Derived equivalences and Serre duality for Gorenstein algebras

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Abstract

We introduce a notion of Gorenstein algebras of codimension c and demonstrate that Serre duality theory plays an essential role in the theory of derived equivalences for Gorenstein algebras.

Let R be a commutative noetherian ring and A a Noether R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. Let $c \geq 0$ be an integer. Assume that $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$ and set

$$\Omega = \mathrm{Ext}^c_R(A, R).$$

We call A a Gorenstein R-algebra of codimension c if $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and Ω is a projective generator for right A-modules. If A is a Gorenstein R-algebra of codimension c, then we will show that Ω lies in the center of the Picard group of A (Proposition 3.7), that Ω is a dualizing complex for A if $\sup\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$ (Proposition 2.6), and that $\mathrm{Ann}_R(A)$ contains an R-regular sequence $x_1, \cdots, x_c$ and A is a Gorenstein S-algebra of codimension 0, where S is the residue ring of R over the ideal generated by $x_1, \cdots, x_c$ (Proposition 2.9). Also, we will see that our Gorenstein algebras are Gorenstein in the sense of [12] (Proposition 2.3). In particular, commutative Gorenstein algebras are Gorenstein rings. We refer to [12] for properties enjoyed by Gorenstein algebras and for the relationship of the notion of Gorenstein algebras to the theory of commutative Gorenstein rings.

Our main aim of this note is to demonstrate that Serre duality theory plays an essential role in the theory of derived equivalences for Gorenstein algebras. In Section 3, we will extend Serre duality theory (cf. [8]) to Noether algebras.

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We will see that for an arbitrary Noether R-algebra A there exists a bifunctorial isomorphism in Mod-R

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(Y^\bullet, X^\bullet \otimes^{\mathbf{L}}_A V^\bullet) \cong \mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet)^*$$

for $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\text{-}A)_{\mathrm{fpd}}$ and $Y^\bullet \in \mathcal{D}(\mathrm{Mod}\text{-}A)$, where $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet)$ with $I^\bullet$ a minimal injective resolution of R and $(-)^* = \mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}R)}(-, R)$ (Proposition 3.3). In particular, a Gorenstein R-algebra A of codimension c with $\Omega \cong A$ as A-bimodules is $(d-c)$-Calabi-Yau$^-$ in the sense of [15] (cf. also [11]) provided $d = \dim R_{\mathfrak{p}}$ for all maximal $\mathfrak{p} \in \mathrm{Supp}_R(A)$ (Corollary 3.4). On the other hand, we know from [1, Theorem 4.7] that if $V^\bullet$ is a dualizing complex for A and if $\mathrm{inj\,dim}\ {}_AA = \mathrm{inj\,dim}\ A_A < \infty$ then $- \otimes^{\mathbf{L}}_A V^\bullet$ induces a self-equivalence of $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\text{-}A)$.

Assume that A is a Gorenstein R-algebra of codimension c. Let $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a tilting complex and $B = \mathrm{End}_{\mathcal{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$. In Section 4, we will ask when B is also a Gorenstein R-algebra of codimension c. Set $\nu = - \otimes^{\mathbf{L}}_A \Omega$. Then by Serre duality theory we have an isomorphism of B-bimodules

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, \nu P^\bullet[i]) \cong \mathrm{Ext}^{i+c}_R(B, R)$$

for all $i \in \mathbb{Z}$. On the other hand, denoting by $\mathcal{S}$ the full subcategory of $\mathcal{D}^-(\mathrm{Mod}\text{-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$, we have an equivalence $\mathrm{Hom}_{\mathcal{D}(\mathrm{Mod}\text{-}A)}(P^\bullet, -) : \mathcal{S} \to \mathrm{Mod}\text{-}B$ (see [20, Section 4]). Thus B is a Gorenstein R-algebra of codimension c if and only if $\mathrm{add}(\nu P^\bullet) = \mathrm{add}(P^\bullet)$ (Corollary 4.4). Unfortunately, this is not the case in general (Example 4.6). However, B is a Gorenstein R-algebra of codimension c with $\mathrm{Ext}^c_R(B, R) \cong B$ as B-bimodules if and only if A is a Gorenstein R-algebra of codimension c with $\Omega \cong A$ as A-bimodules (Corollary 4.5).

We refer to [7], [13] and [22] for basic results in the theory of derived categories and to [20], [21] for definitions and basic properties of tilting complexes and derived equivalences. Also, we refer to [10] for standard homological algebra in module categories and to [18] for standard commutative ring theory.

1 Preliminaries

Notation

For a ring A we denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely presented modules. We denote by Proj-A (resp., Inj-A) the full subcategory of Mod-A consisting of projective (resp., injective) modules and by $\mathcal{P}_A$ the full subcategory of Proj-A consisting of finitely generated projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module X considered is a right (resp., left) A-module. In particular, we denote by proj dim X_A (resp., proj dim ${}_AX$) the projective dimension of a right (resp., left) A-module X. Similar notation is used to denote the injective dimension.

In this note, complexes are cochain complexes of modules and, as usual, modules are considered as complexes concentrated in degree zero. For any $n \in \mathbb{Z}$ we denote by $\mathrm{B}^n(-)$, $\mathrm{Z}^n(-)$, $\mathrm{B}'^n(-)$, $\mathrm{Z}'^n(-)$ and $\mathrm{H}^n(-)$ the n-th boundary, the n-th cycle, the n-th coboundary, the n-th cocycle and the n-th homology of a complex, respectively. For an additive category $\mathcal{B}$, we denote by $\mathcal{K}(\mathcal{B})$ (resp., $\mathcal{K}^+(\mathcal{B})$, $\mathcal{K}^-(\mathcal{B})$, $\mathcal{K}^{\mathrm{b}}(\mathcal{B})$) the homotopy category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{B}$. For an abelian category $\mathcal{A}$, we denote by $\mathcal{D}(\mathcal{A})$ (resp., $\mathcal{D}^-(\mathcal{A})$, $\mathcal{D}^+(\mathcal{A})$, $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$) the derived category of complexes (resp., complexes with bounded above homology, complexes with bounded below homology, complexes with bounded homology) over $\mathcal{A}$. We always consider $\mathcal{K}^*(\mathcal{B})$ (resp., $\mathcal{D}^*(\mathcal{A})$) as a full triangulated subcategory of $\mathcal{K}(\mathcal{B})$ (resp., $\mathcal{D}(\mathcal{A})$) closed under isomorphism classes, where $* = +,\ -$ or b. In particular, for a noetherian ring A, we identify $\mathcal{D}^*(\text{mod-}A)$ with $\mathcal{D}^*_{\text{mod-}A}(\text{Mod-}A)$, the full triangulated subcategory of $\mathcal{D}^*(\text{Mod-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{H}^n(X^\bullet) \in \text{mod-}A$ for all $n \in \mathbb{Z}$, where $* = -$ or b. We denote by $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$) the single complex associated with the double hom (resp., tensor) complex and by $\mathbf{R}\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^{\mathbf{L}} -$) the right (resp., left) derived functor of $\mathrm{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$).

Finally, for an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X.

Gorenstein dimension

Throughout this note, R is a commutative noetherian ring. We denote by $\dim R$ the Krull dimension of R, by $\mathrm{Spec}(R)$ the set of prime ideals of R and by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p} \in \mathrm{Spec}(R)$. For an R-module M, we set $\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$ and $\mathrm{Ann}_R(M) = \{x \in R \mid xM = 0\}$ and we denote by $E_R(M)$ an injective envelope of M in Mod-R. We set

$$D = \mathbf{R}\mathrm{Hom}^\bullet_R(-, R) : \mathcal{D}(\text{Mod-}R) \to \mathcal{D}(\text{Mod-}R).$$

Then for any $X^\bullet, Y^\bullet \in \mathcal{D}(\text{Mod-}R)$ we have a bifunctorial isomorphism

$$\theta_{X^\bullet, Y^\bullet} : \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(X^\bullet, DY^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(Y^\bullet, DX^\bullet).$$

For any $X^\bullet \in \mathcal{D}(\text{Mod-}R)$ we set

$$\xi_{X^\bullet} = \theta^{-1}_{X^\bullet, DX^\bullet}(\mathrm{id}_{DX^\bullet}) : X^\bullet \to D^2 X^\bullet.$$

Also, for any complex $X^\bullet$ and $k \in \mathbb{Z}$, we define the following truncated complexes

$$\begin{aligned}
\sigma'_{\geq k}(X^\bullet) &: \cdots \to 0 \to \mathrm{Z}'^k(X^\bullet) \to X^{k+1} \to X^{k+2} \to \cdots ,\\
\sigma'_{< k}(X^\bullet) &: \cdots \to X^{k-2} \to X^{k-1} \to \mathrm{B}^k(X^\bullet) \to 0 \to \cdots ,\\
\sigma_{\leq k}(X^\bullet) &: \cdots \to X^{k-2} \to X^{k-1} \to \mathrm{Z}^k(X^\bullet) \to 0 \to \cdots ,\\
\sigma_{> k}(X^\bullet) &: \cdots \to 0 \to \mathrm{B}'^k(X^\bullet) \to X^{k+1} \to X^{k+2} \to \cdots .
\end{aligned}$$

In this subsection, we recall several basic results on Gorenstein dimension for finitely generated R-modules and bounded complexes of finitely generated R-modules (see e.g. [9] for details).

Definition 1.1 ([3]). A module $M \in \text{mod-}R$ is said to have Gorenstein dimension zero if the canonical homomorphism

$$M \to \text{Hom}_R(\text{Hom}_R(M, R), R), x \mapsto (f \mapsto f(x))$$

is an isomorphism and $\text{Ext}^i_R(M, R) = \text{Ext}^i_R(\text{Hom}_R(M, R), R) = 0$ for $i > 0$. We denote by $\mathcal{G}_R$ the full additive subcategory of mod-R consisting of modules which have Gorenstein dimension zero. Note that $\mathcal{P}_R \subset \mathcal{G}_R$. Next, a module $M \in \text{mod-}R$ is said to have finite Gorenstein dimension if M has a left resolution $P^\bullet \to M$ with $P^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{G}_R)$.

Definition 1.2. A complex $X^\bullet \in \mathcal{D}^{\text{b}}(\text{mod-}R)$ is said to have finite Gorenstein dimension if $X^\bullet \cong Y^\bullet$ in $\mathcal{D}(\text{Mod-}R)$ for some $Y^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{G}_R)$.

Remark 1.3. For any $M \in \text{mod-}R$ the following are equivalent.

(1) M has finite Gorenstein dimension as a module.

(2) M has finite Gorenstein dimension as a complex.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. Conversely, let $Y^\bullet \cong M$ in $\mathcal{D}(\text{Mod-}R)$ with $Y^\bullet \in \mathcal{K}^{\text{b}}(\mathcal{G}_R)$. Since $\text{H}^i(Y^\bullet) = 0$ for $i > 0$, it follows by [3, Lemma 3.10] that $\text{Z}^0(Y^\bullet) \in \mathcal{G}_R$. Thus we have a left resolution $\sigma_{\leq 0}(Y^\bullet) \to M$ with $\sigma_{\leq 0}(Y^\bullet) \in \mathcal{K}^{\text{b}}(\mathcal{G}_R)$. □

Lemma 1.4 ([16, Proposition 2.10]). *For any $X^\bullet \in \mathcal{D}^{\text{b}}(\text{mod-}R)$ the following are equivalent.*

(1) $X^\bullet$ has finite Gorenstein dimension.

(2) $\text{H}^i(DX^\bullet) = 0$ for $i \gg 0$ and $\xi_{X^\bullet}$ is an isomorphism.

Lemma 1.5 ([17]). *Let $0 \to L \to M \to N \to 0$ be an exact sequence in* mod-R. *Then the following hold.*

(1) If L, M have finite Gorenstein dimension, so does N.

(2) If M, N have finite Gorenstein dimension, so does L.

(3) If N, L have finite Gorenstein dimension, so does M.

Proof. For the benefit of the reader, we include a proof. Since we have a distinguished triangle $DN \to DM \to DL \to$ in $\mathcal{D}(\text{Mod-}R)$, and since we have a homomorphism of distinguished triangles in $\mathcal{D}(\text{Mod-}R)$

$$\begin{tikzcd} L \arrow[r] \arrow[d, "\xi_L"'] & M \arrow[r] \arrow[d, "\xi_M"'] & N \arrow[r] \arrow[d, "\xi_N"'] & {} \\ D^2L \arrow[r] & D^2M \arrow[r] & D^2N \arrow[r] & {} \end{tikzcd},$$

the assertions follow by Lemma 1.4 together with Remark 1.3. □

We refer to [6] for the definition and basic properties of commutative Gorenstein rings.

Remark 1.6. Let $M \in \text{mod-}R$ with $R_\mathfrak{p}$ Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(M)$. Then M has Gorenstein dimension zero if $\mathrm{Ext}^i_R(M,R) = 0$ for $i > 0$. In particular, $M = 0$ if $\mathrm{Ext}^i_R(M,R) = 0$ for all $i \geq 0$.

Proof. It suffices to see that $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R) = 0$ for $i > 0$ and

$$M \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R), x \mapsto (f \mapsto f(x)).$$

So, localizing at each $\mathfrak{p} \in \mathrm{Supp}_R(M) \subset \mathrm{Supp}_R(A)$, we may assume that R is a Gorenstein local ring. Take a projective resolution $P^\bullet \to M$ in mod-R and set $Q^\bullet = \mathrm{Hom}^\bullet_R(P^\bullet, R)$. Then for any $i > 0$ we have

$$\mathrm{Ext}^i_R(\mathrm{Z}'^1(Q^\bullet),R) \cong \mathrm{Ext}^{i+l-1}_R(\mathrm{Z}'^l(Q^\bullet),R) = 0$$

for $l > \dim R$. The assertion follows by [3, Proposition 3.8]. □

Remark 1.7. Assume that $M \in \text{mod-}R$ has finite Gorenstein dimension. Assume that there exists an integer $c \geq 0$ such that $\mathrm{Ext}^i_R(M,R) = 0$ for $i \neq c$ and set $N = \mathrm{Ext}^c_R(M,R)$. Then $\mathrm{Ext}^i_R(N,R) = 0$ for $i \neq c$ and $M \xrightarrow{\sim} \mathrm{Ext}^c_R(N,R)$.

Proof. Since $DM \cong N[-c]$ in $\mathcal{D}(\text{Mod-}R)$, and since $M \xrightarrow{\sim} D^2M$ in $\mathcal{D}(\text{Mod-}R)$, it follows that $\mathrm{Ext}^i_R(N,R) = 0$ for $i \neq c$ and $M \xrightarrow{\sim} \mathrm{Ext}^c_R(N,R)$. □

Dualizing complexes

Throughout the rest of this note, A is a Noether R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. Note that $\mathrm{Ann}_R(A)$ coincides with the kernel of the structure ring homomorphism $R \to A$ and that $\mathrm{Supp}_R(A)$ coincides with the set of prime ideals of R containing $\mathrm{Ann}_R(A)$. We fix a minimal injective resolution $R \to I^\bullet$ in Mod-R and set $V^\bullet = \mathrm{Hom}^\bullet_R(A, I^\bullet) \in \mathcal{K}^+(\text{Mod-}A^{\mathrm{e}})$, where $A^{\mathrm{e}} = A^{\mathrm{op}} \otimes_R A$. Note that $V^\bullet \in \mathcal{K}^+(\text{Inj-}A)$ and $V^\bullet \in \mathcal{K}^+(\text{Inj-}A^{\mathrm{op}})$. We refer to [13] for the definition and basic properties of dualizing complexes.

In the next lemma, A can be replaced by A^{op}.

Lemma 1.8. *As an R-module A has finite Gorenstein dimension if and only if the following conditions are satisfied:*

(1) $\mathrm{H}^i(V^\bullet) = 0$ *for* $i \gg 0$;

(2) $\mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(V^\bullet, V^\bullet[i]) = 0$ *for* $i \neq 0$; *and*

(3) We have an R-algebra isomorphism $A \xrightarrow{\sim} \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(V^\bullet)$ *given by left multiplication.*

Proof. Note first that $DA \cong V^\bullet$ in $\mathcal{D}(\text{Mod-}R)$. We have a cochain map

$$\delta : A \to \text{Hom}^\bullet_A(V^\bullet, V^\bullet)$$

given by the left multiplication map

$$A \to \prod_{i \geq 0} \text{End}_A(V^i), a \mapsto (v_i \mapsto av_i)_{i \geq 0}$$

and an isomorphism of complexes

$$\text{Hom}^\bullet_A(V^\bullet, V^\bullet) \cong \text{Hom}^\bullet_R(\text{Hom}^\bullet_R(A, I^\bullet), I^\bullet).$$

As the composite of them we define a cochain map

$$\eta_A : A \to \text{Hom}^\bullet_R(\text{Hom}^\bullet_R(A, I^\bullet), I^\bullet).$$

It then follows by [1, Lemma 2.3] that ξ_A is an isomorphism if and only if η_A is a quasi-isomorphism. Thus ξ_A is an isomorphism if and only if δ is a quasi-isomorphism. Now, since

$$\text{H}^i(\text{Hom}^\bullet_A(V^\bullet, V^\bullet)) \cong \text{Hom}_{\mathcal{K}(\text{Mod-}A)}(V^\bullet, V^\bullet[i])$$

for all $i \in \mathbb{Z}$, the assertion follows by Lemma 1.4 together with Remark 1.3. □

Lemma 1.9. *The following are equivalent.*

(1) $V^\bullet$ *is a dualizing complex for* A.

(2) $R_\mathfrak{p}$ *is Gorenstein for all* $\mathfrak{p} \in \text{Supp}_R(A)$ *and* $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \text{Supp}_R(A)\} < \infty$.

Proof. See e.g. [1, Propositions 3.7 and 3.8]. □

2 Gorenstein algebras

Throughout the rest of this note, $c \geq 0$ is an integer.

Definition 2.1. We say that A is a Gorenstein R-algebra of codimension c if the following conditions are satisfied:

(1) $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \text{Supp}_R(A)$;

(2) $\text{Ext}^i_R(A, R) = 0$ for $i \neq c$; and

(3) $\text{Ext}^c_R(A, R)$ is a projective generator in Mod-A.

As for the ring structure of a Gorenstein R-algebra A, we may restrict ourselves to the case where $c = 0$ because $\mathrm{Ann}_R(A)$ contains an R-regular sequence $x_1, \cdots, x_c$ and A is a Gorenstein S-algebra of codimension 0, where S is the residue ring of R over the ideal generated by $x_1, \cdots, x_c$ (Proposition 2.9). Also, we will see that our Gorenstein algebras are Gorenstein in the sense of [12] (Proposition 2.3). So we refer to [12] for properties enjoyed by Gorenstein algebras and for the relationship of the notion of Gorenstein algebras to the theory of commutative Gorenstein rings.

There is another notion of Gorenstein algebras. Consider the case where R is an artinian Gorenstein ring. Then an Artin R-algebra A is sometimes called Gorenstein if $\text{inj dim }_A A = \text{inj dim } A_A < \infty$ (see e.g. [4]). It follows by [19, Proposition 1.6] that an Artin R-algebra A is Gorenstein in this sense if and only if $\mathrm{Hom}_R(A, R) \in \text{mod-}A^{\mathrm{e}}$ is a tilting module. We will extend this fact to Noether algebras. Assume that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$, that $\sup\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$, and that $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$. Then $\text{inj dim }_A A = \text{inj dim } A_A < \infty$ if and only if $\mathrm{Ext}^c_R(A, R) \in \text{mod-}A^{\mathrm{e}}$ is a tilting module (Proposition 2.7).

Throughout this section, we assume that $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$ and set

$$\Omega = \mathrm{Ext}^c_R(A, R).$$

Note that $V^\bullet \cong \Omega[-c]$ in $\mathcal{D}(\text{Mod-}A^{\mathrm{e}})$. Also, $\mathrm{H}^i(V^\bullet) \cong \mathrm{Ext}^i_R(A, R)$ for all $i \in \mathbb{Z}$.

Lemma 2.2. *The following hold.*

(1) We have a quasi-isomorphism $V^\bullet \to \sigma'_{\geq c}(V^\bullet)$ *in* $\mathcal{K}(\text{Mod-}A^{\mathrm{e}})$.

(2) $\sigma'_{\geq c}(V^\bullet) \in \mathcal{K}^+(\text{Inj-}A)$ *and* $\sigma'_{\geq c}(V^\bullet) \in \mathcal{K}^+(\text{Inj-}A^{\mathrm{op}})$.

(3) $\mathrm{Ext}^i_A(M, \Omega) \cong \mathrm{Ext}^{i+c}_R(M, R)$ *in* $\text{Mod-}A^{\mathrm{op}}$ *for all* $M \in \text{Mod-}A$ *and* $i \geq 0$.

Proof. The first two assertions are obvious. Then for any $i \geq 0$ and $M \in \text{Mod-}A$ we have functorial isomorphisms in $\text{Mod-}A^{\mathrm{op}}$

$$\begin{aligned}
\mathrm{Ext}^i_A(M, \Omega) &\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(M, (\sigma'_{\geq c}(V^\bullet)[c])[i]) \\
&\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(M, \sigma'_{\geq c}(V^\bullet)[i+c]) \\
&\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}A)}(M, V^\bullet[i+c]) \\
&\cong \mathrm{Hom}_{\mathcal{K}(\text{Mod-}R)}(M, I^\bullet[i+c]) \\
&\cong \mathrm{Ext}^{i+c}_R(M, R).
\end{aligned}$$

□

Proposition 2.3. *For any* $\mathfrak{p} \in \mathrm{Supp}_R(A)$ *with* $R_{\mathfrak{p}}$ *Gorenstein the following hold.*

(1) $\Omega_{\mathfrak{p}} \neq 0$ *and hence* $\dim R_{\mathfrak{p}} \geq c$.

(2) $\mathrm{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, \Omega_{\mathfrak{p}}) = 0$ *for* $i < \dim R_{\mathfrak{p}} - c$.

(3) inj dim ${\Omega_{\mathfrak{p}}}_{A_{\mathfrak{p}}} = \dim R_{\mathfrak{p}} - c$.

Proof. (1) Suppose otherwise. Then $\mathrm{Ext}^i_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$ for all $i \geq 0$ and by Remark 1.6 $A_{\mathfrak{p}} = 0$, a contradiction.

(2) Take a projective resolution $Q^\bullet \to A_{\mathfrak{p}}$ in mod-$R_{\mathfrak{p}}$. Then

$$\begin{aligned} \mathrm{H}^i(\mathrm{Hom}^\bullet_{R_{\mathfrak{p}}}(Q^\bullet, R_{\mathfrak{p}})) &\cong \mathrm{Ext}^i_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}, R_{\mathfrak{p}}) \\ &\cong \mathrm{Ext}^i_R(A, R)_{\mathfrak{p}} \end{aligned}$$

for all $i \geq 0$. Thus $\mathrm{H}^i(\mathrm{Hom}^\bullet_{R_{\mathfrak{p}}}(Q^\bullet, R_{\mathfrak{p}})) = 0$ for $i \neq c$ and $\mathrm{H}^c(\mathrm{Hom}^\bullet_{R_{\mathfrak{p}}}(Q^\bullet, R_{\mathfrak{p}})) \cong \Omega_{\mathfrak{p}}$, so that we have exact sequences in mod-$R_{\mathfrak{p}}$

$$0 \to \mathrm{Hom}_{R_{\mathfrak{p}}}(Q^0, R_{\mathfrak{p}}) \to \cdots \to \mathrm{Hom}_{R_{\mathfrak{p}}}(Q^c, R_{\mathfrak{p}}) \to \mathrm{Z}'^c(\mathrm{Hom}^\bullet_{R_{\mathfrak{p}}}(Q^\bullet, R_{\mathfrak{p}})) \to 0,$$

$$0 \to \Omega_{\mathfrak{p}} \to \mathrm{Z}'^c(\mathrm{Hom}^\bullet_{R_{\mathfrak{p}}}(Q^\bullet, R_{\mathfrak{p}})) \to \mathrm{Hom}_{R_{\mathfrak{p}}}(Q^{-c-1}, R_{\mathfrak{p}}) \to \cdots$$

with the $\mathrm{Hom}_{R_{\mathfrak{p}}}(Q^i, R_{\mathfrak{p}})$ projective. Applying $\mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, -)$, the assertion follows.

(3) By Lemma 2.2(2) we have an injective resolution $\Omega_{\mathfrak{p}} \to \sigma'_{\geq c}(V^\bullet_{\mathfrak{p}})[c]$ in Mod-$A_{\mathfrak{p}}$. Since $V^\bullet_{\mathfrak{p}} \cong \mathrm{Hom}^\bullet_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}, I^\bullet_{\mathfrak{p}})$ with $I^\bullet_{\mathfrak{p}}$ a minimal injective resolution of $R_{\mathfrak{p}}$ in Mod-$R_{\mathfrak{p}}$ (see [5, Corollary 1.3]), it follows that inj dim ${\Omega_{\mathfrak{p}}}_{A_{\mathfrak{p}}} \leq \dim R_{\mathfrak{p}} - c$. Next, by Lemma 2.2(3) we have

$$\begin{aligned} \mathrm{Ext}^i_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}, \Omega_{\mathfrak{p}}) &\cong \mathrm{Ext}^i_A(A/\mathfrak{p}A, \Omega)_{\mathfrak{p}} \\ &\cong \mathrm{Ext}^{i+c}_R(A/\mathfrak{p}A, R)_{\mathfrak{p}} \\ &\cong \mathrm{Ext}^{i+c}_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}, R_{\mathfrak{p}}) \end{aligned}$$

for all $i \geq 0$. Since $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a finite direct sun of copies of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ in Mod-$R_{\mathfrak{p}}$, we have $\mathrm{Ext}^{i+c}_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}, R_{\mathfrak{p}}) \neq 0$ for $i = \dim R_{\mathfrak{p}} - c$. □

Lemma 2.4. *The following are equivalent.*

(1) A has finite Gorenstein dimension as an R-module.

(2) $\mathrm{Ext}^i_A(\Omega, \Omega) = 0$ *for* $i > 0$ *and* $A \xrightarrow{\sim} \mathrm{End}_A(\Omega), a \mapsto (w \mapsto aw)$.

(3) $\mathrm{Ext}^i_{A^{\mathrm{op}}}(\Omega, \Omega) = 0$ *for* $i > 0$ *and* $A \xrightarrow{\sim} \mathrm{End}_{A^{\mathrm{op}}}(\Omega)^{\mathrm{op}}, a \mapsto (w \mapsto wa)$.

Proof. (1) ⇔ (2). Since $\mathrm{H}^i(V^\bullet) = 0$ for $i > c$, this follows by Lemma 1.8.

(1) ⇔ (3). By symmetry. □

Remark 2.5. If $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$, then A has finite Gorenstein dimension as an R-module.

Proof. Take a projective resolution $P^\bullet \to A$ in mod-R and set $M = \mathrm{Z}'^{-c}(P^\bullet)$. Then $\mathrm{Ext}^i_R(M, R) \cong \mathrm{Ext}^{i+c}_R(A, R) = 0$ for $i > 0$ and by Remark 1.6 M has Gorenstein dimension zero. □

Throughout the rest of this section, we assume further that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$. Then by Lemma 2.4 and Remark 2.5 we have $\mathrm{Ext}^i_A(\Omega, \Omega) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\Omega, \Omega) = 0$ for $i > 0$ and $\Omega \in \text{mod-}A^{\mathrm{e}}$ is faithfully balanced, i.e., $A \xrightarrow{\sim} \mathrm{End}_A(\Omega), a \mapsto (w \mapsto aw)$ and $A \xrightarrow{\sim} \mathrm{End}_{A^{\mathrm{op}}}(\Omega)^{\mathrm{op}}, a \mapsto (w \mapsto wa)$.

Proposition 2.6. *The following are equivalent.*

(1) $\sup\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$.

(2) $V^i = 0$ *for* $i \gg 0$.

(3) $\mathrm{inj\ dim}\ \Omega_A < \infty$.

(4) $\mathrm{inj\ dim}\ {}_A\Omega < \infty$.

Proof. By symmetry, it suffices to show (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2). Assume that $V^i \neq 0$. Since $\mathrm{Hom}_R(A, I^i) \neq 0$, there exists $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $E_R(R/\mathfrak{p})$ is a direct summand of I^i and $\mathrm{Hom}_R(A, E_R(R/\mathfrak{p})) \neq 0$. Note that $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and $E_R(R/\mathfrak{p}) \cong E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$ in Mod-$R_{\mathfrak{p}}$. Since $E_R(R/\mathfrak{p})$ is a direct summand of $I^i_{\mathfrak{p}}$ and $R_{\mathfrak{p}} \to I^{\bullet}_{\mathfrak{p}}$ is a minimal injective resolution in Mod-$R_{\mathfrak{p}}$ (see [5, Corollary 1.3]), it follows that $i = \dim R_{\mathfrak{p}}$.

(2) $\Rightarrow$ (3). By Lemma 2.2(2).

(3) $\Rightarrow$ (1). See [1, Proposition 3.7]. □

We refer to [19] for tilting modules. Note however that a module is a tilting module if and only if it is isomorphic to a tilting complex in the derived category (see e.g. [2, Proposition 3.9]).

Proposition 2.7. *Assume that* $\sup\{\dim R_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$. *Then the following are equivalent.*

(1) $\Omega \in \text{mod-}A^{\mathrm{e}}$ *is a tilting module.*

(2) $\mathrm{proj\ dim}\ {}_A\Omega = \mathrm{proj\ dim}\ \Omega_A < \infty$.

(3) $\mathrm{inj\ dim}\ {}_AA = \mathrm{inj\ dim}\ A_A < \infty$.

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (1). See [1, Theorem 3.9].

(1) $\Rightarrow$ (2). See e.g. [2, Lemma 1.5]. □

Proposition 2.8. *The following are equivalent.*

(1) $\Omega \in \mathcal{P}_A$ *and* $\Omega \in \mathcal{P}_{A^{\mathrm{op}}}$.

(2) $\mathrm{add}(\Omega) = \mathcal{P}_A$ *in* Mod-A.

(3) $\mathrm{add}(\Omega) = \mathcal{P}_{A^{\mathrm{op}}}$ *in* Mod-A^{op}.

Proof. Since $\Omega \in \text{mod-}A^{\mathrm{e}}$ is faithfully balanced, (2) $\Leftrightarrow$ (3) follows by Morita theory. Then (3) together with (2) implies (1).

(1) $\Rightarrow$ (3). By Lemmas 2.2(3), 2.4 we have $A \cong \mathrm{Ext}^c_R(\Omega, R)$ in Mod-A^{op} and hence $\Omega \in \mathcal{P}_A$ implies $A \in \mathrm{add}(\Omega)$ in Mod-A^{op}. □

Proposition 2.9. *There exists an R-regular sequence $x_1, \cdots, x_c$ in $\mathrm{Ann}_R(A)$. Set $S = R/(x_1, \cdots, x_c)$ with $(x_1, \cdots, x_c)$ the ideal of R generated by $x_1, \cdots, x_c$. Then the following hold.*

(1) A has Gorenstein dimension zero as an S-module.

(2) $\mathrm{Hom}_S(A, S) \cong \Omega$ in $\text{Mod-}A^{\mathrm{e}}$.

(3) $S_{\mathfrak{q}}$ is Gorenstein for all $\mathfrak{q} \in \mathrm{Supp}_S(A)$.

Proof. Set $\mathfrak{a} = \mathrm{Ann}_R(A)$. Let $i < c$ and $\mathfrak{p} \in \mathrm{Supp}_R(A)$. Note that Ω is faithful as an $R/\mathfrak{a}$-module. Thus $R/\mathfrak{a}$ can be embedded as a submodule in a finite direct sum of copies of Ω. Then $(R/\mathfrak{a})_{\mathfrak{p}}$ can be embedded as a submodule in a finite direct sum of copies of $\Omega_{\mathfrak{p}} \cong \mathrm{Ext}^c_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}, R_{\mathfrak{p}})$ and hence $\mathrm{Ext}^i_R(R/\mathfrak{a}, R)_{\mathfrak{p}} \cong \mathrm{Ext}^i_{R_{\mathfrak{p}}}((R/\mathfrak{a})_{\mathfrak{p}}, R_{\mathfrak{p}}) = 0$. Thus $\mathrm{Ext}^i_R(R/\mathfrak{a}, R) = 0$ for $i < c$ and the first assertion follows (see [6, Corollary 2.11]).

Note that $\mathrm{Hom}_R(S, I^i) = 0$ for $i < c$ and $\mathrm{Hom}^\bullet_R(S, I^\bullet)[c]$ is a minimal injective resolution of S in Mod-S (see [5, Theorem 2.2]). Since

$$\mathrm{Hom}^\bullet_S(A, \mathrm{Hom}^\bullet_R(S, I^\bullet)[c]) \cong \mathrm{Hom}^\bullet_R(A, I^\bullet[c]),$$

by setting $W^\bullet = \mathrm{Hom}^\bullet_S(A, \mathrm{Hom}^\bullet_R(S, I^\bullet)[c])$, we have $\mathrm{H}^i(W^\bullet) \cong \mathrm{H}^{i+c}(V^\bullet)$ for all $i \in \mathbb{Z}$. Thus $\mathrm{Ext}^i_S(A, S) = 0$ for $i > 0$ and $\mathrm{Hom}_S(A, S) \cong \Omega$. Then as an S-module A has finite Gorenstein dimension by Lemma 2.4 and hence has Gorenstein dimension zero by Remark 1.7. Finally, it is easy to see that $S_{\mathfrak{q}}$ is Gorenstein for all $\mathfrak{q} \in \mathrm{Supp}_S(A)$. □

3 Serre duality

In this section, we will extend Serre duality theory (cf. [8]) to Noether algebras. We set

$$(-)^* = \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(-, R) : \mathcal{D}(\text{Mod-}R) \to \text{Mod-}R.$$

Note that $(-)^* \cong \mathrm{H}^0(D(-))$.

Recall that a complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ is said to have finite projective dimension if $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet[-i], -)$ vanishes on mod-A for $i \gg 0$. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes which have finite projective dimension. Note that $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ canonically. Similarly, a complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ is said to have finite injective dimension if $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(-, X^\bullet[i])$ vanishes on mod-A for $i \gg 0$. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fid}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes which have finite injective dimension.

Definition 3.1. We say that A has Serre duality if there exist a self-equivalence of a triangulated category $F : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ and a bifunctorial isomorphism in Mod-R

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, FX^\bullet) \cong \mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet)^*$$

for $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ and $Y^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$. If this is the case, we call F a Serre functor for A.

Note that if A has finite global dimension then $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}} = \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ and that if R is selfinjective then we have bifunctorial isomorphisms in Mod-R

$$\begin{aligned}\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet)^* &\cong \mathrm{H}^0(D\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet))\\ &\cong D\mathrm{H}^0(\mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet))\\ &\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, Y^\bullet)^*\end{aligned}$$

for $X^\bullet, Y^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$. These facts would justify the definition above.

Remark 3.2. Assume that there exists a Serre functor $F : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ for A. Then the restriction of F to $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ is unique up to isomorphism and the following hold.

(1) F induces a self-equivalence of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ and there exists a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $FA \cong P^\bullet$ in $\mathcal{D}(\text{Mod-}A)$ and $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$.

(2) For any $i \in \mathbb{Z}$ we have a functorial isomorphism in Mod-A^{op}

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(M, FA[i]) \cong \mathrm{Ext}^i_R(M, R)$$

for $M \in \text{mod-}A$. In particular, $\mathrm{H}^i(FA) \cong \mathrm{Ext}^i_R(A, R)$ in Mod-A^{e} for all $i \in \mathbb{Z}$.

(3) Assume that $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$ and set $\Omega = \mathrm{Ext}^c_R(A, R)$. Then $FA \cong \Omega[-c]$ in $\mathcal{D}(\text{Mod-}A)$ and $\Omega \in \text{mod-}A^{\mathrm{e}}$ is a tilting module.

Proof. The first assertion follows by Yoneda's lemma.
(1) See [20, Proposition 8.2].
(2) Note first that the isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, FX^\bullet) \cong \mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet)^*$$

in the definition above is an isomorphism of $\mathrm{End}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet)^{\mathrm{op}}$-modules. Thus for any $M \in \text{mod-}A$ and $i \in \mathbb{Z}$ we have isomorphisms in Mod-A^{op}

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(M, FA[i]) &\cong \mathbf{R}\mathrm{Hom}^\bullet_A(A[i], M)^*\\ &\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(M[-i], R)\\ &\cong \mathrm{Ext}^i_R(M, R).\end{aligned}$$

(3) This follows by (1), (2) above. □

Proposition 3.3. *We have a bifunctorial isomorphism in* Mod-R

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, X^\bullet \otimes^{\mathbf{L}}_A V^\bullet) \cong \mathbf{R}\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet)^*$$

for $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ *and* $Y^\bullet \in \mathcal{D}(\text{Mod-}A)$.

Proof. For any $P \in \mathcal{P}_A$, $Q \in \text{Mod-}A$ and $I \in \text{Inj-}R$, since we have functorial isomorphisms in Mod-R

$$Q \otimes_A \mathrm{Hom}_A(P, A) \xrightarrow{\sim} \mathrm{Hom}_A(P, Q), x \otimes f \mapsto (a \mapsto xf(a))$$

and

$$P \otimes_A \mathrm{Hom}_R(A, I) \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_A(P, A), I), a \otimes g \mapsto (f \mapsto g(f(a))),$$

we have functorial isomorphisms in Mod-R

$$\begin{aligned} \mathrm{Hom}_A(Q, P \otimes_A \mathrm{Hom}_R(A, I)) &\cong \mathrm{Hom}_A(Q, \mathrm{Hom}_R(\mathrm{Hom}_A(P, A), I)) \\ &\cong \mathrm{Hom}_R(Q \otimes_A \mathrm{Hom}_A(P, A), I) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_A(P, Q), I). \end{aligned}$$

It is not difficult to see that the functorial isomorphism in Mod-R

$$\mathrm{Hom}_A(Q, P \otimes_A \mathrm{Hom}_R(A, I)) \cong \mathrm{Hom}_R(\mathrm{Hom}_A(P, Q), I)$$

for $P \in \mathcal{P}_A$, $Q \in \text{Mod-}A$ and $I \in \text{Inj-}R$ can be extended to a bifunctorial isomorphism in $\mathcal{K}(\text{Mod-}R)$

$$\mathrm{Hom}^\bullet_A(Q^\bullet, P^\bullet \otimes^\bullet_A V^\bullet) \cong \mathrm{Him}^\bullet_R(\mathrm{Hom}^\bullet_A(P^\bullet, Q^\bullet), I^\bullet)$$

for $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Q^\bullet \in \mathcal{K}(\text{Mod-}A)$. Note that $P^\bullet \otimes^\bullet_A V^\bullet \in \mathcal{K}^+(\text{Inj-}A)$. Applying $\mathrm{H}^0(-)$, the assertion follows. □

Corollary 3.4. *Assume that A is a Gorenstein R-algebra of codimension c with $\mathrm{Ext}^c_R(A, R) \cong A$ as A-bimodules and that $d = \dim R_{\mathfrak{p}}$ for all maximal $\mathfrak{p} \in \mathrm{Supp}_R(A)$. Then, denoting by E the direct sum of all $E_R(R/\mathfrak{p})$ with $\mathfrak{p} \in \mathrm{Supp}_R(A)$ maximal, we have a bifunctorial isomorphism in* Mod-R

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, X^\bullet[d-c]) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, Y^\bullet), E)$$

for $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ and $Y^\bullet \in \mathcal{D}(\text{mod-}A)$ with the Y^i of finite length.

Proof. For any $X^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $Y^\bullet \in \mathcal{D}(\text{mod-}A)$ with the Y^i of finite length, since $V^\bullet \cong A[-c]$ in $\mathcal{D}(\text{Mod-}A^{\mathrm{e}})$, and since $\mathrm{Hom}_R(\mathrm{Hom}^i_A(X^\bullet, Y^\bullet), I^j) = 0$ unless $j = d$, by Proposition 3.3 we have bifunctorial isomorphisms in Mod-R

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, X^\bullet[d-c]) &\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(Y^\bullet, X^\bullet[d] \otimes^{\mathbf{L}}_A V^\bullet) \\ &\cong \mathrm{H}^0(\mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_A(X^\bullet[d], Y^\bullet), I^\bullet)) \\ &\cong \mathrm{H}^0(\mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet), I^\bullet[d])) \\ &\cong \mathrm{H}^0(\mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet), I^d)) \\ &\cong \mathrm{H}^0(\mathrm{Hom}^\bullet_R(\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet), E)) \\ &\cong \mathrm{Hom}_R(\mathrm{H}^0(\mathrm{Hom}^\bullet_A(X^\bullet, Y^\bullet)), E) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, Y^\bullet), E). \end{aligned}$$

□

Theorem 3.5. *Assume that $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and that $\sup\{\dim R_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Supp}_R(A)\} < \infty$. Then $V^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{e}})$ and the following are equivalent.*

(1) A has Serre duality with a Serre functor

$$- \otimes^{\mathbf{L}}_A V^\bullet : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A).$$

(2) A and A^{op} have Serre duality.

(3) $\text{inj dim }{}_A A = \text{inj dim } A_A < \infty$.

Proof. See [1, Proposition 3.8] for the first assertion.

(1) ⇒ (3). By Remark 3.2(1) there exists a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $V^\bullet \cong P^\bullet$ in $\mathcal{D}(\text{Mod-}A)$ and $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. The assertion follows by [1, Theorem 3.9].

(3) ⇒ (1) and (2). By [1, Theorem 4.7] we have a self-equivalence

$$- \otimes^{\mathbf{L}}_A V^\bullet : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$$

which is a Serre functor for A by Proposition 3.3. By symmetry, we also have a Serre functor for A^{op}

$$V^\bullet \otimes^{\mathbf{L}}_A - : \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}}).$$

(2) ⇒ (3). Let $F : \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ be a Serre functor for A. Then by Remark 3.2(1) there exists a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $FA \cong P^\bullet$ in $\mathcal{D}(\text{Mod-}A)$. Take an integer $d \geq 1$ such that $\dim R_\mathfrak{p} < d$ for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$. Then for any $i \geq d$ and $M \in \text{mod-}A$ we have $\mathrm{Ext}^i_R(M, R)_\mathfrak{p} \cong \mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and hence by Remark 3.2(2) we have $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(M, P^\bullet[i]) \cong \mathrm{Ext}^i_R(M, R) = 0$. Thus $P^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fid}}$. Since $\mathrm{add}(P^\bullet)$ generates $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ as a triangulated category, it follows that $A \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}} \subset \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fid}}$ and $\text{inj dim } A_A < \infty$. By symmetry, we also have $\text{inj dim }{}_A A < \infty$. The assertion follows by [23, Lemma A]. □

A complex $\Delta^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{e}})$ is said to be invertible if there exists a complex $\widetilde{\Delta}^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{e}})$, called the inverse of $\Delta^\bullet$, such that $\Delta^\bullet \otimes^{\mathbf{L}}_A \widetilde{\Delta}^\bullet \cong \widetilde{\Delta}^\bullet \otimes^{\mathbf{L}}_A \Delta^\bullet \cong A$ in $\mathcal{D}(\text{Mod-}A^{\mathrm{e}})$. Note that $\widetilde{\Delta}^\bullet \cong \mathbf{R}\mathrm{Hom}^\bullet_A(\Delta^\bullet, A) \cong \mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\Delta^\bullet, A)$. Also, an invertible complex is a special type of two-sided tilting complex (see [21]).

Lemma 3.6. *Let $\Delta^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{e}})$ be an invertible complex and $\widetilde{\Delta}^\bullet$ the inverse of $\Delta^\bullet$. Then $V^\bullet \otimes^{\mathbf{L}}_A \Delta^\bullet \cong \Delta^\bullet \otimes^{\mathbf{L}}_A V^\bullet \cong \mathrm{Hom}^\bullet_R(\widetilde{\Delta}^\bullet, I^\bullet)$ in $\mathcal{D}(\text{Mod-}A^{\mathrm{e}})$.*

Proof. We have isomorphisms in $\mathcal{D}(\text{Mod-}A^{\mathrm{e}})$

$$\begin{aligned} V^\bullet \otimes^{\mathbf{L}}_A \Delta^\bullet &\cong \mathbf{R}\mathrm{Hom}^\bullet_A(\widetilde{\Delta}^\bullet, V^\bullet) \\ &\cong \mathrm{Hom}^\bullet_A(\widetilde{\Delta}^\bullet, V^\bullet) \\ &\cong \mathrm{Hom}^\bullet_R(\widetilde{\Delta}^\bullet, I^\bullet), \end{aligned}$$

$$\begin{aligned}\Delta^\bullet \otimes^{\mathbf{L}}_A V^\bullet &\cong \mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\widetilde{\Delta}^\bullet, V^\bullet)\\ &\cong \mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(\widetilde{\Delta}^\bullet, V^\bullet)\\ &\cong \mathrm{Hom}^\bullet_R(\widetilde{\Delta}^\bullet, I^\bullet).\end{aligned}$$

□

Proposition 3.7. *Assume that A is a Gorenstein R-algebra of codimension c. Then $\Omega = \mathrm{Ext}^c_R(A, R)$ lies in the center of the Picard group of A.*

Proof. It follows by Proposition 2.8 that Ω lies in the Picard group of A. Since $\Omega \cong V^\bullet[c]$ in $\mathcal{D}(\text{Mod-}A^{\mathrm{e}})$, the assertion follows by Lemma 3.6. □

4 Derived equivalences

Throughout this section, we fix a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and set $B = \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. Note that B is a Noether R-algebra and that there exists a tilting complex $Q^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_B)$ such that $A \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}B)}(Q^\bullet)$.

Proposition 4.1. *The following hold.*

(1) $\mathrm{Ann}_R(A) = \mathrm{Ann}_R(B)$ *and hence* $\mathrm{Supp}_R(A) = \mathrm{Supp}_R(B)$.

(2) If A has finite Gorenstein dimension as an R-module, then so does B.

(3) If $\mathrm{inj\ dim}\ {}_AA = \mathrm{inj\ dim}\ A_A < \infty$, *then* $\mathrm{inj\ dim}\ {}_BB = \mathrm{inj\ dim}\ B_B < \infty$.

Proof. Set $X^\bullet = \mathrm{Hom}^\bullet_A(P^\bullet, P^\bullet)$. Then $\mathrm{H}^i(X^\bullet) = 0$ for $i \neq 0$ and $\mathrm{H}^0(X^\bullet) \cong B$. Thus we have exact sequences in mod-R

$$0 \to X^{-l} \to \cdots \to X^0 \to \mathrm{Z}'^0(X^\bullet) \to 0,$$

$$0 \to B \to \mathrm{Z}'^0(X^\bullet) \to X^1 \to \cdots \to X^l \to 0$$

for some $l \geq 0$ with $X^i \in \mathrm{add}(A_R)$ for all $i \in \mathbb{Z}$.

(1) Since every X^i is annihilated by $\mathrm{Ann}_R(A)$, it follows that B is annihilated by $\mathrm{Ann}_R(A)$. By symmetry, A is annihilated by $\mathrm{Ann}_R(B)$.

(2) This follows by Lemma 1.5.

(3) See e.g. [16, Proposition 1.7]. □

Throughout the rest of this section, we assume that $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq c$. We set $\Omega = \mathrm{Ext}^c_R(A, R)$ and

$$\nu = - \otimes^{\mathbf{L}}_A \Omega : \mathcal{D}^-(\text{mod-}A) \to \mathcal{D}^-(\text{mod-}A).$$

We denote by $\mathcal{S}$ the full subcategory of $\mathcal{D}^-(\text{Mod-}A)$ consisting of complexes $X^\bullet$ with $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, X^\bullet[i]) = 0$ for $i \neq 0$. In the following, we define $\mathrm{add}(P^\bullet)$ as a full subcategory of $\mathcal{D}^-(\text{Mod-}A)$. However, the canonical functor $\mathcal{K}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$ induces an equivalence between $\mathrm{add}(P^\bullet)$ defined in $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ and $\mathrm{add}(P^\bullet)$ defined in $\mathcal{D}^-(\text{Mod-}A)$ (cf. [14, Remark 1.7]).

Remark 4.2. Assume that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and $\mathrm{add}(\Omega) = \mathcal{P}_A$ in Mod-A. Then by Proposition 2.8 we have a self-equivalence $\nu : \mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_A$.

Theorem 4.3. *The following hold.*

(1) $\mathrm{Ext}^i_R(B, R) = 0$ *for* $i \neq c$ *if and only if* $\nu P^\bullet \in \mathcal{S}$.

(2) Assume that $\nu P^\bullet \in \mathcal{S}$. *Then* $\mathrm{Ext}^c_R(B, R)$ *is a projective generator in* Mod-B *if and only if* $\mathrm{add}(\nu P^\bullet) = \mathrm{add}(P^\bullet)$.

(3) If $\Omega \cong A$ *in* Mod-A^{e}, *then* $\mathrm{Ext}^i_R(B, R) = 0$ *for* $i \neq c$ *and* $\mathrm{Ext}^c_R(B, R) \cong B$ *in* Mod-B^{e}.

Proof. (1) Since $V^\bullet \cong \Omega[-c]$ in $\mathcal{D}(\text{Mod-}A^{\mathrm{e}})$, and since $B \cong \mathbf{R}\mathrm{Hom}^\bullet_A(P^\bullet, P^\bullet)$ in $\mathcal{D}(\text{Mod-}R)$, by Proposition 3.3 we have

$$\begin{aligned}
\mathrm{Ext}^i_R(B, R) &\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}R)}(B[-i], R)\\
&\cong \mathbf{R}\mathrm{Hom}^\bullet_A(P^\bullet[i], P^\bullet)^*\\
&\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, P^\bullet \otimes^{\mathbf{L}}_A V^\bullet[i])\\
&\cong \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet[i - c])
\end{aligned}$$

for all $i \in \mathbb{Z}$.

(2) We know from [20, Section 4] that the functor

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, -) : \mathcal{S} \to \text{Mod-}B$$

is an equivalence. Since we have isomorphisms in Mod-B^{e}

$$\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, \nu P^\bullet) \cong \mathrm{Ext}^c_R(B, R) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(P^\bullet, P^\bullet) \cong B,$$

the assertion follows.

(3) If $\Omega \cong A$ in Mod-A^{e}, then $\nu P^\bullet \cong P^\bullet$ as complexes and the assertion follows. □

Corollary 4.4. *Assume that A is a Gorenstein R-algebra of codimension c. Then B is a Gorenstein R-algebra of codimension c if and only if* $\mathrm{add}(\nu P^\bullet) = \mathrm{add}(P^\bullet)$.

Proof. By (1), (2) of Proposition 4.1 and (1), (2) of Theorem 4.3. □

Corollary 4.5. *The following are equivalent.*

(1) A is a Gorenstein R-algebra of codimension c with $\mathrm{Ext}^c_R(A, R) \cong A$ *in* Mod-A^{e}.

(2) B is a Gorenstein R-algebra of codimension c with $\mathrm{Ext}^c_R(B, R) \cong B$ *in* Mod-B^{e}.

Proof. By (1), (2) of Proposition 4.1 and Theorem 4.3(3). □

Example 4.6. Assume that R is a Gorenstein ring containing an R-regular sequence $x_1, \cdots, x_c, x$. Set $S = R/(x_1, \cdots, x_c)$ with $(x_1, \cdots, x_c)$ the ideal of R generated by $x_1, \cdots, x_c$ and define Noether R-algebras A, B as follows:

$$A = \begin{pmatrix} S & S \\ xS & S \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} S & S/xS \\ 0 & S/xS \end{pmatrix}.$$

In [2, Example 4.7], we have constructed a tilting complex $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ such that $B \cong \mathrm{End}_{\mathcal{K}(\text{Mod-}A)}(P^\bullet)$. Also, we have seen that A is a Gorenstein S-algebra of codimension 0. Thus A is a Gorenstein R-algebra of codimension c. On the other hand, $\mathrm{Ext}^i_R(B, R) \neq 0$ for $i = c$ and $c + 1$, so that $\nu P^\bullet \notin \mathcal{S}$.

Consider the case where A is a Gorenstein R-algebra of codimension c and $\mathrm{Ext}^i_R(B, R) = 0$ for $i \neq c$. At present, we do not know whether or not B is a Gorenstein R-algebra of codimension c. The example above does not tell us anything about this question.

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