INVERSE BOUNDARY VALUE PROBLEMS IN THE HOROSPHERE — A LINK BETWEEN HYPERBOLIC GEOMETRY AND ELECTRICAL IMPEDANCE TOMOGRAPHY

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Abstract. We consider a boundary value problem for the Schrödinger operator $-\Delta + q(x)$ in a ball $\Omega : (x_1 + R)^2 + x_2^2 + (x_3 - r)^2 < r^2$, whose boundary we regard as a horosphere in the hyperbolic space $\mathbf{H}^3$ realized in the upper half space. Let $S = \{|x| = R, x_3 > 0\}$ be a hemisphere, which is generated by a family of geodesics in $\mathbf{H}^3$. By imposing a suitable boundary condition on $\partial\Omega$ in terms of a pseudo-differential operator, we compute the integral mean of $q(x)$ over $S \cap \Omega$ from the local knowledge of the associated (generalized) Robin-to-Dirichlet map for $-\Delta + q(x)$ around $S \cap \partial\Omega$. The potential $q(x)$ is then reconstructed by virtue of the inverse Radon transform on hyperbolic space. This justifies the well-known Barber-Brown algorithm in electrical impedance tomography.

1. Introduction

1.1. **Inverse problem for electric conductivity.** In electrical impedance tomography (EIT), one seeks to reconstruct the conductivity of a body $\Omega \subset \mathbf{R}^n$ from the boundary measurement of voltage potentials and corresponding current fluxes. Mathematically the problem amounts to reconstructing a positive function $\gamma(x)$ from the Neumann-to-Dirichlet map, called the ND map hereafter,

$$\Lambda : f \to u|_{\partial\Omega}, \tag{1.1}$$

where u is a solution to the boundary value problem

$$\begin{cases} \nabla \cdot (\gamma(x)\nabla u) = 0 & \text{in} \quad \Omega, \\ \gamma(x)\dfrac{\partial u}{\partial n} = f & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

n is the outer unit normal to $\partial\Omega$ and the condition that $\int_{\partial\Omega} f dS = 0$ is assumed. Let us remark that instead of the Neumann-to-Dirichlet map, one often uses the Dirichlet-to-Neumann map which assigns the Neumann data $\gamma(x)\partial u/\partial n$ to the prescribed Dirichlet data $u\big|_{\partial\Omega}$. There is already an extensive literature dealing with this problem. In the late 1980's, it was proved that $\gamma(x)$ is uniquely determined from Λ (see Sylvester-Uhlmann [29], Nachman [25], [26], Khenkin-Novikov [21]) by using the method of complex geometrical optics or the $\overline{\partial}$-theory, and the numerical implementation of this idea has been tried by Siltanen, Müller, Isaacson, Newell [28], [14] and Knudsen [22]. Besides these theoretical developments, approximate

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reconstruction procedures for $\gamma(x)$ had already been widely studied because of their practical importance.

1.2. **Hyperbolic space structure in the background.** Among them, we are interested in the approach proposed by Barber and Brown [1], [24], the applied potential tomography system. This method is known to be efficient despite its low numerical cost and is regarded as the most practical commercial EIT system so far. (See e.g. Cheney-Isaacson-Newell [8]. For the review of recent developments of EIT technique, see also Borcea [6] and Holder [13].) Moreover from the theoretical view point, it was noticed by Santosa and Vogelius [27] that this Barber-Brown algorithm is a sort of inverse Radon transform on hyperbolic space. Let us briefly recall their arguments. They consider the 2-dimensional case, assuming that Ω is a unit disc : $\Omega = \{|x| < 1\}$, and that $\gamma(x)$ is a small perturbation of a constant $\gamma_0 > 0$:

$$\gamma(x) = \gamma_0 + \epsilon\gamma_1(x) + \cdots .$$

Taking a point $\omega = (\omega_1, \omega_2) \in \partial\Omega$ and letting $\omega^\perp = (-\omega_2, \omega_1)$, they linearize the equation (1.2) around a solution

$$u_0(x) = \frac{\omega^\perp \cdot x}{(\omega^\perp \cdot x)^2 + (1 - \omega \cdot x)^2}, \tag{1.3}$$

which solves (1.2) with $\gamma(x) = \gamma_0$. Namely they look for the solution of the form

$$u(x) = u_0(x) + \epsilon u_1(x) + \cdots .$$

The Barber-Brown algorithm proposes as an approximation of the conductivity increment $\gamma_1(x)$ an integral mean of some quantity $\varphi(x, \omega)$, which is computed from the measured data of $u_1(x)$, with respect to $\omega \in S^1$:

$$\gamma_1(x) \approx \int_{S^1} \varphi(x, \omega)\rho_1 d\omega \tag{1.4}$$

with a suitable density ρ_1. Santosa and Vogelius observe that $\varphi(x, \omega)$ is written, in a crude sense, by a convolution operator K and an integral of $\gamma_1(x)$ along a circle C orthogonal to $\partial\Omega$ at ω :

$$\varphi(x, \omega) \approx K\left(\int_C \gamma_1(x)\rho_2 d\sigma\right) \tag{1.5}$$

ρ_2 being a suitable density. Plugging these two formulas (1.4) and (1.5) into the form

$$\gamma_1 \approx R^* K R \gamma_1, \tag{1.6}$$

where R is an integral opetrator

$$Rf = \int_C f(x)\rho_2 d\sigma, \tag{1.7}$$

they conclude that this procedure is essentially an inversion formula for the generalized Radon transform in the sense of Beylkin [5]. Let us also notice that the background solution $u_0(x)$ is singular at $\omega \in \partial\Omega$, in particular $u_0 \notin L^2(\Omega)$.

If we regard Ω as the Poincaré disc, the circle C is a geodesic in hyperbolic space. Therefore the observation of Santosa and Vogelius suggests a deep connection between hyperbolic geometry and the inverse boundary value problem for electric conductivity (1.2). Indeed, Berenstein and Tarabusi [4] analyzed the argument of Santosa and Vogelius further and found that the Barber-Brown procedure could be derived by modifying the exact inversion formula of the Radon transform on hyperbolic space. This settles the relation between the Barber-Brown algorithm and the hyperbolic Radon transform. However the whole mathematical background of the above procedure, especially its relation to the partial differential equation (1.2) itself, has not yet been clarified so far. The aim of the present paper is to study the full non-linear inverse problem for (1.2) in a ball in $\mathbf{R}^3$, by modifying the boundary condition in a suitable manner, and to elucidate the role of hyperbolic geometry in the inverse boundary value problem.

We start with the well-known remark : By the substitution $u = \gamma^{-1/2}v$, the inverse problem for the conductivity equation (1.2) is transformed into the one for the Schrödinger operator $-\Delta + q$ with $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. Therefore we shall consider the inverse boundary value problem for the Schrödinger operator in a ball in $\mathbf{R}^3$.

1.3. **Sketch of results.** First let us briefly summarize the results of this paper ignoring the details. Throughout the paper the potential $q(x)$ is allowed to be complex-valued. In the following Theorems 1.1 and 1.2, we think of a pair of a ball Ω and a sphere S, the latter being orthogonal to $\partial\Omega$ (see Figure 1).

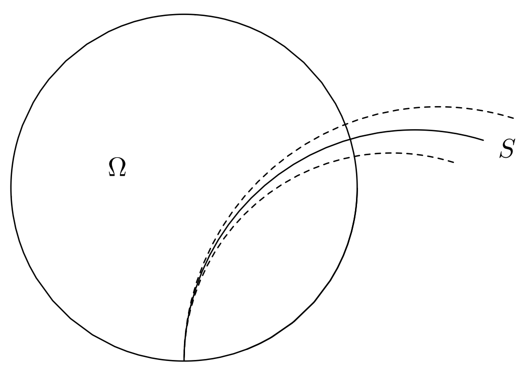

FIGURE 1

Theorem 1.1. *There exists a pseudo-differential operator $P(\tau)$ on $\partial\Omega$ depending on a large real parameter τ such that the boundary value problem*

$$\begin{cases} (-\Delta + q)u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial n} = P(\tau)u + f & \text{on} \quad \partial\Omega \end{cases}$$

has a unique solution u.

Let

$$\mathcal{R}_q(\tau) : f \to u\big|_{\partial\Omega}$$

be the associated generalized Robin-to-Dirichlet (GRD) map.

Theorem 1.2. *There exists a boundary data $f(\tau)$ such that along some sequence $\{\tau_n\}_{n\geq 1}$*

$$\lim_{n\to\infty}\Big(\big(\mathcal{R}_q(\tau_n) - \mathcal{R}_0(\tau_n)\big)f(\tau_n), f(-\tau_n)\Big)_{\partial\Omega} = i\int_{S\cap\Omega} q(x)dS$$

holds, where $(\ ,\ )_{\partial\Omega}$ denotes the inner product on $L^2(\partial\Omega)$ and S is a sphere intersecting orthogonally with $\partial\Omega$.

The boundary operator $P(\tau)$ is written explicitly in terms of hyperbolic functions ((2.19), (4.7), (4.9)). The sequence $\{\tau_n\}_{n\geq 1}$ is also explicit (Theorem 4.8) and so is the boundary data $f(\tau)$ (Theorem 4.7 and (4.54)). The crucial fact is that $P(\tau), \{\tau_n\}_{n\geq 1}$ and $f(\tau)$ do not depend on the potential $q(x)$. Moreover the support of $f(\tau)$ concentrates around the circle $S\cap\partial\Omega$ (Corollary 4.10), and $|f(\tau)|^2 d\Sigma$, $d\Sigma$ being a measure on $\partial\Omega$, converges to a measure supported on $S\cap\partial\Omega$ ((4.43)). This assures that, in spite of the non-local property of $P(\tau)$, the local knowledge around $S\cap\partial\Omega$ of the GRD map is sufficient to compute the integral mean of $q(x)$. In fact, we have the following theorem.

Theorem 1.3. *Let $\chi_\epsilon \in C^\infty(\mathbf{R}^3)$ be such that its support is contained in an ϵ-neighborhood of S and $\chi_\epsilon = 1$ on a smaller neighborhood of S. Then we have*

$$\lim_{n\to\infty}\Big(\big(\mathcal{R}_q(\tau_n) - \mathcal{R}_0(\tau_n)\big)\chi_\epsilon f(\tau_n), \chi_\epsilon f(-\tau_n)\Big)_{\partial\Omega} = i\int_{S\cap\Omega} q(x)dS$$

One can also take χ_ϵ depending on n so that its support is contained in the curved sector between the dashed spheres in Figure 1 and shrinks to the sphere S as $n\to\infty$ (Theorem 5.5 and (5.24)).

Theorem 1.4. *One can reconstruct $q(x)$ from its spherical mean by virtue of the inversion formula of the Radon transform on hyperbolic space.*

1.4. **Converting the problem into the horosphere.** Although the above results are stated in a Euclidean ball, the idea used in the proof is considerably different from the usual analysis for $-\Delta$ in a bounded domain. The main tactics are :

(i) We embed the problem in $\mathbf{H}^3$ and consider the boundary value problem in the horosphere.

(ii) Using hyperbolic isometry, we convert the problem into a half-space.

(iii) By a gauge transformation, we introduce a large parameter τ in the equation.

(iv) We construct special solutions of the Schrödiger equation adapted to our purpose, and then look for the boundary operator and function spaces appropriate to deal with them.

Let us enter into more details. The ND map depends largely on the Hilbert space structure in which the operator $-\Delta + q$ is defined. This Hilbert space structure is not given a-priori, but should be chosen in such a way that the measurement we are trying, more exactly the boundary data, is realized in a proper mathematical setting. Therefore our strategy is as follows. We first construct a sequence of solutions of Schrödinger equation having certain properties appropriate for the reconstruction and then introduce a suitable function space to deal with them. The counter part of the ND map, i.e. the measurement, is then defined in a way adapted to the boundary data.

Consequently, our basic framework is different from the standard one in the following respects :

(a) We deal with the equation $(-\Delta + q(x))u = 0$ not in the usual $L^2(\Omega; dx)$ but in $L^2(\Omega; \rho_\tau(x)dx)$, namely in the L^2-space equipped with a τ-depedent measure.

(b) The boundary condition is not the standard Neumann condition.

As a first step, we assume that our domain Ω is defined by

$$\Omega = \left\{ x \in \mathbf{R}^3; (x_1 + R)^2 + x_2^2 + \left(x_3 - \frac{R}{\delta}\right)^2 < \left(\frac{R}{\delta}\right)^2 \right\} \tag{1.8}$$

with arbitrarily given positive constants $R, \delta > 0$. For this domain we associate the sphere S given by

$$S = \{x \in \mathbf{R}^3; |x| = R\}. \tag{1.9}$$

Regarding $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3; x_3 > 0\}$ as the 3-dimensional hyperbolic space $\mathbf{H}^3$, we see that $\partial\Omega$ is a *horosphere* (for this and related terminologies from hyperbolic geometry, see e.g. [3]) and $S \cap \{x_3 > 0\}$ is generated by a family of *geodesics*. Namely S is a totally geodesic submanifold of $\mathbf{H}^3$. Let $\Delta = \sum_{i=1}^3 (\partial/\partial x_i)^2$ be the Euclidean Laplacian. For a solution u of the equation $(-\Delta + q)u = 0$ in Ω, we put $v = x_3^{1/2} u$. Then v satisfies

$$\big(- x_3^2 \Delta + x_3 \partial_3 - \frac{3}{4} + x_3^2 q(x) \big) v = 0, \quad \partial_3 = \partial/\partial x_3. \tag{1.10}$$

Since $-x_3^2\Delta + x_3\partial_3$ is the Laplace-Beltrami operator on $\mathbf{H}^3$, we are now led to a boundary value problem for the Schrödinger operator in the horosphere.

In the next step we use a suitable hyperbolic isometry to transform Ω into the half-space $D_\delta = \{(y_1, y_2, y_3)\, ;\, y_3 > \delta\}$ and S into the plane $\Pi = \{y_1 = 0\}$. The equation (1.10) is invariant under hyperbolic isometry.

In the third step we consider a gauge transformation of the equation (1.10) in D_δ, which is equivalent to introducing a function space with exponential weight and realizing the differential operator on the resulting Hilbert space. We then construct a solution $u(\tau)$ of the Schrödinger equation (1.10) containing a large parameter $\tau \in \mathbf{R}$, which is exponentially increasing with respect to τ in the half-space $\{\mathrm{sgn}(\tau) y_2 < 0\}$, and exponentially decreasing in the opposite half-space $\{\mathrm{sgn}(\tau) y_2 > 0\}$.

The key fact is that this solution $u(\tau)$ satisfies the boundary condition

$$\frac{\partial}{\partial y_3}u(\tau) = P(\tau)u(\tau) + f(\tau) \quad \text{on} \quad \partial D_\delta = \{y_3 = \delta\}, \tag{1.11}$$

where $P(\tau)$ is a pseudo-differential operator on the boundary. Furthermore, the operator $P(\tau)$ and the function $f(\tau)$ do not depend on the potential $q(x)$. We have thus arrived at the boundary value problem for the Schrödinger equation (1.10) in D_δ with boundary condition (1.11). This problem is uniquely solvable for large $|\tau|$, hence we can regard the above $u(\tau)$ as a unique solution of the boundary value problem with $f(\tau)$ as the inhomogeneous term. We can then define the (generalized) Robin to Dirichlet map and use $u(\tau)\big|_{\partial D_\delta}$ as the result of the measurement. By observing $u(\tau)\big|_{\partial D_\delta}$ we get the integral of the potential $q(x)$ over the plane Π. By transforming back to Ω, we obtain the corresponding result in the horosphere.

The precise conditions and conclusions will be stated first in the half-space case in §4 (Theorems 4.2, 4.8, 4.11). We next rewrite them in the case of horosphere in §5 (Theorems 5.3, 5.4, 5.5).

Returning to our original Schrödinger operator $-\Delta + q(x)$ defined in a ball $B \subset \mathbf{R}^3$, we have now obtained the integral

$$\int_{S\cap B} q(x)dS_E, \tag{1.12}$$

where S is an arbitrary sphere which intersects orthogonally with ∂B, and dS_E is the measure on S induced from the Euclidean metric $(dx)^2$. The inverse Radon transform on the hyperbolic space then enables us to reconstruct $q(x)$ from the measurement by this hyperbolic space approach. We shall discuss this matter in detail in §6.

1.5. **Related works.** Let us remark that all the reconstruction procedures for the full non-linear inverse boundary value problem known so far pass the problem to the inverse scattering and use the Faddeev scattering amplitude [10] (see e.g [15]), except for the boundary control method established by Belishev and Kurylev [2], which, however, uses more information than the ND map (see also Katchalov-Kurylev-Lassas [20]). Apparently, our approach deals with the problem within the framework of the boundary value problem in a bounded domain. However, converting the problem into the horosphere results in a non-compactification of the domain and causes a big change of the property of the spectrum of the Laplacian. Thus even in this approach we again make use of an analogue of Faddeev type Green operator. The idea of using hyperbolic space as a tool for solving the inverse problem was introduced in [16]. Greenleaf and Uhlmann [11], in the Euclidean case, showed that the coincidence of the DN map implies the coincidence of the integral mean of the potential over a plane using exponentially growing boundary data. This result was extended to the hyperbolic space case in [18]. Although their data depends on the potential, these works inspired our approach. The inverse problem in the Euclidean half-space has been studied extensively by e.g. Cheney-Isaacson

[7], Eskin-Ralston [9], Karamyan [19]. The corresponding problem in the hyperbolic half-space (i.e. horoball) has a different feature in that the exact counterpart of the scattering amplitude does not exist because of the fast decay of the Green operator as $x_3 \to \infty$.

1.6. **Notation.** For a domain D, $L^2(D;d\mu)$ denotes the space of L^2-functions on D with respect to the measure $d\mu$, and $C^n(D;\mathbf{C})$ is the space of complex-valued n-times continuously differentiable functions on D. For two Banach spaces X and Y, $\mathbf{B}(X;Y)$ denotes the totality of bounded operators from X to Y. We put

$$\mathrm{sgn}(\tau) = \tau/|\tau|, \quad 0 \neq \tau \in \mathbf{R}. \tag{1.13}$$

We use one non-standard notation :

$$F(\cdots) = \text{the characteristic function of the set } \{\cdots\}. \tag{1.14}$$

For example, $F(t>0)$ means the function $\chi(t)$ such that $\chi(t)=1$ for $t>0$ and $\chi(t)=0$ for $t \leq 0$.

2. Green operator

In this section, we construct a Green operator for the gauge transformed Laplacian on $\mathbf{H}^3$ restricted to the half space $\{x_3 > \delta\}$. The argument below is a slight modification of the case $\delta = 0$ given in [17].

2.1. **1-dimensional operator.** Let $I_{1/2}(y)$ and $K_{1/2}(y)$ be modified Bessel functions of order 1/2 (see e.g. [23], p. 112), i.e.

$$I_{1/2}(y) = \sqrt{\frac{2}{\pi y}}\sinh y, \tag{2.1}$$

$$K_{1/2}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}. \tag{2.2}$$

They are linearly independent solutions of the equation

$$y^2u'' + yu' - (y^2 + \frac{1}{4})u = 0. \tag{2.3}$$

Throughout the paper, we take the branch of $\sqrt{\cdot}$ so that $\mathrm{Re}\sqrt{\cdot} \geq 0$ with cut along the negative real axis, i.e. $\sqrt{z} = \sqrt{|z|}e^{i\varphi/2}$ for $z = |z|e^{i\varphi}$, $-\pi < \varphi < \pi$. For a complex parameter $\zeta \neq 0$ satisfying $\mathrm{Re}\,\zeta \geq 0$, consider the differential operator

$$L_0(\zeta) = y^2(-\partial_y^2 + \zeta^2) + y\partial_y - \frac{3}{4} \tag{2.4}$$

on $(0,\infty)$, where $\partial_y = \partial/\partial y$. We put

$$\widetilde{I}(y,\zeta) = yI_{1/2}(\zeta y), \quad \widetilde{K}(y,\zeta) = yK_{1/2}(\zeta y). \tag{2.5}$$

By virtue of (2.3) they satisfy the following equation

$$L_0(\zeta)\widetilde{I}(y,\zeta) = 0, \quad L_0(\zeta)\widetilde{K}(y,\zeta) = 0. \tag{2.6}$$

We fix $\delta > 0$ arbitrarily and define a Green kernel of the 1-dimensional operator (2.4) defined on (δ, ∞) by

$$G_0(y, y'; \zeta) = \frac{\sqrt{yy'}}{2\zeta}\Big(e^{-\zeta|y-y'|} - e^{-\zeta(y+y')}\Big), \tag{2.7}$$

and introduce the Green operator

$$\Big(G_0(\zeta) f\Big)(y) = \int_\delta^\infty G_0(y, y'; \zeta) f(y') \frac{dy'}{(y')^3}. \tag{2.8}$$

Lemma 2.1. *(1) For any* $f \in C_0^\infty([\delta, \infty))$*, we have*

$$L_0(\zeta) G_0(\zeta) f = f. \tag{2.9}$$

(2) For any $u \in C_0^\infty([\delta, \infty))$*, we have*

$$G_0(\zeta) L_0(\zeta) u(y) = u(y) + \frac{\widetilde{K}(y, \zeta)}{\delta}\Big(\widetilde{I}(\delta, \zeta) u'(\delta) - \widetilde{I}'(\delta, \zeta) u(\delta)\Big), \tag{2.10}$$

where $'$ *denotes* ∂_y*.*

Proof. Note that

$$G_0(y, y'; \zeta) = \begin{cases} \widetilde{K}(y, \zeta)\widetilde{I}(y', \zeta) & (y > y'), \\ \widetilde{I}(y, \zeta)\widetilde{K}(y', \zeta) & (y' > y). \end{cases} \tag{2.11}$$

The lemma then follows from a direct computation using

$$\widetilde{I}(y, \zeta)\left(\widetilde{K}(y, \zeta)\right)' - \left(\widetilde{I}(y, \zeta)\right)' \widetilde{K}(y, \zeta) = -y \tag{2.12}$$

for (2.9), and

$$\int_a^b (L_0(\zeta) u)\, v \frac{dy}{y^3} = -\left[\frac{u'v - uv'}{y}\right]_a^b + \int_a^b u\, (L_0(\zeta) v)\, \frac{dy}{y^3} \tag{2.13}$$

for (2.10). ♢

Lemma 2.2. *The Green function* $G_0(y, y'; \zeta)$ *is analytic in* ζ*. There exists a constant* $C > 0$ *such that the inequalities*

$$|G_0(y, y'; \zeta)| \leq C yy', \tag{2.14}$$

$$|G_0(y, y'; \zeta)| \leq \frac{C}{|\zeta|}\sqrt{yy'}, \tag{2.15}$$

$$\Big|\frac{\partial}{\partial\zeta} G_0(y, y'; \zeta)\Big| \leq \frac{C}{|\zeta|}(y + y')\sqrt{yy'}, \tag{2.16}$$

hold for $y,\ y' > \delta$ *and* ζ *such that* $\mathrm{Re}\,\zeta \geq 0$*.*

Proof. We put $Y = y + y' - |y - y'|$. Then since $|(1 - e^{-z})/z| \leq C$ for $\mathrm{Re}\, z \geq 0$, we have

$$\begin{aligned} \left|\frac{1}{\zeta}\Big(e^{-\zeta|y-y'|} - e^{-\zeta(y+y')}\Big)\right| &= e^{-\mathrm{Re}\,\zeta|y-y'|}\left|\frac{1 - e^{-\zeta Y}}{\zeta Y}\right| Y \\ &\leq CY \\ &\leq C\min\{y, y'\} \leq C\sqrt{yy'}, \end{aligned}$$

which implies (2.14). The inequalities (2.15), (2.16) follow immediately. ♢

2.2. **Green operator in the half space.** We now let

$$D_\delta = \{(x,y)\,;\, x \in \mathbf{R}^2, y > \delta\}, \quad \delta > 0, \tag{2.17}$$

and construct a Green operator of

$$H_0(\theta) = y^2(-\partial_y^2 + (-i\partial_x + \theta)^2) + y\partial_y - \frac{3}{4} \tag{2.18}$$

defined on D_δ. For $\theta, \theta' \in \mathbf{C}^2$, we put

$$\theta \cdot \theta' = \sum_{i=1}^2 \theta_i {\theta_i}', \quad \theta^2 = \theta\cdot\theta,$$

and define for $\xi \in \mathbf{R}^2$

$$\zeta(\xi,\theta) = \sqrt{(\xi+\theta)^2}. \tag{2.19}$$

We put

$$\Big(\mathbf{G}_0(\theta)f\Big)(x,y) = (2\pi)^{-1}\int_{\mathbf{R}^2} e^{ix\cdot\xi}\Big(G_0(\zeta(\xi,\theta))\widehat{f}(\xi,\cdot)\Big)(y)d\xi. \tag{2.20}$$

Here and in the sequel $\widehat{f}(\xi,y)$ denotes the partial Fourier transform with respect to x :

$$\widehat{f}(\xi,y) = (2\pi)^{-1}\int_{\mathbf{R}^2} e^{-ix\cdot\xi}f(x,y)dx. \tag{2.21}$$

We introduce the following function spaces. For $s \in \mathbf{R}$, we define:

$$L^{2,s} \ni f \Longleftrightarrow \int_{D_\delta}(1+|\log y|)^{2s}|f(x,y)|^2\frac{dxdy}{y^3} < \infty. \tag{2.22}$$

For $t, s \geq 0$, we define

$$\mathcal{X}_s^{(\pm)} \ni f \Longleftrightarrow \int_{D_\delta}\left[\frac{(1+|\log y|)^{2s}}{y}\right]^{\pm1}|f(x,y)|^2\frac{dxdy}{y^3} < \infty, \tag{2.23}$$

$$\mathcal{W}_{t,s}^{(\pm)} \ni f \Longleftrightarrow \int_{D_\delta}\left[(1+|x|)^{2t}\frac{(1+|\log y|)^{2s}}{y}\right]^{\pm1}|f(x,y)|^2\frac{dxdy}{y^3} < \infty. \tag{2.24}$$

We equip these spaces with obvious norms. Note that $\mathcal{X}_s^{(\pm)} = \mathcal{W}_{0,s}^{(\pm)}$. The following theorem is proved in the same way as [17], Lemma 2.2 and Theorem 2.8.

Theorem 2.3. *(1) Let $s > 1/2$. Then there exists a constant $C_s > 0$ such that*

$$\|\mathbf{G}_0(\theta)\|_{\mathbf{B}(L^{2,s},L^{2,-s})} \leq C_s, \quad \forall\theta \in \mathbf{C}^2.$$

(2) Let $s > 1$. Then there exists a constant $C_s > 0$ such that for $0 \leq t \leq s$

$$\|\mathbf{G}_0(\theta)f\|_{\mathcal{W}_{t,s}^{(-)}} \leq C_s\left(\frac{\log|\theta_I|}{|\theta_I|}\right)^{1/2}\|f\|_{\mathcal{W}_{s-t,s}^{(+)}} \quad \text{if} \quad |\theta_I| > 2, \tag{2.25}$$

where θ_I is the imaginary part of θ.

Let us define the perturbed Green operator.

Assumption 2.4. *We assume that* $V \in C^1(\mathbf{H}^n;\mathbf{C})$ *and*

$$|\partial_y^\alpha V(x,y)| \leq C(1+|x|)^{-2s-2}(1+|\log y|)^{-2s}y^{-1/2}, \quad \alpha = 0, 1, \tag{2.26}$$

for some $s > 1$.

Since $V \in \mathbf{B}(\mathcal{W}^{(-)}_{t,s};\mathcal{W}^{(+)}_{s-t,s})$, the following theorem is easily proved by Theorem 2.3.

Theorem 2.5. *Let* $s > 1$ *be the constant in (2.26). Let* $\mathbf{G}_V(\theta)$ *be defined by*

$$\mathbf{G}_V(\theta) = (1 + \mathbf{G}_0(\theta)V)^{-1}\mathbf{G}_0(\theta) \tag{2.27}$$

for sufficiently large $|\theta_I|$. *Then there exists a constant* $C_s > 0$ *such that for* $0 \leq t \leq s$

$$\|\mathbf{G}_V(\theta)\|_{\mathbf{B}(\mathcal{W}^{(+)}_{s-t,s};\mathcal{W}^{(-)}_{t,s})} \leq C_s\left(\frac{\log|\theta_I|}{|\theta_I|}\right)^{1/2}, \quad |\theta_I| > C_s. \tag{2.28}$$

As a matter of fact, this theorem holds under the weaker assumption that

$$|V(x,y)| \leq C(1+|x|)^{-s}(1+|\log y|)^{-2s}y, \quad s > 1. \tag{2.29}$$

3. Plane-pulse waves

3.1. **Construction.** The aim of this section is to construct a solution of the Schrödinger equation which behaves like $\sqrt{y}\sin(\tau y)a_\tau(x_1)$, where $a_\tau(x_1)$ is supported near the plane $\{x_1 = 0\}$. In the following, we put

$$\theta = (0, i\tau), \quad \tau \in \mathbf{R}, \tag{3.1}$$

and $|\tau| > C$, C being a sufficiently large constant.

Definition 3.1. *We put* $a(\theta)$ *as follows :*

$$\widehat{a}(\xi,y;\theta) = C(\tau)\widehat{\chi}\big(\frac{\xi_1}{|\tau|^\epsilon}\big)\widehat{\chi}\big(|\tau|\xi_2\big)\sqrt{y}e^{-\zeta y}\sinh(\zeta\delta), \tag{3.2}$$

where $\chi(t)$ *is a real function in the Schwartz space such that*

$$\chi(t) = \chi(-t), \tag{3.3}$$

$$\chi(0)\|\chi\|_{L^2} = \sqrt{\frac{4}{\pi}}, \tag{3.4}$$

$\zeta = \zeta(\xi,\theta)$ *is defined by (2.19),*

$$C(\tau) = -\sqrt{\frac{\pi}{2}}e^{-\mathrm{sgn}(\tau)\pi i/4}e^{-i\tau\delta}|\tau|^{1-\epsilon/2}, \tag{3.5}$$

and ϵ *is a small positive constant. We define* $u(\theta)$ *by*

$$u(\theta) = a(\theta) - \mathbf{G}_V(\theta)Va(\theta), \tag{3.6}$$

By virtue of (2.6), we have

$$H_0(\theta)a(\theta) = 0. \tag{3.7}$$

Hence $u(\theta)$ satisfies

$$(H_0(\theta) + V)u(\theta) = 0. \tag{3.8}$$

3.2. **Properties of $u(\theta)$.**

Lemma 3.2. *(1) There exists a constant $C > 0$ such that*

$$|a(x,y;\theta)| \leq C|\tau|^{\epsilon/2}y^{1/2}.$$

(2) The following expansion holds :

$$\begin{aligned} a(x,y;\theta) &= i\sqrt{\frac{\pi}{2}}e^{-\mathrm{sgn}(\tau)\pi i/4}(1-e^{-2i\tau\delta})\frac{\chi(0)}{2}|\tau|^{\epsilon/2}\chi(|\tau|^{\epsilon}x_1)\sqrt{y}\sin(\tau y) \\ &+ r(x,y,\tau), \\ |r(x,y,\tau)| &\leq C(1+|x|)y^{3/2}|\tau|^{-1+5\epsilon}. \end{aligned}$$

Proof. Since $\mathrm{Re}\,\zeta(\xi,\theta) \geq 0$, we have $|e^{-\zeta y}\sinh(\zeta\delta)| \leq C$ for $y > \delta$, which implies (1). To prove (2), we first note the following estimates :

$$\Big|\widehat{\chi}\big(\frac{\xi_1}{|\tau|^{\epsilon}}\big)\Big| \leq C_N|\tau|^{-\epsilon N}\,(1+|\xi_1|)^{-2} \quad \text{if} \quad |\xi_1| > |\tau|^{2\epsilon}, \tag{3.9}$$

$$\big|\widehat{\chi}\big(|\tau|\xi_2\big)\big| \leq C_N|\tau|^{-N}\,(1+|\xi_2|)^{-N} \quad \text{if} \quad |\xi_2| > 1. \tag{3.10}$$

We can then cut off the parts $|\xi_1| > |\tau|^{2\epsilon}$ or $|\xi_2| > 1$ so that

$$\begin{aligned} a(x,y;\theta) &= \frac{C(\tau)}{2\pi}\int e^{ix\cdot\xi}F(|\xi_1|<|\tau|^{2\epsilon})F(|\xi_2|<1)\widehat{\chi}\big(\frac{\xi_1}{|\tau|^{\epsilon}}\big)\widehat{\chi}\big(|\tau|\xi_2\big) \\ &\quad \times\sqrt{y}e^{-\zeta y}\sinh(\zeta\delta)d\xi + \ r_1(x,y,\tau), \\ |r_1(x,y,\tau)| &\leq C_N y^{1/2}|\tau|^{-N}. \end{aligned}$$

Next let us take notice of the following Propostion.

Proposition 3.3. *(1) For $|\xi_1| < |\tau|^{2\epsilon}$ and $|\xi_2| < 1$, we put*

$$w_{\pm} = \zeta(\xi,\theta) \mp (i\tau + \xi_2) \quad \text{if} \quad \pm\,\xi_2 > 0.$$

Then we have

$$\mathrm{Re}\,w_{\pm} \geq 0, \quad |w_{\pm}| \leq C|\tau|^{4\epsilon-1},$$

where the constant C is independent of ξ.
(2) For any $a \geq 0$, we have

$$\big|e^{-\zeta a} - e^{\mp(i\tau+\xi_2)a}\big| \leq Ca|\tau|^{4\epsilon-1},$$

if $|\xi_1| < |\tau|^{2\epsilon}$, $|\xi_2| < 1$ and $\pm\xi_2 > 0$, where the constant C is independent of ξ.

Proof. We consider w_+. By (2.19) we have

$$w_+ = \frac{\zeta^2 - (i\tau+\xi_2)^2}{\zeta + i\tau + \xi_2} = \frac{\xi_1^2(\overline{\zeta} - i\tau + \xi_2)}{|\zeta + i\tau + \xi_2|^2},$$

which proves $\mathrm{Re}\,w_+ \geq 0$. By our choice of the branch of $\sqrt{\cdot}$, the imaginary parts of ζ^2 and ζ have the same sign. Since $\zeta^2 = |\xi|^2 - \tau^2 + 2i\tau\xi_2$, the imaginary parts of ζ and $i\tau+\xi_2$ have the same sign. Therefore by an elementary geometric consideration we have $|\zeta + i\tau + \xi_2| \geq C|\tau|$, which implies $|w_+| \leq C|\tau|^{4\epsilon-1}$. The assertion (2) follows from this inequality. $\diamondsuit$

Proof of Lemma 3.2 (continued). Taking note of

$$e^{-\zeta y}\sinh(\zeta\delta) = \frac{1}{2}\left(e^{-\zeta(y-\delta)} - e^{-\zeta(y+\delta)}\right),$$

we replace ζ by $\pm(i\tau+\xi_2)$ by using this Proposition. Then we have

$$\begin{aligned}
a(x,y;\theta) &= \frac{C(\tau)}{2\pi}\int_{\mathbf{R}\times\{\xi_2>0\}} e^{ix\cdot\xi}F(|\xi_1|<|\tau|^{2\epsilon})F(|\xi_2|<1)\widehat{\chi}\big(\frac{\xi_1}{|\tau|^{\epsilon}}\big)\widehat{\chi}\big(|\tau|\xi_2\big)\\
&\quad\times\sqrt{y}e^{-(i\tau+\xi_2)y}\frac{1}{2}\left(e^{(i\tau+\xi_2)\delta} - e^{-(i\tau+\xi_2)\delta}\right)d\xi\\
&+ \frac{C(\tau)}{2\pi}\int_{\mathbf{R}\times\{\xi_2<0\}} e^{ix\cdot\xi}F(|\xi_1|<|\tau|^{2\epsilon})F(|\xi_2|<1)\widehat{\chi}\big(\frac{\xi_1}{|\tau|^{\epsilon}}\big)\widehat{\chi}\big(|\tau|\xi_2\big)\\
&\quad\times\sqrt{y}e^{(i\tau+\xi_2)y}\frac{1}{2}\left(e^{-(i\tau+\xi_2)\delta} - e^{(i\tau+\xi_2)\delta}\right)d\xi\\
&+ r_1(x,y,\tau) + r_2(x,y,\tau),
\end{aligned}$$

where the remainder term is estimated as follows :

$$\begin{aligned}
|r_2(x,y,\tau)| &\leq C|\tau|^{1-\epsilon/2}y^{3/2}|\tau|^{4\epsilon-1}\int\big|\widehat{\chi}\big(\frac{\xi_1}{|\tau|^{\epsilon}}\big)\widehat{\chi}\big(|\tau|\xi_2\big)\big|d\xi\\
&\leq Cy^{3/2}|\tau|^{-1+5\epsilon}.
\end{aligned}$$

In the above expression of $a(x,y;\theta)$, we replace the term $F(|\xi_1|<|\tau|^{2\epsilon})F(|\xi_2|<1)$ by 1 with the rapidly decreasing error in $|\tau|$. We next make the change of variable $\xi_2 = \eta_2/|\tau|$ and replace $e^{ix_2\xi_2}$, $e^{\pm\xi_2(y-\delta)}$, $e^{\pm\xi_2(y+\delta)}$,, $e^{\pm i\xi_2\delta}$ by 1. In view of the inequality

$$|e^{z\eta_2/|\tau|} - 1| \leq C\frac{|z\eta_2|}{|\tau|} \quad \text{for} \quad \mathrm{Re}\, z \leq 0,$$

we see that the error is estimated by $C(1+|x|)y^{3/2}|\tau|^{\epsilon/2-1}$. Then we have

$$\begin{aligned}
a(x,y;\theta) &= \frac{C_1(\tau)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix_1\xi_1}\widehat{\chi}\big(\frac{\xi_1}{|\tau|^{\epsilon}}\big)d\xi_1\sqrt{y}e^{-i\tau y}\frac{1}{2}\left(e^{i\tau\delta} - e^{-i\tau\delta}\right)\frac{\chi(0)}{2}\\
&+ \frac{C_1(\tau)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix_1\xi_1}\widehat{\chi}\big(\frac{\xi_1}{|\tau|^{\epsilon}}\big)d\xi_1\sqrt{y}e^{i\tau y}\frac{1}{2}\left(e^{-i\tau\delta} - e^{i\tau\delta}\right)\frac{\chi(0)}{2}\\
&+ r_3(x,y,\tau),\\
C_1(\tau) &= \frac{C(\tau)}{|\tau|} = -\sqrt{\frac{\pi}{2}}e^{-\mathrm{sgn}(\tau)\pi i/4}e^{-i\tau\delta}|\tau|^{-\epsilon/2},
\end{aligned}$$

where we use the fact that

$$\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\widehat{\chi}(k)dk = \frac{\chi(0)}{2},$$

since $\widehat{\chi}$ is an even function. The remainder term is estimated as

$$|r_3(x,y,\tau)| \leq C(1+|x|)y^{3/2}|\tau|^{-1+5\epsilon}.$$

By computing the above integral, we complete the proof of Lemma 3.2 (2). ♢

For two potentials $V_i, i = 1,2$, satisfying (2.26), let $u^{(i)}(\theta)$ be the solution of $(H_0(\theta)+V_i)u^{(i)}(\theta) = 0$ constructed as above.

Theorem 3.4. *Let $f \in C^1(D_\delta)$ be such that*

$$|\partial_y^\alpha f(x,y)| \leq C(1+|x|)^{-2s-2}(1+|\log y|)^{-2s}y, \quad \alpha = 0,1 \tag{3.11}$$

for $s > 1$. We put $\tau_n = (n+1/2)\pi/\delta$, $\theta_n = (0, i\tau_n)$. Then we have as $n \to \infty$

$$\int_{D_\delta} f u^{(1)}(\theta_n)\overline{u^{(2)}(-\theta_n)}\frac{dxdy}{y^3} \to i\int_\Pi f(0,x_2,y)\frac{dx_2dy}{y^2}, \tag{3.12}$$

where $\Pi = \mathbf{R} \times (\delta, \infty)$.

Proof. We have by virtue of Theorem 2.5, Assumption 2.4 and Lemma 3.2 (1),

$$\|\mathbf{G}_V(\theta_n)Va(\theta_n)\|_{\mathcal{W}^{(-)}_{0,s}} \leq C\tau_n^{-1/2+\epsilon},$$

$$\|fu\|_{W^{(+)}_{0,s}} \leq C\|u\|_{W^{(-)}_{0,s}},$$

$$\|fa(\theta_n)\|_{W^{(+)}_{0,s}} \leq C|\tau|^{\epsilon/2}.$$

Therefore to compute the limit in question we have only to replace $u^{(i)}(\theta)$ by $a(\theta)$ and use Lemma 3.2 (2). We then use $\big(\sin(\tau y)\big)^2 = \big(1-\cos(2\tau y)\big)/2$ and integrate by parts to see that the term containing $\cos(2\tau y)$ tends to 0. Finally we use the normalization condition (3.4). ♢

4. The half-space problem

4.1. **Gauge transformation.** We begin with a simple remark. Let Ω be an open set in $\mathbf{R}^n$, A a differential operator on Ω, and $\rho(x)$ a positive function on Ω. Then solving the boundary value problem

$$\left\{ \begin{array}{lll} Au = 0 & \text{in} & \Omega, \\ Bu = f & \text{on} & \partial\Omega \end{array} \right. \tag{4.1}$$

in $L^2(\Omega;\rho^2dx)$ is equivalent to solving the gauge transformed equation

$$\left\{ \begin{array}{lll} A_\rho v = 0 & \text{in} & \Omega, \\ B_\rho v = \rho f & \text{on} & \partial\Omega \end{array} \right. \tag{4.2}$$

in $L^2(\Omega;dx)$, where $A_\rho = \rho A\rho^{-1}, B_\rho = \rho B\rho^{-1}$. Let $\mathcal{R}$ and $\mathcal{R}_\rho$ be the associated GRD maps for (4.1) and (4.2), which assign the solution $u\big|_{\partial\Omega}$, $v\big|_{\partial\Omega}$ to the respective boundary data. Then we have

$$\mathcal{R}_\rho = \rho\mathcal{R}\rho^{-1}. \tag{4.3}$$

Under the assumption of the unique solvability, $\mathcal{R}$ and $\mathcal{R}_\rho$ will be densely defined closed operators on $L^2(\partial\Omega;\rho^2dS)$ and $L^2(\partial\Omega;dS)$ respectively, where dS is the measure on $\partial\Omega$ induced from the Lebesgue measure. Let us note that even when f in (4.1) belongs to $L^2(\partial\Omega;dS)$, u does not necessary belong to $L^2(\Omega;dx)$, and $\mathcal{R}$ may be different from the GRD map defined in $L^2(\Omega;dx)$-setting.

4.2. **Existence and uniqueness.** With the above remark in mind, we consider the operator $-y^2\Delta + y\partial_y$ in the space $L^2(D_\delta; e^{2\tau x_2}dxdy/y^3)$, where $\tau > 0$ is a large parameter. By the unitary transformation

$$L^2\Big(D_\delta; e^{2\tau x_2}\frac{dxdy}{y^3}\Big) \ni u \to e^{\tau x_2}u \in L^2\Big(D_\delta; \frac{dxdy}{y^3}\Big),$$

we have

$$e^{\tau x_2}(-y^2\Delta + y\partial_y)e^{-\tau x_2} = y^2\Big(-\partial_y^2 + (-i\partial_x + \theta)^2\Big) + y\partial_y. \tag{4.4}$$

Here and in the following, we put

$$\theta = (0, i\tau). \tag{4.5}$$

We are thus led to consider $H_0(\theta)$ defined by (2.18) in $L^2(D_\delta; dxdy/y^3)$. Since $\widetilde{I}(y,\zeta)$ satisfies

$$\frac{\partial}{\partial y}\widetilde{I}(y,\zeta)\Big|_{y=\delta} = p(\zeta)\widetilde{I}(\delta,\zeta), \tag{4.6}$$

$$p(\zeta) = \zeta\coth(\zeta\delta) + \frac{1}{2\delta}, \tag{4.7}$$

we impose the following boundary condition

$$B(\theta)u := \Big(\partial_y - P(\theta)\Big)u(y)\Big|_{y=\delta} = 0, \tag{4.8}$$

where $P(\theta)$ is a pseudo-differential operator defined by

$$P(\theta)\varphi = (2\pi)^{-1}\int_{\mathbf{R}^2} e^{ix\cdot\xi}p(\zeta(\xi,\theta))\widehat{\varphi}(\xi)d\xi, \tag{4.9}$$

and we naturally identify ∂D_δ with $\mathbf{R}^2$.

Here we must note that $p(\zeta(\xi,\theta))$ is singular at $\delta\zeta(\xi,\theta) = n\pi i$ $(n \neq 0)$, i.e.

$$\xi = \Big(\pm\sqrt{\tau^2 - \big(\frac{n\pi}{\delta}\big)^2}, 0\Big), \quad n \in \mathbf{Z}\setminus\{0\}, \quad |n| \leq \frac{\delta|\tau|}{\pi}. \tag{4.10}$$

Therefore $P(\theta)\varphi \in L^2(\mathbf{R}^2)$ only if $\widehat{\varphi}$ vanishes on these singularities. This suggests us to introduce the following space of functions on the boundary :

$$\mathcal{B}_\theta = \{u \in L^2(\mathbf{R}^2)\,;\, \zeta(\xi,\theta)q(\zeta(\xi,\theta))\widehat{u}(\xi) \in L^2(\mathbf{R}^2)\}, \tag{4.11}$$

$$q(\zeta) = \frac{e^{\zeta\delta}}{\sinh(\zeta\delta)}. \tag{4.12}$$

Lemma 4.1. *Let $f \in L^{2,s}, s > 1$. Then*

$$\mathbf{G}_V(\theta)f\Big|_{\partial D_\delta} \in \mathcal{B}_\theta, \tag{4.13}$$

$$B(\theta)\mathbf{G}_V(\theta)f\Big|_{\partial D_\delta} = 0. \tag{4.14}$$

Proof. By the resolvent equation $\mathbf{G}_V(\theta) = \mathbf{G}_0(\theta) - \mathbf{G}_0(\theta)V\mathbf{G}_V(\theta)$, we have only to prove this lemma when $V = 0$. Let $u = \mathbf{G}_0(\theta)f$. Then

$$\widehat{u}(\xi,\delta) = \widetilde{I}(\delta,\zeta)\int_\delta^\infty \widetilde{K}(y,\zeta)\widehat{f}(\xi,y)\frac{dy}{y^3}, \tag{4.15}$$

$$\partial_y\widehat{u}(\xi,y)\Big|_{y=\delta} = \partial_y\widetilde{I}(y,\zeta)\Big|_{y=\delta}\int_\delta^\infty \widetilde{K}(y,\zeta)\widehat{f}(\xi,y)\frac{dy}{y^3}. \tag{4.16}$$

The lemma then follows from (2.1), (2.2) and (4.6) immediately. ♢

Now we can solve the boundary value problem.

Theorem 4.2. *Let $s > 1$. Take $T_0 > 0$ large enough. Then for any $\pm\tau > T_0$ there exists a unique solution $u \in \mathcal{X}_s^{(-)}$ of the boundary value problem*

$$\begin{cases} (H_0(\theta) + V)u = 0 \quad \text{in} \quad D_\delta, \\ \quad u\big|_{\partial D_\delta} \in \mathcal{B}_\theta, \\ B(\theta)u = f \quad \text{on} \quad \partial D_\delta, \end{cases} \tag{4.17}$$

with any boundary data $f \in L^2(\mathbf{R}^2)$ satisfying

$$\widehat{f}(\xi)/\zeta(\xi,\theta) \in L^2(\mathbf{R}^2). \tag{4.18}$$

This solution u is written as

$$u = -b(\theta) + \mathbf{G}_V(\theta)Vb(\theta), \tag{4.19}$$

$$b(\theta) = \frac{1}{2\pi}\sqrt{\frac{y}{\delta}}\int_{\mathbf{R}^2} e^{ix\cdot\xi}\frac{e^{-\zeta y}\sinh(\zeta\delta)}{\zeta}\widehat{f}(\xi)d\xi, \quad \zeta = \zeta(\xi,\theta). \tag{4.20}$$

Proof. We first prove the uniqueness. Let $u \in \mathcal{X}_s^{(-)}$ satisfy (4.17) with $f = 0$. Since $B(\theta)u = 0$, u satisfies $\widetilde{I}(\delta,\zeta)\widehat{u}'(\delta) - \widetilde{I}'(\delta,\zeta)\widehat{u}(\delta) = 0$. Then in view of (2.10), we have $u = -\mathbf{G}_0(\theta)Vu$. Theorem 2.3 (2) then implies

$$\|u\|_{\mathcal{X}_s^{(-)}} \leq C\left(\frac{\log|\tau|}{|\tau|}\right)^{1/2}\|Vu\|_{\mathcal{W}_{s,s}^{(+)}} \leq C\left(\frac{\log|\tau|}{|\tau|}\right)^{1/2}\|u\|_{\mathcal{X}_s^{(-)}}.$$

Therefore $u = 0$ for large $\pm\tau > 0$.

We next prove the existence. Since $|\widehat{b}(\xi,y;\theta)| \leq C\sqrt{y}|g(\xi)|$ with $g \in L^2(\mathbf{R}^2)$ by the condition (4.18), we have $b(\theta) \in \mathcal{X}_s^{(-)}$. By a direct computation, we have $b(\theta)\big|_{\partial D_\delta} \in \mathcal{B}_\theta$ and $B(\theta)b(\theta) = -f$. This and Lemma 4.1 prove that u defined by (4.18) solves the equation (4.17). ♢

Let us remark that the condition (4.18) is satisfied if

$$(1 + |x|)^s f(x) \in L^2(\mathbf{R}^2), \quad s > 1. \tag{4.21}$$

In fact, this follows from the estimate $|\zeta|^{-2} \leq C(|\xi^2 - \tau^2| + |\tau\xi_2|)^{-1}$ and the fact that $\widehat{f}(\xi) \in L^\infty(\mathbf{R}^2)$.

Definition 4.3. *For $f \in L^2(\mathbf{R}^2)$ satisfying (4.18), we define the generalized Robin - to - Dirichlet map $\mathcal{R}(\theta)$ by*

$$\mathcal{R}(\theta)f = u\big|_{\partial D_\delta}, \tag{4.22}$$

where u is the solution to the equation (4.17). We also write $\mathcal{R}(\theta, V)$ or $\mathcal{R}_V(\theta)$ instead of $\mathcal{R}(\theta)$ in order to specify the dependence on V.

Lemma 4.4. *There exists a constant $C > 0$ such that*

$$\|\mathcal{R}(\theta)f\|_{L^2(\partial D_\delta)} \leq C\|f\|_{L^2(\mathbf{R}^2)}$$

uniformly in θ.

Proof. Since $|\widehat{b}(\xi, y, \theta)| \leq Cy^{3/2}|\widehat{f}(\xi)|$, we have $\|b(\cdot, y, \theta)\|_{L^2(\mathbf{R}^2)} \leq Cy^{3/2}\|f\|_{L^2(\mathbf{R}^2)}$. Using (2.14), one can easily check that for $t > 1/2$

$$\|\mathbf{G}_0(\theta)F\|_{L^2(\partial D_\delta)} \leq C\|F\|_{L^{2,t}}.$$

Therefore we have by using (2.26)

$$\|\mathbf{G}_V(\theta)Vb(\theta)\|_{L^2(\partial D_\delta)} \leq C\|f\|_{L^2(\mathbf{R}^2)}, \tag{4.23}$$

where we have used the resolvent equation $\mathbf{G}_V(\theta) = \mathbf{G}_0(\theta) - \mathbf{G}_0(\theta)V\mathbf{G}_V(\theta)$ and

$$\begin{aligned} \|V\mathbf{G}_V(\theta)Vb(\theta)\|_{L^{2,t}} &\leq C\|\mathbf{G}_V(\theta)Vb(\theta)\|_{\mathcal{W}^{(-)}_{s,s}} \\ &\leq C\|Vb(\theta)\|_{\mathcal{W}^{(+)}_{0,s}} \leq C\|f\|_{L^2(\mathbf{R}^2)}. \end{aligned}$$

We also have by (4.20)

$$\|b(\theta)\|_{L^2(\partial D_\delta)} \leq C\|f\|_{L^2(\mathbf{R}^2)}. \tag{4.24}$$

The lemma then follows from (4.23) and (4.24). ♢

We need one more lemma.

Lemma 4.5. *(1) Suppose $f \in \mathcal{W}^{(+)}_{2s,s}, s > 1/2$. Let $u = \mathbf{G}_0(\theta)f$. Then we have as $y \to \infty$*

$$\frac{1}{y}\left(\|u(\cdot, y)\|^2_{L^2(\mathbf{R}^2)} + \|\partial_y u(\cdot, y)\|^2_{L^2(\mathbf{R}^2)}\right) \to 0. \tag{4.25}$$

(2) If $f \in L^2(\mathbf{R}^2)$ satisfies (4.18), $b(\theta)$ defined by (4.20) satisfies

$$\frac{1}{y}\left(\|b(\cdot, y, \theta)\|^2_{L^2(\mathbf{R}^2)} + \|\partial_y b(\cdot, y, \theta)\|^2_{L^2(\mathbf{R}^2)}\right) \to 0. \tag{4.26}$$

Proof. By (2.7), we have

$$|G_0(y, y'; \zeta)| \leq C\frac{\sqrt{yy'}}{|\zeta|}e^{-\mathrm{Re}\,\zeta|y-y'|}, \tag{4.27}$$

which yields

$$\frac{1}{y}\|\widehat{u}(\cdot, y)\|^2_{L^2} \leq C\int_{\mathbf{R}^2\times(\delta,\infty)} \frac{e^{-2\mathrm{Re}\,\zeta|y-y'|}}{|\zeta|^2}\frac{(1 + |\log y'|)^{2s}}{(y')^4}|\widehat{f}(\xi, y')|^2 d\xi dy'.$$

Recall that $1/|\zeta|^2 \leq C(|\xi^2-\tau^2|+|\tau\xi_2|)^{-1}$. On the region $\{\xi\,;\,|\xi^2-\tau^2|<1,\ |\xi_2|<1\}$, the integrand is dominated by

$$\frac{1}{|\xi^2-\tau^2|+|\tau\xi_2|}\frac{(1+|\log y'|)^{2s}}{(y')^4}\|(1+|\cdot|)^{2s}f(\cdot,y')\|_{L^2}^2,$$

which is integrable with respect to ξ and y'. On the region $\{\xi\,;\,|\xi^2-\tau^2|>1 \text{ or } |\xi_2|>1\}$, the integrand is dominated by

$$\frac{(1+|\log y'|)^{2s}}{(y')^4}|\widehat{f}(\xi,y')|^2,$$

which is integrable with respect to ξ and y'. Therefore $\|u(\cdot,y)\|_{L^2}^2/y \to 0$ by Lebesgue's convergence theorem. Similarly one can prove $\|\partial_y u(\cdot,y)\|_{L^2}^2/y \to 0$ if we note that the integral kernel of $\partial_y\mathbf{G}_0(\theta)$ is dominated by

$$C\left(1+\frac{1}{|\zeta|}\right)\sqrt{yy'}e^{-\mathrm{Re}\,\zeta|y-y'|}.$$

The assertion (2) can be easily proved by Lebesgue's convergence theorem. ♢

Let (,) and $(\ ,\ )_{\partial D_\delta}$ be the inner products of $L^2(D_\delta)$ and $L^2(\partial D_\delta)$, respectively. One can easily show that for any $\varphi \in \mathcal{B}_\theta, \psi \in \mathcal{B}_{\overline{\theta}}$

$$(P(\theta)\varphi,\psi)_{\partial D_\delta} = (\varphi, P(\overline{\theta})\psi)_{\partial D_\delta}. \tag{4.28}$$

Lemma 4.6. *(1) Suppose u, v satisfy (4.25) and $u\big|_{\partial D_\delta} \in \mathcal{B}_\theta, v\big|_{\partial D_\delta} \in \mathcal{B}_{\overline{\theta}}$. Then we have*

$$(H_0(\theta)u,v)-(u,H_0(\overline{\theta})v) = (B(\theta)u,v)_{\partial D_\delta} - (u,B(\overline{\theta})v)_{\partial D_\delta}.$$

(2) For any $\varphi,\psi \in L^2(\mathbf{R}^2)$ satisfying (4.18), we have

$$(\mathcal{R}(\theta,V)\varphi,\psi)_{\partial D_\delta} = (\varphi,\mathcal{R}(\overline{\theta},\overline{V})\psi)_{\partial D_\delta}.$$

Proof. By Green's formula, we have

$$\begin{aligned}&\int_{\delta<y<r}\Big[(H_0(\theta)u)\overline{v} - u\overline{(H_0(\overline{\theta})v)}\Big]\frac{dxdy}{y^3}\\ = \quad & -\int_{y=r}\Big[(\partial_y u)\overline{v} - u\overline{(\partial_y v)}\Big]\frac{dx}{r} + \int_{y=\delta}\Big[(\partial_y u)\overline{v} - u\overline{(\partial_y v)}\Big]\frac{dx}{\delta}.\end{aligned}$$

By (4.25), the first term of the right-hand side vanishes as $r \to \infty$. By virtue of (4.28), the second term is equal to $(B(\theta)u,v)_{\partial D_\delta} - (u,B(\overline{\theta})v)_{\partial D_\delta}$. The assertion (2) follows from (1). ♢

4.3. **Integral mean of the potential.** We now state the main result for the half-space case. Let $f(\theta)$ be defined by

$$f(\theta) = B(\theta)a(\theta), \tag{4.29}$$

where $a(\theta)$ is defined by Definition 3.1. Note that $u(\theta) \in \mathcal{X}_s^{(-)}$, and $u(\theta)\big|_{\partial D_\delta} \in \mathcal{B}_\theta$.

Theorem 4.7. *The function $u(\theta)$ of Definition 3.1 is a unique solution to the problem (4.17) associated with the boundary data $f(\theta)$. Moreover $f(\theta)$ is written as*

$$\widehat{f}(\xi,\theta) = C_0(\tau)\widehat{\chi}\big(\frac{\xi_1}{|\tau|^{\epsilon}}\big)\widehat{\chi}\big(|\tau|\xi_2\big)\zeta(\xi,\theta), \tag{4.30}$$

$$C_0(\tau) = -\sqrt{\frac{\pi}{2}}e^{-\mathrm{sgn}(\tau)\pi i/4}e^{-i\tau\delta}|\tau|^{1-\epsilon/2}\sqrt{\delta}. \tag{4.31}$$

Proof. The first assertion is a consequence of the uniqueness in Theorem 4.2, and the formula (4.30) follows from a straightforward calculation. ♢

Theorem 4.8. *Let $\mathcal{R}_0(\theta)$ and $\mathcal{R}_V(\theta)$ be the GRD maps associated with $H_0(\theta)$ and $H_0(\theta)+V$, respectively. Let $\tau_n = (n+1/2)\pi/\delta$, and put $\theta_n = (0, i\tau_n)$. Then we have*

$$\begin{aligned}&\lim_{n\to\infty}\big((\mathcal{R}_V(\theta_n)-\mathcal{R}_0(\theta_n))f(\theta_n), f(\overline{\theta}_n)\big)_{\partial D_\delta}\\ &\qquad = i\int_{\mathbf{R}\times(\delta,\infty)} V(0,x_2,y)\frac{dx_2dy}{y^2}.\end{aligned} \tag{4.32}$$

Proof. Let $u^{(V)}(\theta)$ and $u^{(0)}(\overline{\theta})$ be the solutions to the equations

$$\left\{\begin{array}{lll}(H_0(\theta)+V)\,u^{(V)}(\theta) = 0 & \text{in} & D_\delta,\\ B(\theta)u^{(V)}(\theta) = f(\theta) & \text{on} & \partial D_\delta,\end{array}\right. \tag{4.33}$$

$$\left\{\begin{array}{lll}H_0(\overline{\theta})u^{(0)}(\overline{\theta}) = 0 & \text{in} & D_\delta,\\ B(\overline{\theta})u^{(0)}(\overline{\theta}) = f(\overline{\theta}) & \text{on} & \partial D_\delta,\end{array}\right. \tag{4.34}$$

respectively. We let $u = u^{(V)}(\theta), v = u^{(0)}(\overline{\theta})$ in Lemma 4.6 (1). Using Lemma 4.6 (2) we have

$$\Big(Vu^{(V)}(\theta), u^{(0)}(\overline{\theta})\Big) = \big(\big(\mathcal{R}_V(\theta)-\mathcal{R}_0(\theta)\big)f(\theta), f(\overline{\theta})\big)_{\partial D_\delta}. \tag{4.35}$$

Applying Theorem 3.4, we get the theorem. ♢

4.4. **Asymptotic expansion of the boundary data.** Let us compute the asymptotic form of $f(\theta)$. We put

$$C_1(\tau) = -\sqrt{\frac{\pi}{2}}e^{-\mathrm{sgn}\,(\tau)\pi i/4}e^{-i\tau\delta}\sqrt{\delta}, \tag{4.36}$$

$$\chi_s(t) = i\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(t\lambda)\widehat{\chi}(\lambda)d\lambda. \tag{4.37}$$

In the following

$$a_\epsilon(x,\tau) = O_{L^2}(|\tau|^{-N-1/2}) \tag{4.38}$$

means that there exists $\epsilon_N > 0$ such that for $0<\epsilon<\epsilon_N$ one can find $C_{\epsilon,N} > 0$ for which

$$\|a_\epsilon(\cdot,\tau)\|_{L^2(\mathbf{R}^2)} \le C_{\epsilon,N}|\tau|^{-N-1/2}, \quad \forall|\tau| > C_{\epsilon,N}. \tag{4.39}$$

If (4.39) holds for all N, we write $a_\epsilon(x,\tau)\approx 0$.

Theorem 4.9. *The following asymptotic expansion holds :*

$$f(x,\theta) \approx C_1(\tau)\left[i\tau - i\partial_{x_2} + \sum_{n=1}^{\infty}\frac{P_n(-i\partial_x)}{(i\tau)^n}\right]|\tau|^{\epsilon/2}\chi\big(|\tau|^{\epsilon}x_1\big)\chi_s\big(\frac{x_2}{|\tau|}\big), \tag{4.40}$$

where $P_n(\xi)$ is a polynomial of order $n+1$.

If $\chi(t)$ is a Gaussian :

$$\chi(t) = e^{-t^2/2}, \tag{4.41}$$

$\chi_s(t)$ is, up to a constant multiple, a Gauss error function :

$$\chi_s(t) = e^{-t^2/2}\int_0^t e^{s^2/2}ds = \frac{1}{t} + \frac{1}{t^3} + \cdots \quad (|t| \to \infty). \tag{4.42}$$

In fact, letting $\varphi(t) = \displaystyle\int_0^{\infty}\sin(t\lambda)e^{-\lambda^2/2}d\lambda$, we have the differential equation $\varphi'(t) = 1 - \varphi(t)$, from which we get (4.42). Since $\varphi(t) = t + O(t^2)$ as $t \to 0$, we have $\tau\varphi(x_2/|\tau|) \to \mathrm{sgn}\,(\tau)x_2$, which implies

$$\int_{\mathbf{R}^2}|f(x,\theta)|^2\psi(x)dx \to C\int_{-\infty}^{\infty}x_2^2\psi(0,x_2)dx_2 \tag{4.43}$$

for any $\psi \in C_0^{\infty}(\mathbf{R}^2)$.

Proof of Theorem 4.9. We first prove that $\zeta(\xi,\theta)$ admits the following asymptotic expansion for $|\xi_1| < |\tau|^{2\epsilon}, 0 < \xi_2 < 1$:

$$\zeta(\xi,\theta) \sim i\tau + \xi_2 + \sum_{n=1}^{\infty}\frac{P_n(\xi)}{(i\tau)^n}, \tag{4.44}$$

where $P_n(\xi)$ is a polynomial of order $n+1$. More precisely, for any $N \geq 1$ there exists $\epsilon_N > 0$ such that for $0 < \epsilon < \epsilon_N$ one can find a constant $C_{\epsilon,N} > 0$ for which

$$\Big|\zeta(\xi,\theta) - i\tau - \xi_2 - \sum_{n=1}^{N}\frac{P_n(\xi)}{(i\tau)^n}\Big| \leq C_{\epsilon,N}|\tau|^{-N-1/2} \tag{4.45}$$

holds for $|\tau| > C_{\epsilon,N}, |\xi_1| < |\tau|^{2\epsilon}, 0 < \xi_2 < 1$.

To prove (4.45), we put $A = i\tau + \xi_2$ and $B = \xi_1^2$. Then since $\zeta^2 = A^2 + B$, we have

$$\zeta = A + \frac{B}{\zeta + A} =: A + \kappa. \tag{4.46}$$

When $|\xi_1| < |\tau|^{2\epsilon}, 0 < \xi_2 < 1$, we have shown in Proposition 3.3 (1) that $|\kappa| \leq C|\tau|^{4\epsilon-1}$. Then we have

$$\kappa = \frac{B}{2A+\kappa} = \frac{B}{2A}\sum_{n=0}^{\infty}\Big(-\frac{\kappa}{2A}\Big)^n. \tag{4.47}$$

Note that

$$\frac{1}{A} = \sum_{n=0}^{\infty}(-\xi_2)^n(i\tau)^{-n-1}. \tag{4.48}$$

We first drop the terms with $n \geq 1$ in the right-hand side of (4.47) and use (4.48). Then we get the expansion (4.45) with $N = 1$. Using this to the term in the right-hands side of (4.47) with $n = 1$ and droping ther terms with $n \geq 2$, we get (4.45) with $N = 2$. Repeating this procedure, we can prove (4.45). In particular we have

$$\zeta = i\tau + \xi_2 + \frac{\xi_1^2}{2}\frac{1}{i\tau} - \frac{\xi_1^2\xi_2}{2}\frac{1}{(i\tau)^2} + \left(\frac{\xi_1^2\xi_2^2}{2} - \frac{\xi_1^4}{8}\right)\frac{1}{(i\tau)^3} + O(|\tau|^{-3-1/2}). \tag{4.49}$$

The polynomial $P_n(\xi)$ has the following property :

(4.50) *When n is odd (even), $P_n(\xi)$ is of even (odd) order with respect to ξ_2.*

In fact, putting $K = -\kappa/(i\tau)$, $b = B/\tau^2$ and $a = -\xi_2/(i\tau)$, we have by (4.47)

$$K = \frac{b}{2}\sum_{n=0}^{\infty} a^n \left(\frac{K}{2}\sum_{m=0}^{\infty} a^m\right)^n. \tag{4.51}$$

Therefore K is a power series of a and b, which means that κ/τ is a power series of ξ_1^2/τ^2 and ξ_2/τ. Hence, the assertion (4.50) follows.

If $|\xi_1| > |\tau|^{2\epsilon}$ or $|\xi_2| > 1$, $\widehat{f}(\xi,\theta)$ is rapidly decreasing in τ. Hence by using (4.30) and (4.44) we have the following expansion

$$\begin{aligned}&(2\pi)^{-1}\int_{\xi_2>0} e^{ix\cdot\xi}\widehat{f}(\xi,\theta)d\xi \\ &\approx \frac{C_0(\tau)}{2\pi}\int_{\xi_2>0} e^{ix\cdot\xi}\widehat{\chi}\Big(\frac{\xi_1}{|\tau|^{\epsilon}}\Big)\widehat{\chi}\Big(|\tau|\xi_2\Big)\left(i\tau + \xi_2 + \sum_{n=1}^{\infty}\frac{P_n(\xi)}{(i\tau)^n}\right)d\xi.\end{aligned} \tag{4.52}$$

For $|\xi_1| < |\tau|^{2\epsilon}$, $-1 < \xi_2 < 0$, instead of (4.46), one should start with

$$\zeta = -A + \frac{B}{\zeta - A}.$$

This meas that one should replace τ and ξ_2 by $-\tau$ and $-\xi_2$ in the argument to derive (4.44). Thus one gets

(4.53)

$$\begin{aligned}&(2\pi)^{-1}\int_{\xi_2<0} e^{ix\cdot\xi}\widehat{f}(\xi,\theta)d\xi \\ &\approx \frac{C_0(\tau)}{2\pi}\int_{\xi_2<0} e^{ix\cdot\xi}\widehat{\chi}\Big(\frac{\xi_1}{|\tau|^{\epsilon}}\Big)\widehat{\chi}\Big(|\tau|\xi_2\Big)\left(-i\tau - \xi_2 + \sum_{n=1}^{\infty}\frac{P_n(\xi_1,-\xi_2)}{(-i\tau)^n}\right)d\xi \\ &\approx \frac{C_0(\tau)}{2\pi}\int_{\xi_2>0} e^{ix_1\cdot\xi_1}e^{-ix_2\cdot\xi_2}\widehat{\chi}\Big(\frac{\xi_1}{|\tau|^{\epsilon}}\Big)\widehat{\chi}\Big(|\tau|\xi_2\Big)\left(-i\tau + \xi_2 + \sum_{n=1}^{\infty}\frac{P_n(\xi_1,\xi_2)}{(-i\tau)^n}\right)d\xi,\end{aligned}$$

where one has used the fact that $\widehat{\chi}(\lambda)$ is an even function.

Adding (4.52) and (4.53), and using

$$(-i\partial_t)^{2m+1}\int_0^{\infty}\sin(t\lambda)\widehat{\psi}(\lambda)d\lambda = \frac{1}{2i}\int_0^{\infty}(e^{it\lambda} + e^{-it\lambda})\lambda^{2m+1}\widehat{\psi}(\lambda)d\lambda,$$

$$(-i\partial_t)^{2m}\int_0^{\infty}\sin(t\lambda)\widehat{\psi}(\lambda)d\lambda = \frac{1}{2i}\int_0^{\infty}(e^{it\lambda} - e^{-it\lambda})\lambda^{2m}\widehat{\psi}(\lambda)d\lambda,$$

we have completed the proof of the theorem. ♢

Theorem 4.9 simplifies the computation of the integral mean (4.32) in Theorem 4.8. For example, we have from (4.49)

$$
\begin{aligned}
f(x,\theta) = iC_1(\tau)\left[\tau - \partial_{x_2} + \frac{(\partial_{x_1})^2}{2\tau} + \frac{(\partial_{x_1})^2\partial_{x_2}}{2\tau^2}\right]|\tau|^{\epsilon/2}\chi(|\tau|^{\epsilon}x_1)\chi_s\big(\frac{x_2}{|\tau|}\big) \\
+ O_{L^2}(|\tau|^{-2-1/2}).
\end{aligned} \tag{4.54}
$$

Since $f(x,\theta) = O_{L^2}(|\tau|^{3/2})$, one can replace $f(\theta_n)$ by the first term of the right-hand side of (4.54). Another application is the following.

Corollary 4.10.

$$
F(|x_1| > |\tau|^{-\epsilon/2})f(x,\theta) \approx 0. \tag{4.55}
$$

Theorem 4.8 and Corollary 4.10 then imply the following theorem.

Theorem 4.11. *Let $\chi \in C^\infty(\mathbf{R})$ be such that $\chi(t) = 1$ if $|t| < 1$, $\chi(t) = 0$ if $|t| > 2$. We put $\chi_n(x_1) = \chi(\tau_n^{\epsilon/2}x_1)$. Then we have*

$$
\begin{aligned}
\lim_{n\to\infty}\big((\mathcal{R}_V(\theta_n) - \mathcal{R}_0(\theta_n))\chi_n(x_1)f(\theta_n), \chi_n(x_1)f(\overline{\theta}_n)\big)_{\partial D_\delta} \\
= i\int_{\mathbf{R}\times(\delta,\infty)} V(0,x_2,y)\frac{dx_2dy}{y^2}.
\end{aligned} \tag{4.56}
$$

5. Horosphere boundary value problem

5.1. **Hyperbolic isometry on $\mathbf{H}^3$.** We represent $(x_1,x_2,x_3) \in \mathbf{R}^3_+ = \mathbf{H}^3$ by quarternions : $\mathbf{z} = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j}$, which is also represented by a 2×2 matrix :

$$
\mathbf{z} = x_1\mathbf{1} + x_2\mathbf{i} + x_3\mathbf{j} = \begin{pmatrix} x_1 + ix_3 & x_2 \\ -x_2 & x_1 - ix_3 \end{pmatrix}. \tag{5.1}
$$

It is well-known that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{C})$, the action

$$
\mathbf{z} \to \gamma\cdot\mathbf{z} = (a\mathbf{z} + b)(c\mathbf{z} + d)^{-1} \tag{5.2}
$$

is an isometry on $\mathbf{H}^3$. When $x_3 = 0$, this is a linear fractional transformation on $\mathbf{R}^2\times\{0\} \simeq \mathbf{C}$. We choose γ in such a way that this induced transformation maps the circle $\{|x| = R,\ x_3 = 0\}$ to the line $x_1 = x_3 = 0$, i.e.

$$
\gamma = \begin{pmatrix} 1/\sqrt{2R} & -\sqrt{R/2} \\ 1/\sqrt{2R} & \sqrt{R/2} \end{pmatrix}. \tag{5.3}
$$

Then by the action (5.2), the hemi-sphere $\{|x| = R,\ x_3 > 0\}$ is mapped to the semi-plane $\{x_1 = 0,\ x_3 > 0\}$. We show these facts by a direct computation.

Lemma 5.1. *The map $x \to y$ defined by*

$$
\mathbf{w} = (\mathbf{z} - R\mathbf{1})(\mathbf{z} + R\mathbf{1})^{-1} = y_1\mathbf{1} + y_2\mathbf{i} + y_3\mathbf{j} \tag{5.4}
$$

is an isometry on $\mathbf{H}^3$, which maps
(1) the hemisphere $\{|x| = R,\ x_3 > 0\}$ to the semi-plane $\{y_1 = 0,\ y_3 > 0\}$,
(2) the half-ball $\{|x| < R,\ x_3 > 0\}$ to the quarter region $\{y_1 < 0,\ y_3 > 0\}$,
(3) the semi-plane $\{x_2 = 0,\ x_3 > 0\}$ to the semi-plane $\{y_2 = 0,\ y_3 > 0\}$,

(4) the horosphere $\left\{(x_1+R)^2+x_2^2+(x_3-R/\delta)^2=(R/\delta)^2\right\}$ *to the horizontal plane* $\{y_3=\delta\}$*,* $\delta>0$*, and the horoball* $\left\{(x_1+R)^2+x_2^2+(x_3-R/\delta)^2<(R/\delta)^2\right\}$ *to the half-space* $\{y_3>\delta\}$*,*
(5) the measure $dS_E/(x_3)^2$ *on the hemisphere* $\{|x|=R,\ x_3>0\}$ *to the measure* $dy_2dy_3/(y_3)^2$ *on the semi-plane* $\{y_1=0,\ y_3>0\}$*, where* dS_E *is the measure on the sphere* $\{|x|=R\}$ *induced from the Euclidean metric* $(dx)^2$*.*

Proof. By using (5.1), we have

$$\left\{\begin{aligned} y_1 &= \frac{x_1^2+x_2^2+x_3^2-R^2}{(x_1+R)^2+x_2^2+x_3^2},\\ y_2 &= \frac{2x_2R}{(x_1+R)^2+x_2^2+x_3^2},\\ y_3 &= \frac{2x_3R}{(x_1+R)^2+x_2^2+x_3^2}. \end{aligned}\right. \tag{5.5}$$

This is a composition of isometries on $\mathbf{H}^3$, translation : $(x_1,x_2,x_3)\to(x_1+a,x_2,x_3)$, refelection : $(x_1,x_2,x_3)\to(-x_1,x_2,x_3)$, and inversion with respect to the sphere : $x\to a^2x/|x|^2$ ([3], p. 24). The assertions (1) $\sim$ (4) are straightforward consequences of (5.5). Let us note that the inverse transform

$$\mathbf{z}=R(\mathbf{1}-\mathbf{w})^{-1}(\mathbf{1}+\mathbf{w}) \tag{5.6}$$

is written as

$$\left\{\begin{aligned} x_1 &= R\frac{1-y_1^2-y_2^2-y_3^2}{(y_1-1)^2+y_2^2+y_3^2},\\ x_2 &= R\frac{2y_2}{(y_1-1)^2+y_2^2+y_3^2},\\ x_3 &= R\frac{2y_3}{(y_1-1)^2+y_2^2+y_3^2}. \end{aligned}\right. \tag{5.7}$$

We prove (5). Letting $y_1=0, y_2=r\cos\theta, y_3=r\sin\theta$ in (5.7), we have

$$(dx_1)^2+(dx_2)^2+(dx_3)^2=\frac{4R^2}{(1+r^2)^2}\left((dr)^2+(rd\theta)^2\right). \tag{5.8}$$

Therefore the measure dS_E on the sphere $\{|x|=R\}$ induced from the Euclidean metric $(dx)^2$ is written as

$$dS_E=\frac{4R^2}{(1+r^2)^2}rdrd\theta=\frac{4R^2}{(1+r^2)^2}dy_2dy_3.$$

The assertion (5) then follows from this. $\diamondsuit$

5.2. **Main Theorems.** We are now in a position to solve the inverse problem in the horosphere. Suppose we are given a bounded open ball in $\mathbf{R}^3$. Without loss of generality, we assume that this ball is defined by

$$\Omega=\{(x_1+R)^2+x_2^2+(x_3-R/\delta)^2<(R/\delta)^2\}. \tag{5.9}$$

Assumption 5.2. *We assume that the potential* $q(x) \in C^1(\Omega\,;\mathbf{C})$ *satisfies for some* $d > 5/2$

$$|\partial_x^\alpha q(x)| \leq C r^{d-|\alpha|}, \quad |\alpha| \leq 1, \tag{5.10}$$

where C *is a constant and*

$$r = \big((x_1+R)^2 + x_2^2 + x_3^2)\big)^{1/2}. \tag{5.11}$$

Let us consider the boundary value problem

$$\begin{cases} (-\Delta + q(x))w = 0 & \text{in} \quad \Omega, \\ Bw = f & \text{on} \quad \partial\Omega, \end{cases} \tag{5.12}$$

B being an operator on the boundary to be explained below. Then $v = x_3^{1/2} w$ satisfies the following equation

$$(A_0 + V(x))v = 0 \quad \text{in} \quad \Omega, \tag{5.13}$$

where

$$A_0 = -x_3^2\Delta + x_3\partial_3 - E, \quad V(x) = x_3^2 q(x), \quad E = \frac{3}{4}. \tag{5.14}$$

We consider the equation (5.13) under the measure containing a large parameter $\tau > 0$, namely in $L^2(\Omega\,;\rho_\tau(x)dx)$, where

$$\rho_\tau(x) = \frac{e^{2\tau y_2(x)}}{(x_3)^3}, \quad y_2(x) = \frac{2x_2R}{(x_1+R)^2 + x_2^2 + x_3^2}. \tag{5.15}$$

By the gauge transformation $v \to u = e^{\tau y_2(x)}v$, this is equivalent to considering the boundary value problem for the operator $A_0(\tau) + V(x)$, where

$$\begin{aligned} A_0(\tau) &= e^{\tau y_2(x)} A_0 e^{-\tau y_2(x)} \\ &= -x_3^2(\nabla - \tau b(x))^2 + x_3(\partial_3 - \tau b_3(x)) - E, \quad b(x) = \nabla y_2(x), \end{aligned} \tag{5.16}$$

and this operator is defined in $L^2(\Omega\,; dx/(x_3)^3)$. Next we use the hyperbolic isometry in Lemma 5.1 to map the ball Ω to the half-space $D_\delta = \{y_3 > \delta\}$. Then the operator $A_0(\tau) + V$ is mapped to $H_0(\theta) + V$ studied in §4.

By (5.5), we have

$$r = 2R\big((y_1-1)^2 + y_2^2 + y_3^2\big)^{-1/2}. \tag{5.17}$$

Since $V(x)$ satisfies $|\partial_x^\alpha V(x)| \leq C r^{d+2-|\alpha|}$ by (5.10), the formula (5.17) shows that the assumption (2.26) is satisfied for $V(x(y))$. Note that we are now writing $y = (y_1, y_2, y_3)$ instead of (x, y) in (2.26).

Transforming back to Ω, we obtain the following results. Let $\mathcal{X}_s^{(-)}(\Omega)$ be the space defined by (see (2.23))

$$\mathcal{X}_s^{(-)}(\Omega) \ni u \Longleftrightarrow \int_\Omega \frac{y_3(x)}{(1 + |\log y_3(x)|)^{2s}} |u(x)|^2 \frac{dx}{(x_3)^3} < \infty, \tag{5.18}$$

where $y_3(x)$ is defined by (5.5), and $s > 1$ is chosen sufficiently close to 1. Let

$$L^2_\theta(\mathbf{R}^2) \ni g \Longleftrightarrow g \in L^2(\mathbf{R}^2), \quad \widehat{g}(\xi)/\zeta(\xi,\theta) \in L^2(\mathbf{R}^2), \tag{5.19}$$

and $L^2_\theta(\partial\Omega)$ be the pull-back of $L^2_\theta(\mathbf{R}^2)$.

Theorem 5.3. *Take $T_0 > 0$ large enough. Then for any $\pm\tau > T_0$ there exists a unique solution $u \in \mathcal{X}_s^{(-)}(\Omega)$ of the boundary value problem*

$$\left\{ \begin{array}{lll} (A_0(\tau) + V)u = 0 & \text{in} & \Omega, \\ B(\tau)u = f & \text{on} & \partial\Omega \end{array} \right. \tag{5.20}$$

with any boundary data $f \in L^2_\theta(\partial\Omega)$. Here $B(\tau) = \partial/\partial n - P(\tau)$, n is the outer unit normal to $\partial\Omega$ with respect to the hyperbolic metric and $P(\tau)/\delta$ is the push-forward of $P(\theta)$ in (4.9).

Let us define the generalized Robin-to-Dirichlet map by

$$\mathcal{R}(\tau)f = u\big|_{\partial\Omega}, \tag{5.21}$$

where u is a solution to the boundary value problem (5.20). Theorems 4.8 and 4.11 are transferred in the horosphere as follows.

Theorem 5.4. *Let $\mathcal{R}_0(\tau)$ and $\mathcal{R}_V(\tau)$ be the GRD maps associated with $A_0(\tau)$ and $A_0(\tau) + V$, respectively. Then there exists a boundary data $f(\pm\tau)$ defined on $\partial\Omega$, which does not depend on V, having the following property :*

$$\lim_{n\to\infty} ((\mathcal{R}_V(\tau_n) - \mathcal{R}_0(\tau_n))f(\tau_n), f(-\tau_n))_{\partial\Omega} = i\int_{S\cap\Omega} V(x)dS, \tag{5.22}$$

where $S = \{|x| = R\}$, $dS = dS_E/(x_3)^2$, dS_E is the measure on S induced from the Euclidean metric $(dx)^2$, $\tau_n = (n + 1/2)\pi/\delta$.

Theorem 5.5. *Let $\chi \in C^\infty(\partial\Omega)$ be such that $\chi(t) = 1$ if $|t| < 1/2$, $\chi(t) = 0$ if $|t| > 1$. We put $\chi_n(x) = \chi(\tau_n^{\epsilon/2} y_1)$, where y_1 is defined by (5.5). Then we have*

$$\lim_{n\to\infty} ((\mathcal{R}_V(\tau_n) - \mathcal{R}_0(\tau_n))\chi_n f(\tau_n), \chi_n f(-\tau_n))_{\partial\Omega} = i\int_{S\cap\Omega} V(x)dS, \tag{5.23}$$

Since $|y_1| < \tau_n^{-\epsilon/2}$, the support of $\chi_n(x)$ is contained in the curved sector

$$\begin{aligned} S = \Big\{ x \in \mathbf{R}^3_+ ; \Big(x_1 - \frac{tR}{1-t}\Big)^2 + x_2^2 + x_3^2 < \frac{R^2}{(1-t)^2}, \\ \Big(x_1 + \frac{tR}{1+t}\Big)^2 + x_2^2 + x_3^2 > \frac{R^2}{(1+t)^2} \Big\} \end{aligned} \tag{5.24}$$

where $t = \tau_n^{-\epsilon/2}$ (see Figure 1).

6. Reconstruction of the potential

6.1. **Radon transform.** Let us first recall the Radon transform on $\mathbf{H}^3$. Let $B = \{|x| < 1\}$ be the unit ball in $\mathbf{R}^3$ and regard it as the hyperbolic space equipped with the metric

$$ds^2 = \frac{4(dx)^2}{(1 - |x|^2)^2}. \tag{6.1}$$

Let Ξ be the set of all spheres which intersect orthogonally with ∂B (more precisely the intersection of the sphere and B). The Radon transform is defined by

$$Rf(\xi) = \int_\xi f(x)dS(x), \quad \xi \in \Xi, \tag{6.2}$$

where $dS(x)$ is the measure on ξ induced from the hyperbolic metric. The adjoint Radon transform is defined by

$$R^*\varphi(x) = \int_{x\in\xi} \varphi(\xi)d\mu(\xi) = \int_K \varphi(gk\cdot\eta)dk, \tag{6.3}$$

where $K = SO(3), g \in SO(2,1)$ such that $g\cdot o = x$, o being the origin and η is an arbitrary fixed element in Ξ passing through o. Then the following inversion formula holds (see e.g. [12] p. 159) :

Theorem 6.1. *For any rapidly decreasing function f, we have*

$$f = -\frac{1}{2\pi}(1+\Delta_g)R^*Rf, \tag{6.4}$$

where Δ_g is the Laplace-Beltrami operator on $\mathbf{H}^3$.

6.2. **Reconstruction procedure.** Let us return to our original Schrödinger operator $-\Delta + q(x)$ defined in a ball $B \subset \mathbf{R}^3$. We stress here that Δ is the Euclidean Laplacian. Without loss of generality we assume that $B = \{|x| < 1\}$. Suppose $q(x) \in C^1(B;\mathbf{C})$ and for some $d > 5/2$

$$|\partial_x^\alpha q(x)| \le C(1-|x|)^{d-|\alpha|}, \quad |\alpha| \le 1, \tag{6.5}$$

where $|x|$ is the Euclidean length of x. Given any sphere S which intersects orthogonally with ∂B, we take a point $p \in \partial B\cap S$ arbitrarily and rotate p to $(0,0,-1)$. We next translate the whole system so that B lies in the upper half space, which is denoted by B', p is on the horizontal plane $\{x_3 = 0\}$, and S becomes the sphere centered at the origin, which is denoted by S'. Then the Schrödinger operator $-\Delta + q$ is transformed to $-\Delta + \widetilde{q}$, where $\widetilde{q}$ is obtained from q by rotation and translation. We imbed this system into the hyperbolic space $\mathbf{H}^3$ realized as the upper-half space $\mathbf{R}^3_+$ and do the measurement in Theorem 5.3, i.e. we consider the GRD map associated with the potential $(x_3)^2\widetilde{q}(x)$. By Theorem 5.4 we get

$$\int_{S'\cap B'} (x_3)^2\widetilde{q}(x)dS' = \int_{S'\cap B'} \widetilde{q}(x)dS'_E, \tag{6.6}$$

where dS'_E is the measure on S' induced from the Euclidean metric $(dx)^2$, since the factor $(x_3)^2$ cancels out. We then translate and rotate back to get

$$\int_{S\cap B} q(x)dS_E. \tag{6.7}$$

Letting $q_0(x) = q(x)(1-|x|^2)^2/4$, we can rewrite this as

$$\int_{S\cap B} q_0(x)\frac{4dS_E}{(1-|x|^2)^2}. \tag{6.8}$$

Regrading B as a ball model of $\mathbf{H}^3$, we have thus obtained the Radon transform of $q_0(x)$ in $\mathbf{H}^3$. The inverse Radon transform enables us to reconstruct $q(x)$.

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