) If $n = 2$, one can uniquely reconstruct

$$\int_{\mathbf{E}\times(0,\infty)} e^{-ik\cdot x} e^{-|k|z} V(x,\log z) \frac{dxdz}{z^2}, \quad \forall k \in \Gamma^* \setminus \{0\}$$

from the scattering amplitudes $\{A(\lambda;\theta)\}_{\theta\in\mathbf{E}^*}$ of arbitrarily fixed energy $\lambda > 0$.

6. Inverse boundary value problem

6.1 *IBVP in* $\Gamma\backslash\mathbf{H}^n$. Let U be a bounded domain in $\mathbf{E}\times\mathbf{R}$ with smooth boundary S. For $q(x,y)$, real-valued, we consider the following boundary value problem

$$(H_0(\theta) + q)u = 0 \quad \text{in} \quad U, \quad u|_S = f \in H^{3/2}(S). \tag{6.1}$$

Let $H_0(\theta)_D$ be $-e^{2y}(\partial_x + i\theta)^2 - \partial_y^2$ with Dirichlet boundary condition. For the solution u of (6.1), the D-N map is defined by

$$\Lambda_q(\theta)f = (e^{2y}\nu_x\cdot(\partial_x + i\theta)u + \nu_y\partial_y u)|_S, \tag{6.2}$$

where $\nu_e = (\nu_s, \nu_y)$ is the outer unit normal to S with respect to the Euclidean metric $(dx)^2 + (dy)^2$.

Lemma 6.1. *Let U be a bounded domain in $\mathbf{E}\times\mathbf{R}$ with smooth boundary such that $(\mathbf{E}\times\mathbf{R})\setminus\overline{U}$ is connected. Suppose $n \geq 3$ and that $q \in L^\infty(U)$. If $0 \notin \sigma(H_0(\theta)_D + q)$ for all $\theta \in \mathbf{E}^*$, then one can uniquely reconstruct*

$$\int_{\mathbf{E}} e^{-ik\cdot x} q(x,y) dx, \quad \forall k \in \Gamma^* \setminus \{0\}$$

from $\Lambda_q(\theta)$ of all $\theta \in \mathbf{E}^*$.

Admitting this lemma for the moment, we consider its applications.

6.2 *IBVP in* $\mathbf{H}^n$. Let U be a bounded domain in $\mathbf{R}^{n-1} \times \mathbf{R}$ with smooth boundary S. Let $H_0 = -e^{2y}\Delta_x - \partial_y^2$. For a real-valued function $q(x,y)$, we study

$$(H_0 + q)u = 0 \quad \text{in} \quad U, \quad u|_S = f \in H^{3/2}(S). \tag{6.3}$$

Let H_{0D} be $-e^{2y}\Delta_x - \partial_y^2$ with Dirichlet boundary condition. For the solution u of (6.3), we define the D-N map Λ_q by

$$\Lambda_q f = (e^{2y}\nu_x \cdot \partial_x u + \nu_y \partial_y u)|_S, \tag{6.4}$$

where $\nu_e = (\nu_x, \nu_y)$ is the outer unit normal to S with respect to the Euclidean metric $(dx)^2 + (dy)^2$.

Lemma 6.2. *Let U be a bounded domain in $\mathbf{R}^{n-1} \times \mathbf{R}$ with smooth boundary such that $(\mathbf{R}^{n-1} \times \mathbf{R}) \setminus \overline{U}$ is connected. Suppose $n \geq 3$ and that $q \in L^\infty(U)$. If $0 \notin \sigma(H_{0D} + q)$, then*

$$\int_{\mathbf{R}^{n-1}} e^{-k\cdot x} V(x,y)dx, \quad \forall k \in \left(\frac{2\pi}{L}\mathbf{Z}\right)^{n-1} \setminus \{0\}$$

is uniquely reconstructed from Λ_q, where $L > 0$ is a constant such that $U \subset [L/3, 2L/3]^{n-1} \times \mathbf{R}$.

Proof. Take the lattice $\Gamma \subset \mathbf{R}^{n-1}$ such that $\mathbf{R}^{n-1}/\Gamma = [0,L]^{n-1}$. We regard $[0,L]^{n-1}$ as a torus. However U is contained in one coordinate patch of $[0,L]^{n-1} \times \mathbf{R}$. Then since $e^{-ix\cdot\theta} H_{0D} e^{ix\cdot\theta} = H_0(\theta)_D$, we have

$$e^{-ix\cdot\theta}\Lambda_q e^{ix\cdot\theta} = \Lambda_q(\theta). \tag{6.5}$$

Therefore, one can construct $\Lambda_q(\theta)$ from Λ_q. Lemma 6.2 then follows from Lemma 6.1. $\diamondsuit$

6.3 *IBVP in* $\mathbf{R}^n$. Let $\widetilde{U}$ be a bounded domain in the Euclidean space $\mathbf{R}^n$ with smooth boundary $\widetilde{S}$. For $V(x, x_n)$, real-valued, we study the boundary value problem

$$(-\Delta + V)u = 0 \quad \text{in} \quad \widetilde{U}, \quad u|_{\widetilde{S}} = f \in H^{3/2}(\widetilde{S}). \tag{6.6}$$

Let Δ_D be $\Delta = \partial_x^2 + \partial_n^2$ with Dirichlet boundary condition. The D-N map is defined by

$$\widetilde{\Lambda}_V f = (\widetilde{\nu}_x \cdot \partial_x u + \widetilde{\nu}_n \partial_n u)|_{\widetilde{S}} \tag{6.7}$$

for the solution u to (6.6). Here $\widetilde{\nu}_e = (\widetilde{\nu}_x, \widetilde{\nu}_n)$ is the outer unit normal to $\widetilde{S}$ with respect to the Euclidean metric $(dx)^2 + (dx_n)^2$ and $\partial_n = \partial/\partial x_n$.

As is stated in the introduction, the following theorem was proved by Sylvester-Uhlmann [36], Nachman [25] and Khenkin-Novikov [19], and the results has been extended to various cases. However they were proved by essentially the same machinery, complex geometrical optics solution or Faddeev's Green function. We shall give here an alternative proof by using hyperbolic manifolds as a tool.

Theorem 6.3. *(IBVP in $\mathbf{R}^n$). Let $\widetilde{U}$ be a bounded domain in $\mathbf{R}^n$ with smooth boundary such that $\mathbf{R}^n \setminus \overline{\widetilde{U}}$ is connected. Suppose $n \geq 3$ and that $V \in L^\infty(\widetilde{U})$. If $0 \notin \sigma(-\Delta_D + V)$, then $V(x, x_n)$ is uniquely reconstructed from $\widetilde{\Lambda}_V$.*

Proof. One can assume that $\widetilde{U} \subset \{(x, x_n) \in \mathbf{R}^n; x_n > 1\}$. Let $y = \log x_n$ and $u = e^{-y/2}v$. Then if u satisfies (6.6), v satisfis

$$(-e^{2y}\partial_x^2 - \partial_y^2)v + (e^{2y}V + \frac{1}{4})v = 0 \quad \text{in} \quad U, \quad v|_S = e^{y/2}f \in H^{3/2}(S), \tag{6.8}$$

where U and S are obtained from $\widetilde{U}$ and $\widetilde{S}$ by the change of variable. We show that

$$(e^{2y}\nu_x \cdot \partial_x + \nu_y\partial_y)|_S = \lambda(x, x_n)x_n^2(\widetilde{\nu}_x \cdot \partial_x + \widetilde{\nu}_n\partial_n)|_{\widetilde{S}} \tag{6.9}$$

for a smooth function $\lambda(x, x_n) > 0$ on $\widetilde{S}$. Here $\nu_e = (\nu_x, \nu_y)$ and $\widetilde{\nu}_e = (\widetilde{\nu}_x, \widetilde{\nu}_n)$ are the outer unit normals to S and $\widetilde{S}$ with respect to the Euclidean metrics $(dx)^2 + (dy)^2$ and $(dx)^2 + (dx_n)^2$.

Suppose that S is locally represented as $\{(x, y); \varphi(x, y) = 0\}$ and put $\widetilde{\varphi}(x, x_n) = \varphi(x, \log x_n)$. Then

$$\nu_e = c(\partial_x\varphi, \partial_y\varphi), \quad c = (|\partial_x\varphi|^2 + |\partial_y\varphi|^2)^{-1/2},$$

$$\widetilde{\nu}_e = \widetilde{c}(\partial_x\widetilde{\varphi}, \partial_n\widetilde{\varphi}), \quad \widetilde{c} = (|\partial_x\widetilde{\varphi}|^2 + |\partial_n\widetilde{\varphi}|^2)^{-1/2}.$$

Then we have

$$\frac{\widetilde{\nu}_x}{\widetilde{c}} = \partial_x\widetilde{\varphi} = \partial_x\varphi = \frac{\nu_x}{c},$$

$$\frac{\widetilde{\nu}_n}{\widetilde{c}} = \partial_n\widetilde{\varphi} = \frac{1}{x_n}\partial_y\varphi = \frac{\nu_y}{x_n c}.$$

By a simple computation, one can show (6.9) with $\lambda(x, x_n)$ replaced by

$$\frac{c}{\widetilde{c}} = \left(\frac{|\partial_x\widetilde{\varphi}|^2 + x_n^2|\partial_n\widetilde{\varphi}|^2}{|\partial_x\widetilde{\varphi}|^2 + |\partial_n\widetilde{\varphi}|^2}\right)^{-1/2}.$$

Note that $c/\widetilde{c}$ does not depend on the choice of φ, since if we have two such φ and ψ, $\nabla\varphi = k\nabla\psi$ for $k > 0$.

Without loss of generality, we assume that $\widetilde{U} \subset [0, \pi]^n$. Then by Lemma 6.2, one can reconstruct

$$\int_{\mathbf{R}^{n-1}} e^{-ik\cdot x}V(x, y)dx, \quad \forall k \in \mathbf{Z}^{n-1} \setminus \{0\}$$

from $\widetilde{\Lambda}_V$.

Now, the idea is to rotate and translate the domain $\widetilde{U}$ arbitrarily. Then $-\Delta + V(X)$, $X = (x, x_n)$, is transformed into $-\Delta + V(RX + X_0)$ with $R \in SO(n)$ and $X_0 \in \mathbf{R}^n$, and we get the associated D-N map as the unitary transform of $\widetilde{\Lambda}_V$. We next imbed this IBVP into $\mathbf{H}^n$. By the above arguments, one can reconstruct $\hat{V}(R\Xi)$, where $\hat{V}(\Xi)$ is the Fourier transform of V, $R \in SO(n)$, $\Xi = (k, \eta)$, $k \in \mathbf{Z}^{n-1} \setminus \{0\}$, $\eta \in \mathbf{R}^1$. Since $SO(n)$ acts transitively on S^{n-1}, one can reconstruct $\hat{V}(\Xi)$ for $|\Xi| > 1$ by varying R and η. Since $\hat{V}(\Xi)$ is analytic, one can reconstruct $\hat{V}(\Xi)$ for all $\Xi \in \mathbf{R}^n$. ♢

As is clear from the above arguments, the IBVP in $\mathbf{R}^n$ and that in $\mathbf{H}^n$ are equivalent. Therefore Theorem 6.3 also holds for a bounded contractible domain in $\mathbf{H}^n$ and also for any hyperbolic manifold.

6.4 *Proof of Lemma 6.1.* We prove Lemma 6.1 by showing the equivalence of IBVP and ISP. We follow the arguments of Isakov-Nachman [10].

Fix a constant $\lambda > 0$ arbitrarily and let

$$V(x,y) = \begin{cases} q(x,y)+\lambda, & (x,y) \in U, \\ \\ 0, & (x,y) \notin U. \end{cases}$$

Let $H(\theta) = H_0(\theta) + V$ be the Schrödinger operator on $\mathbf{E}\times\mathbf{R}$, and let $R(\lambda \pm i0;\theta) = (H(\theta) - \lambda \mp i0)^{-1}$.

First let us note that

$$R(\lambda \pm i0;\theta) \in \mathbf{B}(H^{-s}_{loc}; H^{2-s}_{loc}), \quad 0 \leq s \leq 2, \tag{6.10}$$

which follows from the fact $R(\lambda \pm i0;\theta) \in \mathbf{B}(L^2_{loc}; H^2_{loc}) \cap \mathbf{B}(H^{-2}_{loc}; L^2_{loc})$ and interpolation. For $f \in L^2(S)$, we define $f_S \in H^{-1/2}$ by

$$\langle \varphi, f_S\rangle = \int_S \varphi \overline{f} dS_E, \quad \varphi \in H^{1/2}, \tag{6.11}$$

where dS_E is the surface element induced from the Euclidean metric $(dx)^2 + (dy)^2$. Using (6.10) and (6.11), we have

$$R(\lambda \pm i0;\theta) f_S \in H^{3/2}_{loc}. \tag{6.12}$$

Let us introduce the boundary operator

$$B(\theta)u = (e^{2y}\nu_x \cdot (\partial_x + i\theta)u + \nu_y \partial_y u)|_S, \tag{6.13}$$

where $\nu_e = (\nu_x, \nu_y)$ is the outer unit normal to S with respect to the Euclidean metric $(dx)^2 + (dy)^2$. For $u \in H^{1/2}_{loc}$, let $[u]_{ext}$ and $[u]_{int}$ be the boundary values of u on S from outside and inside of S, respectively.

Lemma 6.4. *Let $f \in L^2(S)$. Then*

$$[B(\theta)R(\lambda + i0;\theta)f_S]_{ext} - [B(\theta)R(\lambda + i0;\theta)f_S]_{int} = -\lambda(x, e^y)f,$$

where $\lambda(x, x_n)$ is the function appearing in (6.9).

Proof. By the resolvent equation, we have

$$R(\lambda + i0;\theta)f_S = R_0(\lambda + i0;\theta)f_S - R_0(\lambda + i0;\theta)VR(\lambda + i0;\theta)f_S.$$

Therefore we have only to consider $u = R_0(\lambda + i0;\theta)f_S$, since $R(\lambda + i0;\theta)f_S - u \in H^2_{loc}$.

Take $\chi_j \in C_0^\infty(\mathbf{E}\times\mathbf{R})$ such that $\sum_j \chi_j = 1$ near S, and each supp χ_j is contained in a local coordinate patch. Let $x_n = e^y, X = (x, x_n)$. Then we have

$$-\Delta_X(\chi_j u) = x_n^{-2}\chi_j f_s + g_j, \quad g_j \in L^2.$$

Letting $F(X)$ be the fundamental solution to $-\Delta_X$, we have

$$\chi_j u = \int F(X - X'){x'_n}^{-2}\chi_j(X')f(X')dS_{X'} + h_j, \quad h_j \in H^2_{loc}.$$

Let $\widetilde{\nu}_e$ be the outer unit normal to S with respect to the Euclidean metric $(dX)^2$. Then by the well-known computation from potential theory

$$\left[\frac{\partial}{\partial \widetilde{\nu}_e}(\chi_j u)\right]_{ext} - \left[\frac{\partial}{\partial \widetilde{\nu}_e}(\chi_j u)\right]_{int} = -x_n^{-2}\chi_j f.$$

Using (6.9), we get the lemma. $\diamondsuit$.

Let U_{ext} be the region exterior to U. Let $H_{ext}(\theta) = H_0(\theta)$ in U_{ext} with Dirichlet boundary condition, and $R_{ext}(z;\theta) = (H_{ext}(\theta) - z)^{-1}$. The limiting absorption principle also holds for $R_{ext}(z;\theta)$:

$$R_{ext}(\lambda \pm i0;\theta) \in \mathbf{B}(L^{2,s};L^{2,-s}), \quad s > 1/2. \tag{6.14}$$

For $f \in L^{2,s}$, $u^{(\pm)}_{ext} = R_{ext}(\lambda \pm i0;\theta)f$ satisfies the radiation condition

$$\begin{cases} u^{(\pm)}_{ext} \in L^{2,-s}, \\ F(\pm y > 0)(i\partial_y \pm \sqrt{\lambda})u^{(+)}_{ext} \in L^{2,-\alpha}, \\ F(\pm y > 0)(i\partial_y \mp \sqrt{\lambda})u^{(-)}_{ext} \in L^{2,-\alpha} \end{cases} \tag{6.15}$$

for some $0 < \alpha < 1/2$. These facts will be proved in §8.

The D-N maps $\Lambda^{(\pm)}_{ext}(\theta)$ for the exterior Dirichlet problem are defined by

$$\Lambda^{(\pm)}_{ext}(\theta)f = B(\theta)v^{(\pm)}_{ext}|_S, \tag{6.16}$$

where $v^{(\pm)}_{ext}$ are the outgoing or incoming solutions of

$$(H_0(\theta) - \lambda)v = 0 \quad \text{in} \quad U_{ext}, \quad v|_S = f \in H^{3/2}(S),$$

and where $B(\theta)$ is defined by (6.13) with the same ν_e as in (6.13). Therefore, viewed from U_{ext}, ν_e is the inner unit normal.

Note that $v^{(\pm)}_{ext}$ exist by virtue of the limiting absorption principle. In fact, take $\tilde{f} \in H^2(U_{ext})$ such that $\tilde{f} = f$ on S and $\tilde{f}$ has compact support. Then $v^{(\pm)}_{ext}$ is given by

$$v^{(\pm)}_{ext} = \tilde{f} + R_{ext}(\lambda \pm i0;\theta)(\partial_y^2 + e^{2y}(\partial_x + i\theta)^2 + \lambda)\tilde{f}.$$

The uniqueness of outgoing or incoming solution will be proved in §8.

We need one more preparation. Let $u_{ext} \in H^2_{loc}(U_{ext})$ and $u_{int} \in H^2(U)$ satisfy the equations

$$(H_0(\theta) - \lambda)u_{ext} = 0 \quad \text{in} \quad U_{ext}, \quad u_{ext}|_S = f, \tag{6.17}$$

$$(H_0(\theta) + V - \lambda)u_{int} = 0 \quad \text{in} \quad U, \quad u_{int}|_S = f. \tag{6.18}$$

Let χ_{ext} and χ_{int} be the characteristic functions of U_{ext} and U, respectively. Letting $u = \chi_{ext}u_{ext} + \chi_{int}u_{int}$, we have for $\varphi \in C_0^\infty(\mathbf{E} \times \mathbf{R})$

$$\int_{\mathbf{E}\times\mathbf{R}} u(H_0(\theta) + V - \lambda)\varphi dxdy = \int_S ([B(\theta)u]_{int} - [B(\theta)u]_{ext})\varphi dS_E. \tag{6.19}$$

Therefore we have

$$u_{ext} = R(\lambda \pm i0;\theta)(\Lambda_{V-\lambda}(\theta) - \Lambda^{(\pm)}_{ext}(\theta))f \tag{6.20}$$

accordingly as u_{ext} is outgoing or incoming.

For $f \in L^2(S)$, we define

$$M^{(\pm)}_S(\theta)f = (R(\lambda \pm i0;\theta)f_S)\big|_S. \tag{6.21}$$

Lemma 6.5. *Suppose $\lambda \notin \sigma(H_0(\theta)_D + V)$. Then :*
(1) $\Lambda_{V-\lambda}(\theta) - \Lambda^{(\pm)}_{ext}(\theta) : H^{3/2}(S) \to H^{1/2}(S)$ is an isomorphism.
(2) $(\Lambda_{V-\lambda}(\theta) - \Lambda^{(\pm)}_{ext}(\theta))^{-1} = M^{(\pm)}_S(\theta)$.

Proof. We show that $M_S^{(\pm)}(\theta) : H^{1/2}(S) \to H^{3/2}(S)$ is an isomorphism and

$$M_S^{(\pm)}(\theta)(\Lambda_{V-\lambda}(\theta) - \Lambda_{ext}^{(\pm)}(\theta)) = 1. \qquad (6.22)$$

Suppose $M_S^{(\pm)}(\theta)f = 0$. Then $R(\lambda \pm i0; \theta)f_S$ is a Dirichlet eigenfunction of $H_0(\theta)+V$ in the interior region, hence $R(\lambda\pm i0;\theta)f = 0$ on U. In the exterior region, $R(\lambda\pm i0;\theta)f_S$ is a solution to the homogeneous Schrödinger equation satisfying the radiation condition. In §8, such a solution is shown to vanish identically. Therefore $R(\lambda\pm i0;\theta)f_S = 0$ on U_{ext}. Hence by Lemma 6.5, $f = 0$, which proves that $M_S^{(\pm)}(\theta)$ is 1 to 1. The formula (6.20) implies (6.22). This proves the lemma. ♢.

Let us take notice of the following fact.

Lemma 6.6. *For* $f, g \in H^{3/2}(S)$

$$(\Lambda_{V-\lambda}(\theta)f, g)_{L^2(S)} = (f, \Lambda_{V-\lambda}(\theta)g)_{L^2(S)}, \qquad (6.23)$$

$$(\Lambda_{ext}^{(\pm)}(\theta)f, g)_{L^2(S)} = (f, \Lambda_{ext}^{(\mp)}(\theta)g)_{L^2(S)}. \qquad (6.24)$$

Proof. We shall prove (6.24). Let u be the outgoing solution of

$$(H_0(\theta) - \lambda)u = 0 \quad \text{in} \quad U_{ext}, \quad u\big|_S = f,$$

and v the incoming solution of

$$(H_0(\theta) - \lambda)v = 0 \quad \text{in} \quad U_{ext}, \quad v\big|_S = g.$$

Let $U_R = \{(x, y); |y| < R\} \cap U_{ext}$. Then we have by integration by parts

$$\begin{aligned}
& \int_{U_R} \Big\{(H_0(\theta) - \lambda)u\overline{v} - u\overline{(H_0(\theta) - \lambda)v}\Big\}\, dxdy \\
= \quad & (\Lambda_{ext}^{(+)}(\theta)f, g)_{L^2(S)} - (f, \Lambda_{ext}^{(-)}(\theta)g)_{L^2(S)} \\
+ \quad & \int_{y=R} \Big\{u\overline{(\partial_y v + i\sqrt{\lambda}v)} - (\partial_y u - i\sqrt{\lambda}u)\overline{v}\Big\}\, dx \\
- \quad & \int_{y=-R} \Big\{(\partial_y u + i\sqrt{\lambda}u)\overline{v} - u\overline{(\partial_y v - i\sqrt{\lambda}v)}\Big\}\, dx.
\end{aligned}$$

Since u, v satisfy the radiation condition, we have

$$F(\pm y > 0)(\partial_y u \mp i\sqrt{\lambda}u) \in L^{2,s-1},$$
$$F(\pm y > 0)(\partial_y v \pm i\sqrt{\lambda}v) \in L^{2,s-1}$$

(see Theorem 7.7). Therefore the integrands in $\int_{y=R} \cdots dx$ and $\int_{y=-R} \cdots dx$ belong to
$L^1(U_{ext}; \langle y\rangle^{-1}dxdy)$. We then see that these integrals vanish as $R \to \infty$ along a suitable sequence $\{R_j\}$. This proves (6.24). The formula (6.23) is proved by the similar and simpler method of integration by parts. ♢

Let us introduce the transformation from near field pattern to far field pattern. Let $\mathcal{F}^{(\pm)}(\lambda;\theta)$ be defined by (4.14). Then one can naturally define

$$\mathcal{G}^{(\pm)}(\lambda;\theta) = \mathcal{F}^{(\pm)}(\lambda;\theta)(\Lambda_{V-\lambda}(\theta) - \Lambda_{ext}^{(\pm)}(\theta)). \qquad (6.25)$$

Using Lemma 6.6 (2), we have

Lemma 6.7.

$$\mathcal{G}^{(\pm)}(\lambda;\theta)M_S^{(\pm)}(\theta) = \mathcal{F}^{(\pm)}(\lambda;\theta).$$

By virtue of (6.20), Theorem 4.4 and Lemma 6.7, we have

Lemma 6.8. *Let* $u^{(\pm)}_{ext}$ *be the outgoing or incoming solution to (6.17). Then as* $y \to -\infty$

$$u^{(\pm)}_{ext} \sim C_{\pm}(\lambda)e^{\mp i\sqrt{\lambda}y}\mathcal{G}^{(\pm)}(\lambda;\theta)f \quad \text{in} \quad L^2(\mathbf{E}),$$

where $C_{\pm}(\lambda)$ *is given in Theorem 4.4.*

Finally we introduce the generalized Fourier transformation associated with $H_0(\theta)$ in the exterior domain. It follows easily from Lemma 6.6 (2) that

$$R_{ext}(\lambda\pm i0;\theta) = R_0(\lambda\pm i0;\theta) - R(\lambda\pm i0;\theta)(\Lambda_{V-\lambda}(\theta) - \Lambda^{(\pm)}_{ext}(\theta))R_0(\lambda\pm i0;\theta). \quad (6.26)$$

Taking the adjoint and using Lemma 6.7, we also have

$$R_{ext}(\lambda\pm i0;\theta) = R_0(\lambda\pm i0;\theta) - R_0(\lambda\pm i0;\theta)(\Lambda_{V-\lambda}(\theta) - \Lambda^{(\pm)}_{ext}(\theta))R(\lambda\pm i0;\theta). \quad (6.27)$$

We put

$$\mathcal{F}^{(-)}_{ext}(\lambda;\theta) = \mathcal{F}^{(-)}_0(\lambda;\theta) - \mathcal{F}^{(-)}_0(\lambda;\theta)(\Lambda_{V-\lambda}(\theta) - \Lambda^{(-)}_{ext}(\theta))R(\lambda - i0;\theta). \quad (6.28)$$

Then by (6.27) we have as $y \to -\infty$

$$R_{ext}(\lambda - i0;\theta)f \sim C_-(\lambda)e^{i\sqrt{\lambda}y}\mathcal{F}^{(-)}_{ext}(\lambda;\theta)f. \quad (6.29)$$

This shows in particular that $\mathcal{F}^{(-)}_{ext}(\lambda;\theta)$ does not depend on V. It is not hard to show that for $\varphi \in L^2(\mathbf{E})$

$$(H_0(\theta) - \lambda)\mathcal{F}^{(-)}_{ext}(\lambda;\theta)^*\varphi = 0 \quad \text{in} \quad U_{ext},$$

$$\mathcal{F}^{(-)}_{ext}(\lambda;\theta)^*\varphi = 0 \quad \text{on} \quad S,$$

$$\mathcal{F}^{(-)}_{ext}(\lambda;\theta)^*\varphi - \mathcal{F}^{(-)}_0(\lambda;\theta)^*\varphi \quad \text{is} \quad \text{outgoing}.$$

We define the geometric scattering amplitude for the exterior domain by

$$\widetilde{A_{ext}}(\lambda;\theta) = \mathcal{F}^{(+)}(\lambda;\theta)(\Lambda_{V-\lambda}(\theta) - \Lambda^{(+)}_{ext}(\theta))\mathcal{F}^{(-)}_0(\lambda;\theta)^*. \quad (6.30)$$

Then we have as $y \to -\infty$

$$\mathcal{F}^{(-)}_{ext}(\lambda;\theta)^*\varphi - \mathcal{F}^{(-)}_0(\lambda;\theta)^*\varphi \sim -C_+(\lambda)e^{-i\sqrt{\lambda}y}\widetilde{A_{ext}}(\lambda;\theta)\varphi. \quad (6.31)$$

Lemma 6.9.

$$\mathcal{G}^{(+)}(\lambda;\theta)M^{(+)}_S(\theta)\mathcal{G}^{(-)}(\lambda;\theta)^* = \widetilde{A_{ext}}(\lambda;\theta) - \widetilde{A}(\lambda;\theta).$$

Proof. Let $u = \mathcal{F}^{(-)}(\lambda;\theta)^*\varphi - \mathcal{F}^{(-)}_{ext}(\lambda;\theta)^*\varphi$. Then since u is the outgoing solution of

$$(H_0(\theta) - \lambda)u = 0 \quad \text{in} \quad U_{ext}, \quad u|_S = \mathcal{F}^{(-)}(\lambda;\theta)^*\varphi,$$

we have by (6.20)

$$\begin{aligned} u &= R(\lambda + i0;\theta)(\Lambda_{V-\lambda}(\theta) - \Lambda^{(+)}_{ext}(\theta))\mathcal{F}^{(-)}(\lambda;\theta)^*\varphi \\ &= R(\lambda + i0;\theta)\mathcal{G}^{(-)}(\lambda;\theta)^*\varphi. \end{aligned}$$

Therefore as $y \to -\infty$

$$u \sim C_+(\lambda)e^{-i\sqrt{\lambda}y}\mathcal{F}^{(+)}(\lambda;\theta)\mathcal{G}^{(-)}(\lambda;\theta)^*\varphi.$$

Inserting $1 = (\Lambda_{V-\lambda}(\theta) - \Lambda^{(+)}_{ext}(\theta))M^{(+)}_S(\theta)$, we obtain

$$u \sim C_+(\lambda)e^{-i\sqrt{\lambda}y}\mathcal{G}^{(+)}(\lambda;\theta)M^{(+)}_S(\theta)\mathcal{G}^{(-)}(\lambda;\theta)^*\varphi. \quad (6.32)$$

On the other hand, we have

$$\mathcal{F}^{(-)}(\lambda;\theta)^*\varphi - \mathcal{F}^{(-)}_{ext}(\lambda;\theta)^*\varphi \sim -C_+(\lambda)e^{-i\sqrt{\lambda}y}\left(\widetilde{A}(\lambda;\theta)\varphi - \widetilde{A_{ext}}(\lambda;\theta)\varphi\right). \quad (6.33)$$

Comparing (6.32) and (6.33), we obtain the lemma. $\diamondsuit$.

We prove the equivalence of IBVP and ISP by showing that $\Lambda_{V-\lambda}(\theta)$ and $A(\lambda;\theta)$ determine each other.

Given $\Lambda_{V-\lambda}(\theta)$, construct $M_S^{(+)}(\theta)$ by Lemma 6.5. Then $A(\lambda;\theta)$ is constructed by Lemma 6.9.

The converse direction is less explict. We show that $\mathcal{G}^{(\pm)}(\lambda;\theta)$ are 1 to 1. Then, since the ranges of $\mathcal{G}^{(\pm)}(\lambda;\theta)^*$ are dense in $L^2(\mathbf{E})$, Lemma 6.10 gives $\Lambda_{V-\lambda}(\theta)$ from $A(\lambda;\theta)$.

Suppose $\mathcal{G}^{(\pm)}(\lambda;\theta)f = 0$. Let u be the (outgoing or incoming) solution to (6.17). By virtue of Lema 6.9, we have $\|u(\cdot,y)\|_{L^2(\mathbf{E})} \to 0$ as $y \to -\infty$. By expanding u into a Fourier series $u = \frac{1}{|\mathbf{E}|}\sum \hat{u}(\gamma^*,y)e^{i\gamma^*\cdot x}$, we see that the Fourier coefficient $\hat{u}(\gamma^*,y)$ satisfies the equation $-v'' + e^{2y}(\gamma^*+\theta)^2 v - \lambda v = 0$. Therefore $\hat{u}(\gamma^*,y)$ is written by a linear combination of modified Bessel functions. Since $\hat{u}(\gamma^*,y) \to 0$ as $y \to -\infty$, we have $\hat{u}(\gamma^*,y) = 0$ by observing the behavior of $K_{i\sqrt{\lambda}}(|\gamma^*+\theta|e^y), I_{i\sqrt{\lambda}}(|\gamma^*+\theta|e^y)$ as $y \to -\infty$. Then $u(x,y) = 0$ for $y < -R$, R being sufficiently large. By the unique continuation theorem, u vanishes identically on U_{ext}. Therefore $f = 0$. $\diamondsuit$

6.5 *ISP at the cusp.* We show that the D-N map and the scattering amplitude at the cusp determine each other. As in (6.25) we introduce a transformation from near field pattern to far field pattern at the cusp. We put

$$\Lambda_{V-\lambda} = \Lambda_{V-\lambda}(0), \quad \Lambda^{(\pm)}_{ext} = \Lambda^{(\pm)}_{ext}(0), \quad R(\lambda \pm i0) = R(\lambda \pm i0;0) \quad (6.34)$$

and define

$$\mathcal{G}^{(\pm)}_{c\gamma^*}(\lambda) = \mathcal{F}^{(\pm)}_{c\gamma^*}(\lambda)(\Lambda_{V-\lambda} - \Lambda^{(\pm)}_{ext}). \quad (6.35)$$

More precisely, for $f \in H^{3/2}(S)$ we put

$$u_\pm = (1 - VR(\lambda \pm i0))(\Lambda_{V-\lambda} - \Lambda^{(\pm)}_{ext})f,$$

$$\mathcal{G}^{(\pm)}_{c\gamma^*}(\lambda)f = \int_S I_{\mp i\sqrt{\lambda}}(|\gamma^*|e^y)e^{-i\gamma^*\cdot x}u_\pm(x,y)dS \quad (\gamma^* \neq 0),$$

$$\mathcal{G}^{(\pm)}_{c\gamma^*}(\lambda)f = (2\pi)^{-1/2}\int_S e^{\mp i\sqrt{\lambda}y}u_\pm(x,y)dS \quad (\gamma^* = 0).$$

By Lemma 6.5 (2), we have

Lemma 6.10. $\mathcal{G}^{(\pm)}_{c\gamma^*}(\lambda)M_S^{(\pm)} = \mathcal{F}^{(\pm)}_{c\gamma^*}(\lambda)$, *where* $M_S^{(\pm)} = M_S^{(\pm)}(0)$.

By virtue of Theorem 4.11, we also have

Lemma 6.11. *Let* $u^{(\pm)}_{ext}$ *be the outgoing or incoming solution to (6.17). Then as* $y \to \infty$

$$\langle e^{-i\gamma^*\cdot x}, u^{(\pm)}_{ext}\rangle \sim \sqrt{\frac{\pi}{2|\gamma^*|}}\exp(-\frac{y}{2} - |\gamma^*|e^y)\mathcal{G}^{(\pm)}_{c\gamma^*}(\lambda)f \quad (\gamma^* \neq 0),$$

$$\langle e^{-i\gamma^*\cdot x}, u^{(\pm)}_{ext}\rangle \sim \mp i\sqrt{\frac{\pi}{2\lambda}}e^{\pm i\sqrt{\lambda}y}\mathcal{G}^{(\pm)}_{c\gamma^*}(\lambda)f \quad (\gamma^* = 0).$$

We also introduce the generlized Fourier transformation associated with the cusp. We put

$$\mathcal{F}^{(-)}_{ext,c\gamma^*}(\lambda) = \mathcal{F}^{(-)}_{0c\gamma^*}(\lambda)\Big[1 - (\Lambda_{V-\lambda} - \Lambda^{(-)}_{ext})R(\lambda - i0)\Big], \tag{6.36}$$

$$\mathcal{F}^{(-)}_{ext,c}(\lambda) = \sum_{\gamma^*\in\Gamma^*} \mu_{\gamma^*} e^{i\gamma^*\cdot x}\mathcal{F}^{(-)}_{ext,c\gamma^*}(\lambda), \tag{6.37}$$

$$\mathcal{G}^{(\pm)}_{c}(\lambda) = \sum_{\gamma^*\in\Gamma^*} \mu_{\gamma^*} e^{i\gamma^*\cdot x}\mathcal{G}^{(\pm)}_{c\gamma^*}(\lambda). \tag{6.38}$$

We finally put

$$\widetilde{A_{ext,c}}(\lambda) = \mathcal{F}^{(+)}_{c}(\lambda)(\Lambda_{V-\lambda} - \Lambda^{(+)}_{ext})\mathcal{F}^{(-)*}_{0c}(\lambda). \tag{6.39}$$

Then as in Lemma 6.9, we have

Lemma 6.12. $\mathcal{G}^{(+)}_{c}(\lambda)M^{(+)}_{S}\mathcal{G}^{(-)}_{c}(\lambda)^* = \widetilde{A_{ext,c}}(\lambda) - \widetilde{A}_c(\lambda)$.

Proof. One can show that for $\varphi \in L^2(\mathbf{E})$

$$(H_0 - \lambda)\mathcal{F}^{(-)}_{ext,c}(\lambda)^*\varphi = 0 \quad \text{in} \quad U_{ext},$$

$$\mathcal{F}^{(-)}_{ext,c}(\lambda)^*\varphi = 0 \quad \text{on} S,$$

$$\mathcal{F}^{(-)}_{ext,c}(\lambda)^*\varphi - \mathcal{F}^{(-)}_{0c}(\lambda)^*\varphi \quad \text{is outgoing.}$$

Let $u = \mathcal{F}^{(-)}_{c}(\lambda)^*\varphi - \mathcal{F}^{(-)}_{ext,c}(\lambda)^*\varphi$. Then as in the proof of Lemma 6.9,

$$\begin{aligned} u &= R(\lambda + i0)(\Lambda_{V-\lambda} - \Lambda^{(+)}_{ext})\mathcal{F}^{(-)}_{c}(\lambda)^*\varphi \\ &= R(\lambda + i0)\mathcal{G}^{(-)}_{c}(\lambda)^*\varphi. \end{aligned}$$

If $0 \neq \gamma^* \in \Gamma^*$, we have as $y \to \infty$

$$\begin{aligned} \langle e^{-i\gamma^*\cdot x}, u\rangle &\sim \sqrt{\frac{\pi}{2|\gamma^*|}}\exp(-\frac{y}{2} - |\gamma^*|e^y)\mathcal{F}^{(+)}_{c\gamma^*}(\lambda)\mathcal{G}^{(-)}_{c}(\lambda)^*\varphi \\ &\sim \sqrt{\frac{\pi}{2|\gamma^*|}}\exp(-\frac{y}{2} - |\gamma^*|e^y)\mathcal{G}^{(+)}_{c\gamma^*}(\lambda)M^{(+)}_{S}\mathcal{G}^{(-)}_{c}(\lambda)^*\varphi. \end{aligned}$$

On the other hand, we have

$$\langle e^{-i\gamma^*\cdot x}, \mathcal{F}^{(-)}_{c}(\lambda)^*\varphi - \mathcal{F}^{(-)}_{ext,c}(\lambda)^*\varphi\rangle$$

$$\sim -\sqrt{\frac{\pi}{2|\gamma^*|}}\exp(-\frac{y}{2} - |\gamma^*|e^y)(\widetilde{A_c}(\lambda)\varphi - \widetilde{A_{ext,c}}(\lambda)\varphi).$$

By computing the case $\gamma^* = 0$ also, and comparing the asmptotic expansions, we get the lemma. $\diamondsuit$

We now show that $\Lambda_{V-\lambda}$ and $\widetilde{A_c}(\lambda)$ determine each other. By Lemma 6.12, $\widetilde{A_c}(\lambda)$ is constructed from $\Lambda_{V-\lambda}$. To show the converse direction, we show that $\mathcal{G}^{(\pm)}_{c}(\lambda)$ are 1 to 1. Suppose $\mathcal{G}^{(\pm)}_{c}(\lambda)f = 0$. Let u be the (outgoing or incoming) solution to $(H_0 - \lambda)u = 0$ in U_{ext}, $u = f$ on S. We expand u into a Fourier series $u = \frac{1}{|\mathbf{E}|}\sum \hat{u}(\gamma^*, y)e^{i\gamma^*\cdot x}$. Then $\hat{u}(\gamma^*, y)$ satisfies $-v'' + e^{2y}|\gamma^*|^2 - \lambda v = 0$ for large $|y|$, hence is written as a linear combination of modified Bessel functions (or of $e^{\pm i\sqrt{\lambda}y}$ if $\gamma^* = 0$). In view of Lemma 6.12 and (2.5), (2.6), $\hat{u}(\gamma^*, y) = 0$ for large y. By the unique continuation theorem $u = 0$ on U_{ext}, hence $f = 0$.

7. Resolvent estimtes for the 1-dimensional problem

We shall prove the limiting absorption principle for $L_0(\zeta)$ defined by (2.11). We prepare a-priori estimates (Lemmas 7.1 $\sim$ 7.4) and the uniqueness result (Lemma 7.6). We fix a compact interval $I \subset (0,\infty)$ and let $J = \{z \in \mathbf{C}; \mathrm{Re}\, z \in I, |\mathrm{Im}\, z| \leq 1\}$. In the following, C denotes a constant independent of $\zeta \geq 0$ and $z \in J$. In this section we take the branch of $\sqrt{z}$ such that Im $\sqrt{z} \geq 0$.

Lemma 7.1. *Let u satisfy*

$$-u'' + (e^{2y}\zeta^2 - z)u = f, \quad \zeta \geq 0.$$

If $u \in L^{2,s}, f \in L^{2,s}$, for some $s \in \mathbf{R}$, we have

$$\|u'\|_s + \|e^y\zeta u\|_s \leq C(\|u\|_s + \|f\|_s).$$

Proof. The proof is based on the following formula

$$\int_a^b \varphi(-u'' + (V_0 - z)u)\overline{u}dy$$

$$= -[\varphi u'\overline{u}]_a^b + \int_a^b \varphi' u'\overline{u}dy + \int_a^b \varphi|u'|^2 dy + \int_a^b \varphi(V_0 - z)|u|^2 dy, \qquad (7.1)$$

where φ is real-valued.

Let $V_0 = e^{2y}\zeta^2$ and take the real part of (7.1). Then we have

$$\int_{-R}^R \varphi|u'|^2 dy + \int_{-R}^R \varphi e^{2y}\zeta^2|u|^2 dy$$

$$= \mathrm{Re} \int_{-R}^R \varphi f\overline{u}dy + \frac{1}{2}\int_{-R}^R \varphi''|u|^2 dy + \mathrm{Re}\, z\int_{-R}^R \varphi|u|^2 dy$$

$$+\mathrm{Re}\, [\varphi u'\overline{u}]_{-R}^R - \frac{1}{2}[\varphi'|u|^2]_{-R}^R.$$

Taking $\varphi = \langle y\rangle^{2s}$, we have

$$\int_{-R}^R \varphi|u'|^2 dy + \int_{-R}^R \varphi e^{2y}\zeta^2|u|^2 dy$$

$$\leq C(\|f\|_s^2 + \|u\|_s^2) + \mathrm{Re}\, [\varphi u'\overline{u}]_{-R}^R - \frac{1}{2}[\varphi'|u|^2]_{-R}^R.$$

The lemma is then proved if we show

$$u \in L^{2,s} \Longrightarrow \liminf_{R\to\infty} R^{2s+1}(|u(R)|^2 + |u(-R)|^2) = 0, \qquad (7.2)$$

$$u \in L^{2,s} \Longrightarrow \liminf_{R\to\infty} [\varphi u'\overline{u}]_{-R}^R \leq 0. \qquad (7.3)$$

The fisrt assertion is obvious. To show (7.3), first note

$$\mathrm{Re}\, [\varphi u'\overline{u}]_{-R}^R = \frac{1}{2}[(\varphi|u|^2)']_{-R}^R - \frac{1}{2}[\varphi'|u|^2]_{-R}^R.$$

As above $\liminf_{R\to\infty} |[\varphi'|u|^2]_{-R}^R| = 0$. If $\liminf_{R\to\infty}[(\varphi|u|^2)']_{-R}^R \geq 2\delta > 0$, there exists $R_0 > 0$ such that

$$[(\varphi|u|^2)']_{-R}^R \geq \delta \quad \text{for} \quad R \geq R_0.$$

Integrate over (R_0, R) to get

$$\varphi(R)|u(R)|^2 + \varphi(-R)|u(-R)|^2 - \varphi(R_0)|u(R_0)|^2 - \varphi(-R_0)|u(-R_0)|^2 \geq \delta(R - R_0).$$

Take $\liminf_{R\to\infty}$ to arrive at a contradiction. ♢

Lemma 7.2. *Let u satisfy*

$$-u'' + (e^{2y}\zeta^2 - z)u = f, \quad \zeta \geq 0.$$

If $u \in L^{2,s}, f \in L^{2,s+1}$, for some $s \in \mathbf{R}$, and $\mathrm{Im}\,\sqrt{z} > 0$, we have

$$\mathrm{Im}\,\sqrt{z}\|u\|_{s+1/2} \leq C(\|u\|_s + \|f\|_{s+1}).$$

Proof. Take $\varphi = \langle y\rangle^{2s+1}$ and compute

$$\mathrm{Im}\int_{-R}^{R}\varphi f\overline{u}dy = -\mathrm{Im}\,[\varphi u'\overline{u}]_{-R}^{R} + \mathrm{Im}\int_{-R}^{R}\varphi' u'\overline{u}dy - \mathrm{Im}\,z\int_{-R}^{R}\varphi|u|^2dy.$$

The lemma then follows from Lemma 7.1. ♢

Let

$$D_{\pm} = \sqrt{z} \pm i\partial_y. \tag{7.4}$$

Lemma 7.3. *Let u satisfy $-u'' + (e^{2y}\zeta^2 - z)u = f$.*
(1) If $\zeta \geq 0, u \in L^{2,-s}, f \in L^{2,s}$ for some $s > 1/2$, Im $\sqrt{z} \geq 0$, and $F(y < -1)D_-u \in L^{2,s-1}$, we have

$$\|F(y < -1)D_-u\|_{s-1} \leq C(\|u\|_{-s} + \|f\|_s).$$

(2) If $\zeta = 0, u \in L^{2,-s}, f \in L^{2,s}$ for some $s > 1/2$, Im $\sqrt{z} \geq 0$, and $F(y > 1)D_+u \in L^{2,s-1}$, we have

$$\|F(y > 1)D_+u\|_{s-1} \leq C(\|u\|_{-s} + \|f\|_s).$$

Proof. The differential equation is rewritten as

$$(D_-u)' = i\sqrt{z}D_-u - ie^{2y}\zeta^2u + if. \tag{7.5}$$

Let $\varphi(y) = 0$ $(y > 0), \varphi(y) = (-y)^{2s-1}$ $(y < -1)$. Multiply (7.5) by $\varphi(y)\overline{D_-u}$, integrate over $(-R, 0)$ and take the real part. Then we have

$$\begin{aligned}-\frac{1}{2}\int_{-R}^{0}\varphi'|D_-u|^2dy &= -\frac{1}{2}[\varphi|D_-u|^2]_{-R}^{0} - \mathrm{Im}\,\sqrt{z}\int_{-R}^{0}\varphi|D_-u|^2dy \\ &\quad -\mathrm{Re}\,i\int_{-R}^{0}\varphi e^{2y}\zeta^2u\overline{D_-u}dy + \mathrm{Re}\,i\int_{-R}^{0}\varphi f\overline{D_-u}dy.\end{aligned}$$

Let $R \to \infty$ (along a suitable sequence). Applying the Schwarz inequality to the last term of the right-hand side, we have

$$C_1\|F(y < -1)D_-u\|_{s-1}^2 \leq C_2(\|f\|_s^2 + \|u\|_{-s}^2) - \mathrm{Re}\,i\int_{-\infty}^{0}\varphi e^{2y}\zeta^2u\overline{D_-u}dy$$

for some constants $C_1, C_2 > 0$. Again by integration by parts we have

$$-\mathrm{Re}\,i\int_{-\infty}^{0}\varphi e^{2y}\lambda u\overline{D_-u}dy = -\mathrm{Im}\,\sqrt{z}\int_{-\infty}^{0}\varphi e^{2y}\zeta^2|u|^2dy - \frac{\zeta^2}{2}\int_{-\infty}^{0}(\varphi e^{2y})'|u|^2dy.$$

There exists $N > 0$ such that $(\varphi e^{2y})' > 0$ for $y < -N$. Therefore this integral is dominated from above by

$$C_3\int_{-\infty}^{0}\zeta^2e^{2y}|u|^2dy \leq C_4(\|u\|_{-s}^2 + \|f\|_s^2),$$

where we have used Lemma 7.1. This proves (1). The assertion (2) is proved similarly. ♢

Lemma 7.4. *Let u satisfy $-u'' + (e^{2y}\zeta^2 - z)u = f$. Suppose* $\mathrm{Im}\ \sqrt{z} > 0$, *and* $u \in L^{2,-s}, f \in L^{2,s}$ *for some* $s > 1/2$. *Let* $s = 1/2 + \epsilon$. *Then*

$$\|F(y < -r)u\|_{-s} + \|F(y > r)u\|_{-s} \leq Cr^{-\epsilon/2}(\|u\|_{-s} + \|f\|_s).$$

Proof. We first show for $y < 0$,

$$-\mathrm{Im}\ u'(y)\overline{u(y)} \leq C\langle y\rangle^{\epsilon}(\|u\|_{-s}^2 + \|f\|_s^2). \tag{7.6}$$

In fact by (7.1)

$$-[u'(\varphi\overline{u})]_{-R}^{y} = \int_{-R}^{y}\varphi f\overline{u}dt - \int_{-R}^{y}(e^{2t}\zeta^2 - z)\varphi|u|^2dt - \int_{-R}^{y}\varphi' u'\overline{u}dt - \int_{-R}^{y}\varphi|u'|^2dt.$$

Taking the imaginary part, we have

$$-\mathrm{Im}\ [u'\varphi\overline{u}]_{-R}^{y} = \mathrm{Im}\int_{-R}^{y}\varphi f\overline{u}dt + \mathrm{Im}\ z\int_{-R}^{y}\varphi|u|^2dt - \mathrm{Im}\int_{-R}^{y}\varphi' u'\overline{u}dt.$$

Put $\varphi = \langle y\rangle^{-1-2\epsilon}$ and let $R \to \infty$. Then by using Lemma 7.2, we have

$$\begin{aligned} & -\mathrm{Im}\ \varphi(y)u'(y)\overline{u(y)} \\ \leq\ & \langle y\rangle^{-1-2\epsilon}\|f\|_s\|u\|_{-s} + \mathrm{Im}\ z\langle y\rangle^{-1-\epsilon}\|u\|_{-\epsilon}^2 + C\langle y\rangle^{-1-\epsilon}\|u'\|_{-1-\epsilon}\|u\|_{-1-\epsilon} \\ \leq\ & C\langle y\rangle^{-1-\epsilon}(\|u\|_{-s}^2 + \|f\|_s^2), \end{aligned}$$

which proves (7.6).

Since

$$(\mathrm{Re}\ \sqrt{z})|u|^2 = \mathrm{Re}\ (D_-u)\overline{u} - \mathrm{Im}\ u'\overline{u},$$

we have by (7.6)

$$(\mathrm{Re}\ \sqrt{z})|u|^2 \leq |D_-u|^2 + C\langle y\rangle^{\epsilon}(\|u\|_{-s}^2 + \|f\|_s^2).$$

Multiply $\langle y\rangle^{-1-2\epsilon}$ and integrate over $(-\infty, -r)$. Then

$$\mathrm{Re}\ \sqrt{z}\int_{-\infty}^{-r}\langle y\rangle^{-1-2\epsilon}|u|^2dy \leq Cr^{-\epsilon}\int_{-\infty}^{-r}\langle y\rangle^{-1-\epsilon}|D_-u|^2dy + Cr^{-\epsilon}(\|u\|_{-s}^2 + \|f\|_s^2).$$

This proves the lemma for $F(y < -r)u$. The estimate for $F(y > r)u$ is proved similarly. ◇

Definition 7.5. A solution u of the equation $-u'' + e^{2y}\zeta^2 u - Eu = f$ $(E > 0)$ is said to satisfy the outgoing radiation condition if for some $0 < \alpha < 1/2 < s$

$$u \in L^{2,-s}, \quad F(\pm y > 0)\Big(i\partial_y \pm \sqrt{E}\Big)u \in L^{2,-\alpha}.$$

Lemma 7.6. *An outgoing solution of $-u'' + e^{2y}\zeta^2 u - Eu = 0$, $E > 0$, vanishes identically.*

Proof. For $R > 2$, pick $\chi_R(y) \in C_0^\infty(\mathbf{R})$ such that $\chi_R(-y) = -\chi_R(y)$, $\chi_R(y) = 0$ if $0 < y < 1$ or $y > R+1$, $\chi_R(y) = 1$ if $2 < y < R$, $\chi_R(y) \geq 0$ for $y \geq 0$. Put

$$\widetilde{\chi}_R(y) = \int_{-\infty}^{y}\chi_R(t)\langle t\rangle^{-2\alpha}dt \in C_0^\infty(\mathbf{R}).$$

Taking the imaginary part of $0 = (-u'' + e^{2y}\zeta^2 u - Eu, \widetilde{\chi}_R u)$, we get

$$\mathrm{Im}\ (u', \chi_R\langle y\rangle^{-2\alpha}u) = 0.$$

Letting $f = u' - i\sqrt{E}(\mathrm{sgn}\ y)u$, we have

$$\sqrt{E}(|\chi_R|\langle y\rangle^{-2\alpha}u, u) = -\mathrm{Im}\ (f, \chi_R\langle y\rangle^{-2\alpha}u).$$

We have, therefore,

$$\sqrt{E}\|\sqrt{|\chi_R|}u\|_{-\alpha} \leq \|\sqrt{|\chi_R|}f\|_{-\alpha}.$$

Letting $R \to \infty$, we get $u \in L^{2,-\alpha}$. Now we use Theorem 9.1 (letting $B(t) = e^{-2t}\zeta^2$ and $V(t) = 0$) to show $u = 0$. ♢

We also say that a solution of the equation $-u'' + e^{2y}\zeta^2 u - Eu = f$ $(E > 0)$ satisfies the *incoming radiation condition* if

$$u \in L^{2,-s}, \quad F(\pm y > 0)\Big(i\partial_y \mp \sqrt{E}\Big)u \in L^{2,-\alpha}$$

for some $0 < \alpha < 1/2 < s$. Lemma 7.6 also holds for the incoming solution.

Theorem 7.7. *For $E > 0, \zeta \geq 0$ and $s > 1/2$, there exists the strong limit*

$$\mathrm{s-}\lim_{\epsilon \downarrow 0} \left(-{\partial_y}^2 + e^{2y}\zeta^2 - E \mp i\epsilon\right)^{-1} \in \mathbf{B}(L^{2,s}; L^{2,-s}). \tag{7.7}$$

Moreover for $f \in L^{2,s}$, $(-{\partial_y}^2 + e^{2y}\zeta^2 - E - i0)^{-1}f$ satisfies the outgoing radiation condition, and $(-{\partial_y}^2 + e^{2y}\zeta^2 - E + i0)^{-1}f$ the incoming radiation condition, with $\alpha = 1 - s$. We have also the uniform estimate

$$\Big\|\Big(- {\partial_y}^2 + e^{2y}\zeta^2 - z\Big)^{-1}\Big\|_{\mathbf{B}(L^{2,s};L^{2,-s})} \leq C, \tag{7.8}$$

$$\Big\|F(\pm y > 0)\Big(i\partial_y \pm \mathrm{Re}\sqrt{z}\Big)\Big(- {\partial_y}^2 + e^{2y}\zeta^2 - z\Big)^{-1}\Big\|_{\mathbf{B}(L^{2,s};L^{2,s-1})} \leq C, \tag{7.9}$$

$$\Big\|F(\pm y > 0)\Big(i\partial_y \mp \mathrm{Re}\sqrt{z}\Big)\Big(- {\partial_y}^2 + e^{2y}\zeta^2 - z\Big)^{-1}\Big\|_{\mathbf{B}(L^{2,s};L^{2,s-1})} \leq C, \tag{7.10}$$

for $\zeta \geq 0$ and $z \in \{0 < a \leq \mathrm{Re}\ z \leq b < \infty,\ 0 \leq \pm\mathrm{Im}\ z \leq 1\}$, a, b being arbitrarily fixed constants, where the inequality (7.9) holds for Im $z \geq 0$ and (7.10) for Im $z \leq 0$.

Proof. Suppose that the uniform bound (7.8) does not hold. Then there exist $\zeta_n \geq 0, z_n \in \mathbf{C}, f_n \in L^{2,s}$ such that $a \leq \mathrm{Re}\ z_n \leq b, 0 \leq \mathrm{Im}\ z_n \leq 1, \|f_n\|_s \to 0, \|u_n\|_{-s} = 1$, where $u_n = (-{\partial_y}^2 + e^{2y}\zeta_n^2 - z_n)^{-1}f_n$. If $\{\zeta_n\}_{n\geq 0}$ is unbounded, one can assume that $\zeta_n \to \infty$. By Lemma 7.1, $u_n \to 0$ in L^2_{loc}. This and Lemma 7.4 imply that $u_n \to 0$ in $L^{2,-s}$. This is a contradiction. Therefore one can assume that $\zeta_n \to \zeta, z_n \to E$. Obviously $E \in \mathbf{R}$. By Lemmas 7.1, 7.4 and Rellich's theorem, one can choose a subsequence u_{n_j} such that $u_{n_j} \to u$ in $L^{2,-s}$. Lemmas 7.1 and 7.3 imply that this u is an outgoing solution of $-u'' + e^{2y}\zeta^2 u - Eu = 0$. Hence $u = 0$ by Lemma 7.6. This is a contradiction.

We next show the existence of the limit (7.7). For $f \in L^{2,s}$, put $u_n = (-{\partial_y}^2 + e^{2y}\zeta^2 - E - i\epsilon_n)^{-1}f_n$ and let $\epsilon_n \to 0$. Arguing as above, one can choose a subsequence $\{u_{n_j}\}$ which converges to u in $L^{2,-s}$. Moreover, this u is an outgoing solution of $-u'' + e^{2y}\zeta^2 u - Eu = f$. Such a solution is unique by virtue of Lemma 7.6. Since any subsequence of $\{u_n\}$ contains a sub-subsequence which converges to one and the same limit, $\{u_n\}$ itself converges in $L^{2,-s}$ without choosing a subsequence. This completes the proof of Theorem 7.7. ♢

8. Limiting Absorption Principle in $\mathbf{H}^n$ and $\Gamma\backslash\mathbf{H}^n$

8.1 *LAP in* $\mathbf{H}^n$. In the first part of this section, we shall study

$$H_0 = -\partial_y^2 - e^{2y}\Delta_x \tag{8.1}$$

in $L^2(\mathbf{R}^{n-1}\times\mathbf{R}^1; dxdy)$. We pass to the Fourier transformation in x and consider

$$L_0(|\xi|^2) = -\partial_y^2 + e^{2y}|\xi|^2, \quad \xi\in\mathbf{R}^{n-1}. \tag{8.2}$$

By virtue of the results in the previous section, we get the estimates uniform in ξ for the resolvent of $L_0(|\xi|^2)$. Let $R_0(z) = (H_0 - z)^{-1}$. For $t, s\in\mathbf{R}$, let $\mathcal{H}^{t,s}$ be the function space defined by

$$u\in\mathcal{H}^{t,s} \Longleftrightarrow \|u\|_{t,s} = \|(1+|x|)^t(1+|y|)^s u(x,y)\|_{L^2(\mathbf{R}^n)} < \infty. \tag{8.3}$$

$\mathcal{H}^{0,s}$ is denoted by $L^{2,s}$. The following theorem follows from Theorem 7.7.

Theorem 8.1. *Let $s > 1/2$. Then*

$$\|R_0(z)\|_{\mathbf{B}(L^{2,s};L^{2,-s})} \leq C,$$

for $z\in\{0 < a < \mathrm{Re}\ z < b, \mathrm{Im}\ z\neq 0\}$, $a, b > 0$ being arbitrarily fixed constants. For any $\lambda > 0$, there exists the weak limit $\mathrm{w}-\lim_{\epsilon\downarrow 0} R_0(\lambda\pm i\epsilon)\in\mathbf{B}(L^{2,s};L^{2,-s})$. *Moreover we have*

$$F(\pm y > 0)(i\partial_y \pm \sqrt{\lambda})R_0(\lambda + i0)\in\mathbf{B}(L^{2,s};L^{2,s-1}),$$
$$F(\pm y > 0)(i\partial_y \mp \sqrt{\lambda})R_0(\lambda - i0)\in\mathbf{B}(L^{2,s};L^{2,s-1}).$$

Let us study the spectral properties of $H = H_0 + V$. We assume V to satisfy (3.6), (3.7), and V_1 to be compactly supported.

Lemma 8.2. *(1) For $z\in\mathbf{C}\setminus\mathbf{R}$, $R_0(z)\in\mathbf{B}(L^2;H^2_{loc})$.*
(2) For any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\|Vu\| \leq \epsilon\|H_0 u\| + C_\epsilon\|u\|, \quad \forall u\in C_0^\infty(\mathbf{R}^n).$$

Proof. The first assertion is obvious. To prove (2), we split $V_1 = V_{1,\epsilon} + V_{2,\epsilon}$ in such a way that $\|V_{1,\epsilon}\|_{L^p} < \epsilon$, $V_{2,\epsilon}\in L^\infty$. Pick $\chi\in C_0^\infty(\mathbf{R}^n)$ such that $\chi = 1$ on supp V_1. Then as is well-known

$$\|V_{1,\epsilon}\chi u\| \leq \epsilon\|(\Delta_x + \partial_y^2)(\chi u)\| + C_\epsilon\|\chi u\|, \quad \forall u\in C_0^\infty(\mathbf{R}^n).$$

To complete the proof we have only to note

$$\|(\Delta_x + \partial_y^2)(\chi u)\| \leq C(\|H_0(\chi u)\| + \|\chi u\|),$$

which follows from the ellipticity of H_0. ♢

By the above lemma, $H\big|_{C_0^\infty}$ is essentially self-adjoint, whose self-adjoint extension is denoted by H again. By Weyl's theorem, we have

Theorem 8.3. $\sigma_e(H) = [0,\infty)$.

Theorem 8.4. *(1) Let $0 < \alpha < 1/2$ and $\lambda > 0$. If $u\in L^{2,-\alpha}$ satisfies $(H-\lambda)u = 0$, then $u = 0$.*
(2) $\sigma_p(H)\cap(0,\infty) = \emptyset$.

Proof. To prove (1), we first note that by applying Theorem 9.1 with $B(t) = -e^{-2t}\Delta_x$, there exists $R_1 > 0$ such that $u = 0$ if $y < -R_1$. Therefore by the unique continuation theorem (see e.g. Jerison-Kenig [15] Theorem 6.3), u vanishes identically. The assertion (2) is a direct cosequence of (1). ♢

Definition 8.5. A solution of u of the equation $(H-\lambda)u = f$ is said to satisfy the outgoing radiation condition if for some $0 < \alpha < 1/2 < s$

$$u \in L^{2,-s}, \quad F(\pm y > 0)(i\partial_y \pm \sqrt{\lambda})u \in L^{2,-\alpha}.$$

The following lemma can be proved in the same way as Lemma 7.6.

Lemma 8.6. *An outgoing solution of $(H-\lambda)u = f, \lambda > 0$, is unique.*

Similary a solution of u of the equation $(H-\lambda)u = f$ is said to satisfy the *incoming* radiation condition if for some $0 < \alpha < 1/2 < s$

$$u \in L^{2,-s}, \quad F(\pm y > 0)(i\partial_y \mp \sqrt{\lambda})u \in L^{2,-\alpha}.$$

Lemma 8.6 also holds for the incoming solution.

Lemma 8.7. *Let $1 < 2s \leq 1+\rho$, and $\epsilon = s-1/2$. Let $u \in L^{2,-s}$ satsify $(H-\zeta^2)u = f \in L^{2,s}$, where $\mathrm{Re}\,\zeta > 0,\ \mathrm{Im}\,\zeta \geq 0$. Then for any $\delta > 0$*
(1) $\|F(\pm y > 0)(i\partial_y \pm \mathrm{Re}\,\zeta)u\|_{s-1} \leq C(\|u\|_{-s} + \|f\|_s)$,

(2) $\|F(|x| + |y| > r)u\|_{\mathcal{H}^{-\delta/2,-s}} \leq Cr^{-\min(\delta,\epsilon)/2}(\|u\|_{-s} + \|f\}_s)$,

where the costants C do not depend on $r > 1$ and ζ when ζ varies over $\{0 < a < \mathrm{Re}\,\zeta < b,\ 0 < \mathrm{Im}\,\zeta < 1\}$, a, b being arbitrarily fixed constants.

Proof. When $V = 0$, the assertion (1) follows from Lemma 7.3 by passing to the Fourier transform in x. When V is present, we have

$$\|F(\pm y > 0)(i\partial_y \pm \mathrm{Re}\,\zeta)u\|_{s-1} \leq C(\|u\|_{-s} + \|f\|_s + \|Vu\|_s).$$

By the decay assumption on V, we have $\|Vu\|_s \leq C\|u\|_{-s}$, which proves (1).

By the same computation as in Lemma 7.4, we have

$$\|F(|y| > r)u\|_{-s} \leq Cr^{-\epsilon/2}(\|u\|_{-s} + \|f\|_s).$$

It is easy to see

$$\|F(|x| > r)\langle x\rangle^{-\delta/2}\langle y\rangle^{-s}u\| \leq Cr^{-\delta/2}\|u\|_{-s}.$$

These two inequalities yield (2). ♢

Theorem 8.8. *For $\lambda > 0, \delta > 0$ and $s > 1/2$, there exists the strong limit*

$$\mathrm{s} - \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon) \quad \text{in} \quad \mathbf{B}(\mathcal{H}^{0,s}; \mathcal{H}^{-\delta,-s}). \tag{8.4}$$

Moreover there exits the weak limt

$$\mathrm{w} - \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon) \quad \text{in} \quad \mathbf{B}(L^{2,s}; L^{2,-s}), \tag{8.5}$$

and for $f \in L^{2,s}$, $R(\lambda+i0)f$ is outgoing and $R(\lambda-i0)f$ is incoming, with $\alpha = 1-s$. For any $0 < a < b < \infty$, there exists a constant $C > 0$ such that

$$\|R(\lambda \pm i0)\|_{\mathbf{B}(L^{2,s};L^{2,-s})} \leq C, \tag{8.6}$$

$$\|F(\pm y > 0)(i\partial_y \pm \sqrt{\lambda})R(\lambda + i0)\|_{\mathbf{B}(L^{2,s};L^{2,s-1})} \leq C, \tag{8.7}$$

$$\|F(\pm y > 0)(i\partial_y \mp \sqrt{\lambda})R(\lambda - i0)\|_{\mathbf{B}(L^{2,s};L^{2,s-1})} \leq C, \tag{8.8}$$

for $a < \lambda < b$.

Proof. The proof of (8.4) is almost the same as Theorem 7.7. The assertion (8.5) follows from Therorem 8.1 and the resolvent equation. The assertion (8.6) is proven in the same way as Theorem 7.7 in $\mathbf{B}(\mathcal{H}^{0,s};\mathcal{H}^{-\delta,-s})$, where δ is chosen sufficiently small. We then use the resolvent equation. ♢

8.2 *LAP in* $\Gamma\backslash\mathbf{H}^n$. Let $H_0(\theta)$ and $H(\theta)$ be as in §4. The limiting absorption principle in $\Gamma\backslash\mathbf{H}^n$ is proven in the same way as above with the Fourier transformation replaced by the Fourier series. In this case we have only to use the space $L^{2,s}$ with the weight $\langle y\rangle^s$ only.

8.3 *LAP in the exterior domain of* $\Gamma\backslash\mathbf{H}^n$. Finally we prove LAP for $H_{ext}(\theta) = H_0(\theta)$ in the exterior domain $U_{ext} = \mathbf{E}\times\mathbf{R}\setminus\overline{U}$ with Dirichlet boundary condition. The proof is the same as above if one notices the following facts.

Let $u = (H_{ext}(\theta) - z)^{-1}$ and let $\chi(x,y) \in C^\infty(\mathbf{E}\times\mathbf{R})$ be such that $\chi = 1$ if $|y| > R+1$, $\chi = 0$ if $|y| < R$, where R is taken so large that $U \subset \{|y| < R + 1/2\}$. Then applying the results in $\mathbf{E}\times\mathbf{R}$ to χu, one can see that Lemma 8.7 also holds for this case. The uniqueness of the outgoing or incoming solution is proven by Theorem 9.1. We have thus shown that LAP also holds in U_{ext}.

9. Growth properties of solutions to the homogeneous equation

Let X be a Hilbert space and consider the following differential equation for an X-valued function $u(t)$ of $t > 0$:

$$-u''(t) + B(t)u(t) + V(t)u(t) - Eu(t) = f(t),$$

E being a positive constant. We assume that

(A-1) For each fixed t, $B(t)$ is non-negative, self-adjoint on X with domain $D(B(t)) = D$ independent of t. For each $x \in D$, the map $(0,\infty) \ni t \to B(t)x \in X$ is C^1 and there exists $t_0 \geq 0$ such that

$$t\frac{dB(t)}{dt} + B(t) \leq 0, \quad \forall t \geq t_0.$$

(A-2) For each t, $V(t)$ is bounded, self-adjoint on X. Further $V(t) = V_L(t) + V_S(t)$, where

$$V_L(t) \in C^1((0,\infty);\mathbf{B}(X)), \quad V_S(t) \in C^0((0,\infty);\mathbf{B}(X)),$$

and there exist $C, \epsilon > 0$ such that

$$\|V_L(t)\| \leq C(1+t)^{-\epsilon}, \quad \|\frac{dV_L(t)}{dt}\| + \|V_S(t)\| \leq C(1+t)^{-1-\epsilon}, \quad \forall t \geq 0.$$

(A-3) $\|f(t)\|_X \leq C(1+t)^{-1-\epsilon}\|u(t)\|_X, \quad \forall t \geq 0.$

Theorem 9.1. *Assume (A-1), (A-2), (A-3). If*

$$\liminf_{t\to\infty}(\|u'(t)\|_X + \|u(t)\|_X) = 0,$$

there exists $t_1 > 0$ *such that* $u(t) = 0$ *for* $\forall t > t_1$.

This theorem is proved in the same way as in pp. 29 - 35 of [30] with a slight modification. By virtue of our assumption (A-1), we have

$$-(\frac{dB(t)}{dt}v, v)_X \geq \frac{1}{t}(B(t)v, v)_X. \tag{9.1}$$

We use the notation of [30] and follow his computation using (9.1). Then the inequality (3.5) in [30] reads

$$\frac{d}{dr}(Kv) \geq -2c_2(1+r)^{-2\delta}|v|_X|v'|_X + \frac{1}{r}(Bv,v)_X - (C_0'v,v)_X.$$

Therefore his Lemma 3.2 also hlods in our case. His formula (3.13) reads

$$\begin{aligned}\frac{d}{dr}(Nv) &= (4mr^{1-\mu}+1)|w'|_X^2 + \{k^2 + (1-2\mu)r^{-2\mu}(m^2 - \log r) - r^{-2\mu}\}|w|_X^2 \\ &- ((C_0 + rC_0')w, w)_X + 2re^d \,\mathrm{Re}\,(C_1 v - f, w')_X \\ &- 2(\mu m r^{-\mu} + r^{1-2\mu}\log r)\,\mathrm{Re}\,(w,w')_X - ((r\frac{dB(r)}{dr} + B(r))w,w)_X.\end{aligned}$$

Dropping the last term, we see that his inequality (3.14) also holds in our case. The remaining argument is entirely the same.

10. Inverse scattering at the cusp

Let $\mathcal{M}$ be an n-dimesional connected Riemannian manifold. Suppose $\mathcal{M}$ consists of two parts : $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_\infty$, where $\overline{\mathcal{M}_0}$ is compact, and $\mathcal{M}_\infty$ is diffeomorphic to $\mathbf{E}\times(1,\infty)$, $\mathbf{E} = \Gamma\backslash\mathbf{R}^{n-1}$, Γ being a lattice of rank $n-1$ in $\mathbf{R}^{n-1}$. We assume that the Riemannian metric g of $\mathcal{M}$, when restricted to $\mathcal{M}_\infty$, takes the following form :

$$g\big|_{\mathcal{M}_\infty} = (dy)^2 + e^{-2y}(dx)^2, \qquad (10.1)$$

where $y \in (1,\infty)$ and $(dx)^2$ is the flat metric on $\mathbf{E}$. We consider the Schrödinger operator

$$H = -\Delta_g + A, \qquad (10.2)$$

where A is a formally self-adjoint 2nd order differential operator. We assume that for $j = 1,2$ the coefficients of j-th covariant derivatives are in C^j, and that the multiplication operator term is bounded. Moreover we assume the following.

The supports of the coefficients of A are contained in a bounded contractible set in $\mathcal{M}$.

We impose a suitable boundary condition on $\partial\mathcal{M}$ (if it exists) so that H is self-adjoint. It can be shown that

Theorem 10.1. *(1)* $\sigma_e(H) = [(n-1)^2/4,\infty)$.
(2) $\sigma_p(H)\cap((n-1)^2/4,\infty)$ *is discrete.*

We fix a point $P_0 \in \mathcal{M}$ arbitrarily and let ρ be the geodesic distance from P_0. For $s \in \mathbf{R}$, we define the function space $L^{2,s}$ by

$$f \in L^{2,s} \Longleftrightarrow \|f\|_s^2 = \int_{\mathcal{M}} \langle\log(1+\rho)\rangle^{2s}|f|^2 d\mathcal{M}. \qquad (10.3)$$

A solution u of the equation $(H-\lambda)u = f$ is said to satisfy the outgoing (or incoming) radiation condition if

$$u \in L^{2,-s}, \quad F(\rho>1)\big(\frac{\partial}{\partial r} \mp ik(\lambda)\big)u \in L^{2,-\alpha}, \quad k(\lambda) = \sqrt{\lambda - \frac{(n-1)^2}{4}} \qquad (10.4)$$

for some $0<\alpha<1/2<s$, where $r = \log\rho$ and we choose the $-$ sign for outgoing and the $+$ sign for incoming radiation condition. Let $R(z) = (H-z)^{-1}$.

Theorem 10.2. *Let I be a compact interval in $((n-1)^2/4,\infty)\setminus\sigma_p(H)$. Then for $1/2 < s < 1$, there exists a constant $C > 0$ such that for $\lambda \in I$*

$$\|R(\lambda \pm i0)f\|_{-s} \leq C\|f\|_s,$$

$$\|F(\rho > 1)(\frac{\partial}{\partial r} \mp ik(\lambda))R(\lambda \pm i0)f\|_{s-1} \leq C\|f\|_s.$$

For $\lambda \in ((n-1)^2/4,\infty)\setminus\sigma_p(H)$, the solution of the equation $(H-\lambda)u = f$ satisfying the outgoing or incoming radiation condition is unique.

Sketch of the proof. We use the coordinates (x, y) in §4. Let $\chi(y) \in C^\infty$ be such that $\chi(y) = 1$ $(y > R_0 + 1)$, $\chi(y) = 0$ $(y < R_0)$. By taking R_0 large enough, $\chi(y)R(z)f$ satisfies

$$(-\partial_y^2 - e^{2y}\Delta_x - z + \frac{(n-1)^2}{4})\chi(y)R(z)f = \chi(y)f - [\partial_y^2, \chi(y)]R(z)f.$$

Therefore one can apply the results in §8 to $\chi(y)R(z)f$. The remaining part is treated as compact perturbation. The first part of the theorem is then reduced to the second half.

To prove the second part, let u be the outgoing solution of the equation $(H-\lambda)u = 0$. Pick $\chi_R(y) \in C^\infty(\mathbf{R})$ such that $\chi_R(y) = 1$ $(y < R), \chi_R(y) = 0$ $(y > R+1)$ and put

$$\widetilde{\chi}_R(y) = \int_y^\infty \chi_R(t)\langle t\rangle^{-2\alpha}dt \in C_0^\infty((1,\infty)),$$

Taking the imaginary part of

$$0 = ((-\partial_y^2 - e^{2y}\Delta_x - k(\lambda)^2)u, \widetilde{\chi}_R u),$$

we have Im $(\partial_y u, \widetilde{\chi}'_R(y)u) = 0$. Hence

$$\text{Im } (\partial_u y, \chi_R(y)\langle y\rangle^{-2\alpha}u) = 0,$$

Since $\partial_y u - ik(\lambda)u \in L^{2,-\alpha}$, we then see that $\|\sqrt{\chi_R(y)}u\|_{-\alpha} \leq C$, with C independent of $R > 0$. Letting $R \to \infty$, we get $u \in L^{2,-\alpha}$.

We next expand u into a Fourier series in x. For $\gamma^* \in \Gamma^*$, $\hat{u}(\gamma^*, y)$ satisfies the equation

$$(-\partial_y^2 + e^{2y}|\gamma^*|^2 - k(\lambda)^2)\hat{u}(\gamma^*, y) = 0$$

for large y. Comparing its behavior with that of modified Bessel functions, we see that $\hat{u}(\gamma^*, y) \in L^2$. Therefore u is an L^2-eigenfunction of H, hence it vanishes identically. $\diamondsuit$.

We next introduce the exponentially growing solution at the cusp. Let $\mathcal{F}_{0c}(\lambda)$ be as in §4. Pick $\chi(y) \in C^\infty$ such that $\chi(y) = 1$ $(y > 3), \chi(y) = 0$ $(y < 2)$ and put

$$\widetilde{\mathcal{F}}_{0c}(\lambda)^* = \chi(y)\mathcal{F}_{0c}^{(\pm)}(k(\lambda)^2)^* - R_0(\lambda \pm i0)(-\Delta_g - \lambda)\chi(y)\mathcal{F}_{0c}^{(\pm)}(k(\lambda)^2)^*, \quad (10.5)$$

where $R_0(z) = (-\Delta_g - z)^{-1}$. Here we must assume that $\lambda \notin \sigma_p(-\Delta_g)$. Then for $\varphi \in L^2(\mathbf{E})$

$$(-\Delta_g - \lambda)\widetilde{\mathcal{F}}_{0c}^{(\pm)}(\lambda)^*\varphi = 0 \quad (10.6)$$

and $\widetilde{\mathcal{F}}_{0c}^{(\pm)}(\lambda)^*\varphi$ is exponentially growing at the cusp. We let

$$u = \widetilde{\mathcal{F}}_{0c}^{(-)}(\lambda)^*\varphi - R(\lambda + i0)(A\widetilde{\mathcal{F}}_{0c}^{(-)}(\lambda)^*\varphi). \quad (10.7)$$

Then u satisfies

$$(H - \lambda)u = 0. \quad (10.8)$$

Moreover we have

$$F(\rho > 1)\big(\frac{\partial}{\partial r} - ik(\lambda)\big)(u - \widetilde{\mathcal{F}}_{0c}(\lambda)^*\varphi) \in L^{2,-\alpha} \tag{10.9}$$

for some $0 < \alpha < 1/2$. Such a solution is unique by virtue of Theorem 10.2. As in Theorem 4.11, we have the following asymptotic expansion.

Theorem 10.3. *For $\varphi \in L^2(\mathbf{E})$, we have as $\rho \to \infty$*

$$\langle e^{-i\gamma^*\cdot x}, u - \widetilde{\mathcal{F}}^{(-)}_{0c}(\lambda)^*\varphi\rangle \sim -\sqrt{\frac{\pi}{2}}|\gamma^*|^{-1/2}\rho^{-1/2}e^{-|\gamma^*|\rho}\mu_{\gamma^*}\mathbf{A}_{c\gamma^*}(\lambda)\varphi \quad (\gamma^* \neq 0)$$

$$\langle e^{-i\gamma^*\cdot x}, u - \widetilde{\mathcal{F}}^{(-)}_{0c}(\lambda)^*\varphi\rangle \sim i\frac{\sqrt{2\pi}}{k(\lambda)}\rho^{iyk(\lambda)}\mathbf{A}_{c\gamma^*}(\lambda)\varphi \quad (\gamma^* = 0),$$

$\{\mathbf{A}_{c\gamma^*}(\lambda)\}_{\gamma^*\in\Gamma^*}$ being bounded in $\mathbf{B}(L^2(\mathbf{E}), \mathbf{C})$.

As in §4

$$\mathbf{A}_c(\lambda) = \sum_{\gamma^*\in\Gamma^*} e^{i\gamma^*\cdot x}\mathbf{A}_{c\gamma^*}(\lambda) \tag{10.9}$$

defines the *scattering amplitude at the cusp*.

Take a bounded contractible domain $\Omega \subset \mathcal{M}$ such that $A = 0$ outside Ω, and define the D-N map $\Lambda(A)$ for $H_D = -\Delta_g + A$ in Ω with Dirichlet boundary condition. Then by the same arguments as in §6, we have

Theorem 10.4. *Suppose $\lambda \notin \sigma_p(H) \cup \sigma_p(-\Delta_g) \cup \sigma_p(H_D)$. Then the scattering amplitude at the cusp $\mathbf{A}_c(\lambda)$ and the D-N map $\Lambda(A)$ determine each other.*

With the aid of Theorem 10.4, one can argue the reconstruction of the local perturbation of the metric. Let $\mathcal{M}$ be an n-dimensional hyperbolic manifold with a cusp. Take a bounded contractible domain Ω in $\mathcal{M}$. If $n \geq 3$, the conformal deformation of the metric in Ω can be reconstructed from the scattering amplitude at the cusp by using the result of Sylvester-Uhlmann [36] and Nachman [25]. If $n = 2$, one can deal with the general perturbation of the metric and reconstruct $\sqrt{\det(g_{ij})}g^{ij}$ by virtue of the result of Nachman [26]. For two metrics g and $\overline{g}$, $\sqrt{\det(g_{ij})}g^{ij} = \sqrt{\det(\overline{g}_{ij})}\overline{g}^{ij}$ is equivalent to that g and $\overline{g}$ are conformal. Therefore the coincidence of the scattering amplitudes assocaited with g and $\overline{g}$ is equivalent to the conformality of g and $\overline{g}$ (see also [23]). One can also treat the case of many cusps.

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