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PRODUCTS OF k-SPACES, AND SPECIAL COUNTABLE SPACES

By

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Introduction

We recall that every k-space is characterized as a quotient space of a locally compact space. As is well-known, not every product of a metric space with a k-space is a k-space.

In this paper, we give special countable spaces, such as the sequential fan, Arens' space, their certain subspaces, and modifications of these countable spaces. In terms of these spaces, we give necessary conditions (resp. necessary and sufficient conditions) for the product $X \times Y$ to be a k-space when X is a certain non-locally compact space (resp. a bi-k-space), and Y is a sequential space which is one of the following spaces $(A_1)\sim(A_8)$.

(A_1) Fréchet space.

(A_2) Space in which every point is a G_δ-set.

(A_3) Hereditarily normal space.

(A_4) Space having a point-countable k-network.

(A_5) Quotient s-image of a *paracompact* countably bi-k-space.

(A_6) Closed image of a countably bi-k-space.

(A_7) Closed image of a *normal* countably bi-k-space.

(A_8) Closed image of an M-space.

We assume that all spaces are regular and T_1, and all maps are continuous surjections.

Let us recall some definitions used in this paper. A space X is *a sequential space*, if $A \subset X$ is open in X if and only if, for any $x \in A$ and any sequence $\{x_n : n \in N\}$ converging to x, $x_n \in A$ except at most finitely many x_n.

A space X is *Fréchet* if for any $A \subset X$ and any $x \in \bar{A}$, there exist points $x_n \in A$ such that $\{x_n : n \in N\}$ converges to x. Also, a space X is *strongly Fréchet*

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[14] (= *countably bi-sequential* in the sence of E. Micheal [7]), if for every decreasing sequence $\{A_n : n \in N\}$ of subsets of X with $x \in \overline{A_n}$ for any $n \in N$, then there exist points $x_n \in A_n$ $(n \in N)$ such that $\{x_n : n \in N\}$ converges to the point x.

Recall that a *filter base* $\mathscr{F}$ is a non-empty collection of non-empty sets such that $F_1, F_2 \in \mathscr{F}$ implies $F_3 \subset F_1 \cap F_2$ for some $F_3 \in \mathscr{F}$. A filter base $\mathscr{F}$ *accumulates* at $x \in X$ if $x \in \bar{F}$ for all $F \in \mathscr{F}$. Also, a sequence $\{A_n : n \in N\}$ in X is *k-sequence* (resp. *q-sequence*) [7] if it is a decreasing sequence such that $A = \bigcap\{A_n : n \in N\}$ is compact (resp. countably compact), and any open set $U \supset A$ contains some A_n.

A space X is a *bi-k-space* [7] if, whenever a filter base $\mathscr{F}$ accumulates at $x \in X$, then there exists a k-sequence $\{A_n : n \in N\}$ such that $x \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathscr{F}$. When the filter base $\mathscr{F}$ is a decreasing sequence, then such a space X is a *countably bi-k-space* [7]. Every bi-k-space (resp. countably bi-k-space) is characterized as the bi-quotient (resp. countably bi-quotient) image of a paracompact M-space; see [7]. Here, a map $f : X \to Y$ is *bi-quotient* [6] if, whenever $y \in Y$ and $\mathscr{U}$ is a cover of $f^{-1}(y)$ by open subsets of X, then finitely many $f(U)$, with $U \in \mathscr{U}$, cover some nbd of y in Y. When the cover $\mathscr{U}$ is countable, then such a map f is *countably bi-quotient*; see [14], for example. We recall that a space is an M-space if and only if it is the inverse image of a metric space under a quasi-perfect map (i.e., a closed map such that the inverse image of each point is countably compact). Also, we recall that a space is of *pointwise countable type* (resp. *q-space*) if each point has a k-sequence (resp. q-sequence) of nbds. Spaces of poinwise countable type, or M-spaces are q-spaces. For spaces, the implications below hold. Also, strongly Fréchet spaces are countably bi-k, and the converse holds among Fréchet spaces ([13]).

first countable $\to$ strongly Fréchet $\to$ Fréchet $\to$ sequential $\to$ k. Also,
paracompact M or first countable $\to$ pointwise countable type $\to$ bi-k $\to$
countably bi-k $\to$ k.

We recall that a cover $\mathscr{P}$ of a space X is a *k-network* for X if, for any compact subset K, and any open set V with $K \subset V$, $K \subset \bigcup \mathscr{F} \subset V$ for some finite $\mathscr{F} \subset \mathscr{P}$. Every quotient s-image of a metric space has a point-countable k-network. Here, an *s-image* is the image under an s-map (i.e., a map such that the inverse image of each point is separable). Every k-space with a point-countable k-network is sequential, and every countably bi-k-space with a point-countable k-network has a point-countable base ([3]).

Let X be a space. For a (not necessarily open or closed) cover $\mathscr{P}$ of X, X *is determined by* a cover $\mathscr{P}$, if $U \subset X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathscr{P}$. Here, we can replace "open" by "closed". (Following [3], we shall use "X is determined by $\mathscr{P}$" instead of the usual "X has the *weak topology* with respect to $\mathscr{P}$"). Obviously, every space is determined by its open cover. As is well-known, a space is called a *k-space* it is determined by the cover of *all* compact subsets, equivalently, it is detemined by some cover of compact subsets. Recall that a space is sequential if and only if it is determined by *some* cover of (compact) metric subsets. Every sequential space is precisely a quotient space of a (locally compact) metric space ([1]). Sequential spaces, or countably bi-k-spaces are k-spaces. Every k-space in which each point is a G_δ-set is sequential ([7]).

We conclude this section by recording two elementary facts which will be used. (These are easily or routinely shown).

Fact 1. Let X be a space determined by a cover $\mathscr{P}$, and let $\mathscr{C}$ be a cover of X. If each element of $\mathscr{P}$ is contained in some element of $\mathscr{C}$, then X is also determined by $\mathscr{C}$.

Fact 2. Let X be a space determined by a cover $\{X_\alpha : \alpha\}$. If each X_α is determined by a cover $\mathscr{P}_\alpha$, then X is determined by a cover $\bigcup\{\mathscr{P}_\alpha : \alpha\}$.

1. Special Countable Spaces

We define some special countable spaces, including canonical sequential spaces, *Arens' space* and *the sequential fan*.

Let $T = \{\infty\} \cup \{p_n : n \in N\} \cup \{p_{nm} : n, m \in N\}$ be the countable space such that each p_{nm} is isolated in T, $K = \{p_n : n \in N\}$ converges to $\infty \notin K$, and each $L_n = \{p_{nm} : m \in N\}$ converges to $p_n \notin L_n$.

For the space T, let us consider the following compact space T^*, and the non-compact spaces T_1^* & T_2^*.

T^*: For every $x_n \in L_n$ $(n \in N)$, $\{x_n : n \in N\}$ converges to ∞.

T_1^*: For any $x_{n_i} \in L_{n_i}$ $(i \in N)$, $\{x_{n_i} : i \in N\}$ does not converge to ∞.

T_2^*: For every finite $F_n \subset L_n$ $(n \in N)$, $\bigcup\{F_n : n \in N\}$ is closed in T.

The space T_1^* is not Fréchet nor locally compact, neither is the space T_2^*. The space T_2^* is called the *Arens' space*, and it is denoted by S_2. The quotient space $T_2^*/(K \cup \{\infty\})$ is, so-called, the *sequential fan*, and it is denoted by S_ω (that is, S_ω is the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points).

Remark 1.1. The space T_2^* implies the space T_1^*. When T_1^* is sequential, $T_1^* = T_2^* = S_2$. But, the space T_1^* need not imply the space T_2^*. (Indeed, there exists a non-Fréchet, compact sequential space Ψ^* ([2; Example 7.1]). Thus, Ψ^*

contains a copy of T_1^* by Proposition 1.3(1) below. If the space T_1^* is the space T_2^*, then Ψ^* contains S_2. Then Ψ^* contains a closed copy of S_2 by Lemma 1.2 below. But, Ψ^* is compact, then it does not contain a closed copy of S_2, a contradiction).

LEMMA 1.2. *Let X be a sequential space. Then X contains a copy of S_ω (resp. S_2) if and only if X contains a closed copy of S_ω (resp. S_2); see* [18] (*resp.* [12]), *or* [19].

Concerning the spaces (A_i) $(i = 1, 2, 3, 4)$ in the previous section, we recall the following proposition. (1); and (2) is respectively due to [2]; and [18] or [19]. For (3), the result for the space (A_2); (A_3); and (A_4) is respectively due to [18] or [19]; [4] or [19]; and [5]. Analogous results on the spaces (A_i) $(i = 5, 6, 7, 8)$ will be obtained later on; see Propositions 2.8 & 2.14.

PROPOSITION 1.3. (1) *A sequential space is the space* (A_1) *if and only if it contains no copy of the space T_1^*.*

(2) *The space* (A_1) *is strongly Fréchet if and only if it contains no* (*closed*) *copy of S_ω.*

(3) *Let X be a sequential space which is the space* (A_2), (A_3), *or* (A_4). *Then X is the space* (A_1) *if and only if X contains no* (*closed*) *copy of S_2.*

2. Products

PROPOSITION 2.1. *The following property* (a), (b), *or* (c) *implies that $X \times Y$ is a k-space.*

(a) *X is a locally countably compact k-space, and Y is sequential.*

(b) *X is bi-k, and Y is countably bi-k.*

(c) *X and Y are spaces determined by a closed* (*or increasing*) *countable cover of locally compact subsets.*

PROOF. The result for property (a) follows from Theorems 2.1 and 2.5, and Remark (after Theorem 2.5) in [15]. The result for property (b) is due to [17; Proposition 4.6]. For property (c), we will refer to the proof of [9: Lemma 2.1] (for the products of countable CW-complxes). Let X (resp. Y) be determined by a closed cover $\{X_n : n \in N\}$ (resp. $\{Y_n : n \in N\}$) of locally compact subsets. Here, we can assume that $X_n \subset X_{n+1}$, and $Y_n \subset Y_{n+1}$ by *Fact* 1. For $G \subset (X \times Y)$, let $G \cap (X_n \times Y_n)$ is open in $X_n \times Y_n$ for each $n \in N$. Let $(x, y) \in G \cap (X_k \times Y_k)$. For each $n \geq k$, since X_n; Y_n is locally compact, by induction, there exists a nbd U_n of x in X_n; V_n of y in Y_n such that $\overline{U_n} \times \overline{V_n} \subset G$, but $\overline{U_n}$ and $\overline{V_n}$ are compact subsets with $\overline{U_n} \subset U_{n+1}$ and $\overline{V_n} \subset V_{n+1}$. Let $U = \bigcup\{U_n : n \in N\}$, and $V = \bigcup\{V_n : n \in N\}$. Then, each $U \cap X_n$; $V \cap Y_n$ is open in X_n; Y_n. Thus, U; V is open in X; Y. Then $U \times V$ is a nbd of (x, y) with $U \times V \subset G$. Thus G is open in $X \times Y$. This shows that $X \times Y$ is determined by a cover $\{X_n \times Y_n : n \in N\}$ of locally compact subsets. But, each locally compact subset is determined by a cover of compact subsets. Thus $X \times Y$ is determined by a cover of compact subsets by *Fact* 2. Hence, $X \times Y$ is a k-space.

Remark 2.2. We recall that, for sequential spaces X and Y, $X \times Y$ is sequential if and only if it is a k-space ([15]). Thus, for sequential spaces X and Y, property (a), (b), or (c) in Proposition 2.1 implies that $X \times Y$ is sequential.

Remark 2.3. (1) Every product of a separable metric space with a Fréchet space need not be a k-space. Indeed, let $\mathbf{R}; \mathbf{Q}; \mathbf{Z}$ be the usual space of all real numbers; rational numbers; integers. Let $\mathbf{R}^* = \mathbf{R}/\mathbf{Z}$ and $\mathbf{Q}^* = \mathbf{Q}/\mathbf{Z}$ be quotient spaces, then $\mathbf{R}^*$ and $\mathbf{Q}^*$ are Fréchet. However, as is well-known, $\mathbf{Q} \times \mathbf{Q}^*$ is not a k-space ([1]). Similarly, any of $(\mathbf{R} - \mathbf{Q}) \times \mathbf{Q}^*$, $\mathbf{Q} \times \mathbf{R}^*$, $(\mathbf{R} - \{1/n : n \in N\}) \times \mathbf{R}^*$, $\mathbf{Q}^* \times \mathbf{Q}^*$, and $\mathbf{Q}^* \times \mathbf{R}^*$ is not a k-space. While, $\mathbf{R} \times \mathbf{R}^*$ and $\mathbf{R}^* \times \mathbf{R}^*$ are sequential by Remark 2.2.

(2) Every product of sequential countably bi-k-spaces (acutually, strongly Fréchet countable spaces) need not be a k-space under $(2^{\aleph_0} < 2^{\aleph_1})$ ([13]).

In Remark 2.3(1), any of the first cordinate spaces of the products, $\mathbf{Q}$, $\mathbf{R} - \mathbf{Q}$, $\mathbf{R} - \{1/n : n \in N\}$, and $\mathbf{Q}^*$ is not locally compact. Let us introduce a certain general type $X(H)$ of these spaces, using a slight modification T' of the countable space T in the previous section.

Let X be a space which contains a countable subset $T' = \{x_0\} \cup \{x_n : n \in N\} \cup \{x_{nm} : n, m \in N\}$ such that each $L_n = \{x_{nm} : m \in N\}$ *converges* to $x_n \notin L_n$, and $H = \{x_n : n \in N\}$ *accumulates* to $x_0 \notin H$. When H *converges* to x_0, we will use "K" instead of "H". For such a space X, let us define a non-locally countably compact subspace

$$X(H) = X - H.$$

If X is the countable space T, then $X(H) = T(K)$. For any sequential space Y having a countable non-closed subset H of non-isolated points, the subspace $Y(H)$ exists.

Remark 2.4. (1) A first countable space X is locally countably compact if and only if it contains no closed copy of the space $T^*(K)$.

(2) For a locally countably compact first countable space Y, and $X \subset Y$, X is not locally countably compact if and only if $X = Y(K)$.

(Indeed, for the "if" part of (1), let X be not locally countably compact at $x \in X$. Then any $\overline{V_n(x)}$ is not countably compact for a local base $\{V_n(x) : n \in N\}$ at x with $\overline{V_{n+1}(x)} \subset V_n(x)$. Hence, there exist distinct points x_{nm} $(n, m \in N)$ such that for each $n \in N$, $\{x_{nm} : m \in N\} \subset \overline{V_n(x)}$, and $\{x_{nm} : n, m \in N\}$ has no accumulation points. Then $S = \{\infty\} \cup \{x_{nm} : m, n \in N\}$ is a closed copy of the space $T^*(K)$. For (2), the "only if" part is shown by putting $V_n(x) = X \cap G_n(x)$ in the previous proof, where $\{G_n(x) : n \in N\}$ is a local base at x in Y such that each $\overline{G_n(x)}$ is compact with $\overline{G_{n+1}(x)} \subset G_n(x)$).

As a weaker condition than "strongly Fréchet spaces", we recall the following condition (C) introduced in [16], and define condition (C*) as its modification. (A space satisfying (C) is called a *Tanaka space* in [11]).

(C) Let $\{A_n : n \in N\}$ be a *decreasing* sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in N$. Then there exist $x_n \in A_n$ such that $\{x_n : n \in N\}$ *converges* to some point $y \in X$.

(C*) Let $\{A_n : n \in N\}$ be a sequence of *countable* subsets of X with $x \in \overline{A_n}$ for any $n \in N$. Then there exist $x_n \in A_n$ such that $\{x_n : n \in N\}$ *accumulates* to some point $y \in X$.

Remark 2.5. (1) Among sequential spaces, (C*) implies (C). (In fact, among sequential spaces, for $x \in \bar{A}$, there exist a countable set $\{x_n : n \in N\} \subset A$ which accumulates to the point x ([7]); while, every subset having an accumulation point contains a convergent sequence).

(2) q-spaces satisfy (C*). Among sequential spaces, countably bi-k-spaces or q-spaces satisfy (C).

(3) Any space satisfying (C) or (C*) contains no closed copy of S_ω, and no S_2. Among sequential spaces, the closedness of the copies can be omitted by Lemma 1.2.

Lemma 2.6. (1) *If $X(H) \times Y$ is a k-space, then Y satisfies* (C*).

(2) *Let X be a bi-k-space. If $X \times Y$ is a k-space, then X is locally countably compact, or Y satisfies* (C*).

Proof. (1) Suppose that Y does not satisfy (C*). Then there exist a point

$y_0 \in Y$, and a sequence $\{A_n : n \in N\}$ of countable subsets with $y_0 \in \overline{A_n}$ for any $n \in N$, but no $\{y_n : n \in N\}$ with $y_n \in A_n$ has accumulation points in Y. Let $X(H) = X - H \supset \{x_0\} \cup \{x_{nm} : n, m \in N\}$, where $H = \{x_n : n \in N\}$ accumulates to the point x_0, and each x_n is a limit point of the sequence $L_n = \{x_{nm} : m \in N\}$ in $X(H)$. Let $A_n = \{y_{nm} : n, m \in N\}$ for each $n \in N$. Let $F = \{(x_{nm}, y_{nm}) : n, m \in N\}$. Then, $(x_0, y_0) \in \bar{F} - F$. Hence F is not closed in $X(H) \times Y$. Since $X(H) \times Y$ is a k-space, it is determined by the cover of all compact subsets. But, each compact set in $X(H) \times Y$ is contained in some compact set $C \times K$ in $X(H) \times Y$. Thus, by *Fact* 1, $X(H) \times Y$ is determined by $\mathscr{P} = \{C \times K : C \times K$ is compact in $X(H) \times Y\}$. While, each compact set K in Y meets only finitely many A_n, because no $\{x_n : n \in N\}$ with $x_n \in A_n$ has accumulation points in Y. But, each compact set C in $X(H)$ meets only finitely many points in L_n. Hence, for each $C \times K \in \mathscr{P}$, $F \cap (C \times K)$ is finite, hence closed in $C \times K$. Thus, F is closed in $X(H) \times Y$. This is a contradiction. Hence Y satisfies (C*).

(2) Assume that X is not locally countably compact at $x_0 \in X$. Let $\mathscr{F} = \{V - C$: V is a nbd of x_0; C is countably compact in $X\}$. Then $\mathscr{F}$ is a filter base accumulating at x_0. Since X is bi-k, there exists a k-sequence $\{A_n : n \in N\}$ such that $x_0 \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathscr{F}$. Here, we can assume that the A_n are closed in X. Then, any A_n is not countably compact. Thus, for each $n \in N$, there exists an infinite discrete countable closed subset D_n of X with $D_n \subset A_n$. Let $A = \bigcap\{A_n : n \in N\}$, $S = \bigcup\{D_n : n \in N\} \cup A$, and let $M = S/A$. Then, S is closed in X, thus $S \times Y$ is a k-space. While, $M \times Y$ is the perfect (hence, quotient) image of $S \times Y$. Thus, $M \times Y$ is a k-space. But, the countable space M is a copy of $T^*(K)$. Thus $T^*(K) \times Y$ is a k-space. Thus Y satisfies (C*) by (1).

Remark 2.7. $S_\omega \times S_\omega$, $S_\omega \times S_2$, $S_2 \times S_2$, $T^* \times S_\omega$, and $T^* \times S_2$ are all sequential spaces by Remark 2.2. While, neither $T^*(K) \times S_\omega$ nor $T^*(K) \times S_2$ is a k-space by Lemma 2.6(1).

Proposition 2.8. *Let Y be a sequential space. Let Y be the space* (A_5) *under a quotient s-map f, or the space* (A_6) *under a closed map g. Then Y is countably bi-k if and only if Y satisfies* (C). *For the space* (A_6), *it is possible to replace* "(C)" *by* "*Y contains no* (*closed*) *copy of S_ω*".

Proof. The "only if" part holds by Remark 2.5(2)&(3). For the "if" part, first recall that a space X is an *A-space* (resp. *inner-closed A-space*) [8] if, whenever $\{A_n : n \in N\}$ is a decreasing sequence with $x \in \overline{(A_n - \{x\})}$, then there exist $B_n \subset A_n$ (resp. $B_n \subset A_n$ which are closed in X) such that $\bigcup\{\overline{B_n} : n \in N\}$ is not closed in X. Now, let Y satisfy (C). Then Y is an inner-closed A-space, because, among sequential spaces, we can assume that $\bigcap\{A_n : n \in N\} = \emptyset$ in the above definition; cf. [8; Lemma 4.1]. Thus the quotient s-map f is bi-quotient by Theorem 9.5 and Lemmas 8.3 & 9.6 in [7]. For the space (A_6), more generally, let Y contain no closed copy of S_ω. Then the sequential space Y is an A-space by [18; Theorem 1.1]. Thus the closed map g is countably bi-quotient by Proposition 2.4 and Theorem 6.3 in [8]. While, every countably bi-quotient image of a countably bi-k-space is countably bi-k; see [7]. Thus, Y is countably bi-k.

THEOREM 2.9. *Let Y be a sequential space, and let $X(H) \times Y$ be a k-space. Then the following* (1) *and* (2) *hold. Also, the converses of these hold if X or $X(H)$ is bi-k.*

(1) *Y is strongly Fréchet if Y is the space* (A_1), (A_2), (A_3), *or* (A_4). (*Y has a point-countable base for the space* (A_4)).

(2) *Y is countably bi-k if Y is the space* (A_5) *or* (A_6).

PROOF. By Lemma 2.6(1), Y satisfies (C*), thus, Y also contains no closed copy of S_ω, and no S_2 by Remark 2.5(3). Then (1) holds by Proposition 1.3, and (2) holds by Proposition 2.8 and Remark 2.5(1). For the latter part, let X be bi-k. Since $X(H)$ is a G_δ-subset of X, $X(H)$ is bi-k by [7; Proposition 3.E.4]. Thus the latter part holds by Proposition 2.1.

REMARK 2.10. For the latter part of Theorem 2.9, the bi-k-ness of X or $X(H)$ is essential even if $X(H)$ is Fréchet, and Y is metric. (Indeed, let $X = \mathbf{R}/\mathbf{Z}$, and let $Y = \mathbf{Q} = S(K)$ for the obvious countable subspace S of $\mathbf{R}$. Thus $X(K) \times \mathbf{Q} (= S(K) \times X(K))$ is not a k-space by Theorem 2.9, because the sequential space $X(K)$ which is the spaces $(A_1) \sim (A_6)$ is not strongly Fréchet, nor countably bi-k).

Similarly as in the proof of Theorem 2.9, using Lemma 2.6(2), we have the following characterization for the products of bi-k-spaces with certain sequential spaces to be k-spaces. (1) is due to [16], where X is first countable, and Y is the space (A_1) or (A_2).

THEOREM 2.11. *Let X be a bi-k-space, and let Y be sequential. Then the following* (1) *and* (2) *hold.*

(1) *Suppose that Y is the space* (A_1), (A_2), (A_3), *or* (A_4). *Then $X \times Y$ is a k-space if and only if X is locally countably compact, or Y is strongly Fréchet.* (*Y has a point-countable base for the space* (A_4)).

(2) *Suppose that* Y *is the space* (A_5) *or* (A_6). *Then* $X \times Y$ *is a* k*-space if and only if* X *is locally countably compact*, *or* Y *is countably bi-*k.

LEMMA 2.12. *Let* $f : X \to Y$ *be a closed map such that* X *is a normal space or an* M*-space*, *and* Y *is sequential. If* Y *contains no closed copy of* S_ω, *then every boundary* $\partial f^{-1}(y)$ *is countably compact.*

PROOF. If X is normal, then the result holds by [19; Proposition 1.13]. If X is an M-space, then the result also holds by the same way. (Recall that every discrete closed subset $\{x_n : n \in N\}$ of X, there exists a discrete open collection $\{U_n : n \in N\}$ with $x_n \in U_n$, because X is the inverse image of a metric space under a quasi-perfect map).

LEMMA 2.13. (1) *Every quasi-pefect image of an* M*-space* (*resp. countably bi-*k*-space*) *is a* q*-space* (*resp. countably bi-*k*-space*).

(2) *Every product of a bi-*k*-space with a* k*-and-*q*-space is a* k*-space.*

PROOF. (1) is known or routinely shown. (In fact, let $f : X \to Y$ be a quasi-perfect map with X an M-space. Let $y \in Y$. Then there exist a q-sequence $\{V_n : n \in N\}$ of open sets such that $f^{-1}(y) \subset V_n$, because X is the inverse image of a metric space under a quasi-perfect map. Then the point y has a q-sequence $\{W_n : n \in N\}$ of nbds such that $f^{-1}(W_n) \subset V_n$. Thus Y is a q-space). For (2), recall that every bi-k-space is precisely the bi-quotient image of a paracompact M-space ([7]), and that every product of bi-quotient maps is bi-quotient ([6]), hence, quotient. Now, let X be bi-k, and let Y be a k-and-q-space. Then X is the bi-quotient image of a paracompact M-space Z. Thus $X \times Y$ is the quotient image of $Z \times Y$. While, Z is of pointwise countable type, and Y is a k-and-q-space, then $Z \times Y$ is a k-space by [15; Theorem 2.6]. Thus, $X \times Y$ is also a k-space.

PROPOSITION 2.14. *Let* Y *be a sequential space. If* Y *is the space* (A_7) (*resp.* (A_8)) *under a closed map* f, *then the following are equivalent.*

(a) Y *is a countably bi-*k*-space* (*resp.* q*-space*).

(b) Y *contains no* (*closed*) *copy of* S_ω.

(c) *Every* $\partial f^{-1}(y)$ *is countably compact.*

PROOF. (a) $\Rightarrow$ (b) holds by Remark 2.5(2)&(3). (b) $\Rightarrow$ (c) holds by Lemma 2.12. For (c) $\Rightarrow$ (a), since f is closed, as is well-known, there exists a closed subset

F of the domain such that $f(F) = Y$, $f^{-1}(y) = \partial f^{-1}(y)$ or a singleton on F. Thus we can assume that f is quasi-perfect. Then Y is a countably bi-k-space (resp. q-space) by Lemma 2.13(1).

THEOREM 2.15. *Let Y be a sequential space. If Y is the space* (A_7) (*resp.* (A_8)) *under a closed map f, then* (1) *and* (2) *below hold.*

(1) *The implication* (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) *below holds. If X or $X(H)$ is bi-k, then* (a), (b), *and* (c) *are equivalent.*

(a) *Let $X(H) \times Y$ be a k-space.*

(b) *Every $\partial f^{-1}(y)$ is countably compact.*

(c) *Y is a countably bi-k-space* (*resp. q-space*).

(2) *Let X be a bi-k-space. Then the following are equivalent.*

(a) *$X \times Y$ is a k-space.*

(b) *X is locally countably compact, or every $\partial f^{-1}(y)$ is countably compact.*

(c) *X is locally countably compact, or Y is a countably bi-k-space* (*resp. q-space*).

PROOF. We show that (2) holds since (1) is similarly shown. (a) $\Rightarrow$ (b) holds by Lemma 2.6(2) and Proposition 2.14. (b) $\Rightarrow$ (c) holds by Proposition 2.14. (c) $\Rightarrow$ (a) holds by Proposition 2.1 and Lemma 2.13(2).

Finally, let us give Question 2.16 below in view of the results, Theorems 2.11 & 2.15, Remark 2.5, and Lemma 2.6(2). In this question, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) holds by these results. If Y is one of the spaces (A_1)$\sim$(A_8), then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) holds, and so does (a) $\Leftrightarrow$ (d), but except for the space (A_5) in view of the proof of Theorems 2.11 & 2.15. We do not know whether (a) $\Leftrightarrow$ (d) holds for the space (A_5).

QUESTION 2.16. Let X be a bi-k-space, and let Y be a sequential space. Then, are the following equivalent?

(a) $X \times Y$ is a k-space.

(b) X is locally countably compact, or Y satisfies (C*).

(c) X is locally countably compact, or Y satisfies (C).

(d) X is locally countably compact, or Y contains no (closed) copy of S_ω, and no S_2.

Comment. The equivalence (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) holds. Indeed, when X is first countable, (c) $\Rightarrow$ (a) holds by combining [10] and [11]. Thus, when X is bi-k, this implication also holds as in the proof of Proposition 4.6 in [17]. Thus, (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) holds. However, we do not know whether (a) $\Leftrightarrow$ (d) holds or not.

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