THREE-DISTANCE SEQUENCES WITH THREE SYMBOLS

By

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Abstract. We will show that every 3 dimensional cutting sequence is a three-distance sequence, and there are uncountable many periodic or aperiodic three-distance sequences (with 3-symbols) which are not 3 dimensional cutting sequences.

1 Introduction

W. F. Lunnon and P. A. B. Pleasants [1] defined two-distance sequences and proved that each 2 dimensional (2D) cutting sequence (see below, for the definition) is a two-distance sequence and the converse also holds. The basic framework of their research is traced back to the one by M. Morse and G. A. Hedlund [4].

In this paper, we will discuss the relationships between 3 dimensional (3D) cutting sequences and three-distance sequences. We will show that every 3D cutting sequence is a three-distance sequence, and there are uncountable many periodic or aperiodic three-distance sequences which are not 3D cutting sequences.

First, we recall the definition of 2D cutting sequences. Although the definition given below is slightly different from that described in [1] or [5], the equivalence of 2D cutting sequences and two-distance sequences ([1, theorem 1]) holds by the same proof.

The set of the real numbers and the rational integers, and the non-negative rational integers are denoted by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$, respectively.

We consider the standard orthogonal coordinates $x, y$ in the 2 dimensional Euclidean space $\mathbb{R}^2$, and take a line $L$ in $\mathbb{R}^2$. We assume that the slope of the line $L$ is non-negative, and $L$ is not parallel to either axis. When the line $L$ crosses a
vertical grid line or a horizontal one, we mark the point of the intersection and label it as A and B, respectively.

![Figure 1](image)

In the above labeling, we need to specify the way of labeling the intersection L ∩ Z².

**Type 1:** 
#(L ∩ Z²) = 1. Label the point of the intersection L ∩ Z² by either of the two elements of \( S_2 = \{AB, BA\} \).

**Type 2:** 
#(L ∩ Z²) ≥ 2. Observe that #(L ∩ Z²) = ∞.

1. Label all the points of the intersection L ∩ Z² by one of the two elements of \( S_2 \).
   In this way, we obtain two infinite periodic sequences associated with the line L.
2. Fix an arbitrary point P on L. The point P divides L into two half-lines \( L^+_P \) and \( L^-_P \). We label the integer points on \( L^+_P \setminus \{P\} \) by an element of \( S_2 \), and label the integer points on \( L^-_P \setminus \{P\} \) by another element of \( S_2 \). When P is an integer point, we label P by an element of \( S_2 \).

These give one or more two-way infinite sequences of symbols A and B. Such sequences are called the 2D cutting sequences obtained from L.

**Remark 1.1.** The labeling of Type 2 (2) is introduced to obtain the equivalence between 2D cutting sequences and two-distance sequences ([1]).

### 2 3D Cutting Sequence

In this section, we define 3D cutting sequences as a natural extension of 2D cutting sequences. We consider the standard orthogonal coordinates \( x, y, z \) in the 3 dimensional Euclidean space \( \mathbb{R}^3 \). Let \( P_{uv}(L) \) be the projection of a line L in \( \mathbb{R}^3 \).
on the uv-plane, where \( u, v \in \{ x, y, z \} \). We assume that each projection \( P_{uv}(L) \) has a non-negative slope, and \( L \) does not lie in any uv-hyperplane. Let \( \mathcal{H}_A \) (resp. \( \mathcal{H}_B, \mathcal{H}_C \)) be the collection of hyperplanes in \( \mathbb{R}^3 \) defined by

\[
x = r_x, \quad (\text{resp. } y = r_y, \ z = r_z)
\]

where \( r_x, r_y, r_z \in \mathbb{Z} \).

When \( L \) intersects with a hyperplane \( H_A \in \mathcal{H}_A \) (resp. \( H_B \in \mathcal{H}_B, \ H_C \in \mathcal{H}_C \)), label the point of the intersection \( H_A \cap L \) (resp. \( H_B \cap L, \ H_C \cap L \)) by \( A \) (resp. \( B, C \)).

Let \( \mathcal{L}_x \) (resp. \( \mathcal{L}_y, \mathcal{L}_z \)) be the collection of the lines defined by the equation

\[
y = r_y \quad \text{and} \quad z = r_z, \quad r_y, r_z \in \mathbb{Z}
\]

(resp. \( x = r_x \) and \( z = r_z, \quad r_x, r_z \in \mathbb{Z} \),

\[
x = r_x \quad \text{and} \quad y = r_y, \quad r_x, r_y \in \mathbb{Z}.
\]

We put \( \mathcal{L} = \mathcal{L}_x \cup \mathcal{L}_y \cup \mathcal{L}_z \) and the set \( \Lambda = \bigcup \mathcal{L} \) is called the grid of \( \mathbb{R}^3 \) in the present paper.

As we did in defining the 2D cutting sequences, we need to specify the way of labeling the points of the intersection of \( L \) and \( \Lambda \) or \( \mathbb{Z}^3 \). We divide our consideration into the following three cases. First notice that if \( L \cap \mathbb{Z}^3 \neq \emptyset \) then \( \#(L \cap \mathbb{Z}^3) = 1 \) or \( \infty \).

**Case 1** \( L \cap \mathbb{Z}^3 \neq \emptyset \) and \( L \cap (\Lambda \setminus \mathbb{Z}^3) = \emptyset \).

**Case 2** \( L \cap \mathbb{Z}^3 = \emptyset \) and \( L \cap (\Lambda \setminus \mathbb{Z}^3) \neq \emptyset \) and

**Case 3** \( L \cap \mathbb{Z}^3 \neq \emptyset \) and \( L \cap (\Lambda \setminus \mathbb{Z}^3) \neq \emptyset \).
Case 1:

**type 1:** \#(L ∩ \mathbb{Z}^3) = 1.

Label the point of the intersection L ∩ \mathbb{Z}^3 by an element of S_3, where

$$S_3 = \{ABC, ACB, BAC, BCA, CAB, CBA\}.$$  

In this way, we obtain the six infinite sequences associated with the line L.

**type 2:** \#(L ∩ \mathbb{Z}^3) = \infty.

Fix an arbitrary point P on L. The point P divides L into two half-lines L_+ and L_. Pick up two (possibly equal) elements X_+, X_- of S_3. Then label the points of the intersection (L_+ \setminus \{P\}) ∩ \mathbb{Z}^3 by X_+, and label the points of the intersection (L_- \setminus \{P\}) ∩ \mathbb{Z}^3 by X_-.

In this way, we obtain the 36 infinite periodic sequences associated with the line L.

Case 2:

**type 1:** Suppose that there exists a unique \ell ∈ \mathcal{L} which intersects with L.

We define S_u (u = x, y, z) as follows.

$$S_x = \{BC, CB\}, \quad S_y = \{AC, CA\}, \quad S_z = \{AB, BA\}.$$  

When \ell ∈ \mathcal{L}_u, label the point of the intersection \ell ∩ L by an element of S_u.

In this way, we obtain two infinite periodic sequences associated with the line L.

**type 2:** Suppose that there exist two lines \ell, \ell' ∈ \mathcal{L} such that \ell ∩ L \neq \varnothing and \ell' ∩ L \neq \varnothing, and recall that L does not lie in any uv-hyperplane. Fix an arbitrary point P on L. The point P divides L into two half-lines L_+ and L_. Pick up two (possibly equal) elements X_u^+, X_u^- of S_u. Then label the point of the intersection (L_+ \setminus \{P\}) ∩ \ell, \ell ∈ \mathcal{L}_u by X_u^+, and the point of the intersection (L_- \setminus \{P\}) ∩ \ell', \ell' ∈ \mathcal{L}_u by X_u^-.

When \{P\} = L ∩ \ell, \ell ∈ \mathcal{L}_u, we label P by an element of S_u.

Case 3: First we observe that, \#{\ell ∈ \mathcal{L} : L ∩ (\ell \setminus \mathbb{Z}^3) \neq \varnothing} = \infty.

We define the following notation for the labeling in this case. Let W be the set of all finite sequences with symbols A, B, C. A function

$$D_u : W \to W$$

(u = x, y, z) is defined as follows: for w ∈ W, D_u(w) is a finite sequence with two symbols obtained by removing δ(u) from w, where
Also a function
\[
\delta(u) = \begin{cases} 
A, & \text{if } u = x \\
B, & \text{if } u = y \\
C, & \text{if } u = z.
\end{cases}
\]

is defined as follows: for an element \( w = w_1 \cdots w_l \) of \( W \) (\( \{w_1, \ldots, w_l\} \subset \{A, B, C\} \)), \( F_u(w) = w_l \cdots w_1 \).

We fix an arbitrary point \( P \) on \( L \). The point \( P \) divides \( L \) into two half-lines \( L^+_p \) and \( L^-_p \).

**type 1:** \( \#(L \cap \mathbb{Z}^3) = 1 \).

Label the point of the intersection \( L^+_p \cap \mathbb{Z}^3 \) by an element \( X \) of \( S_3 \). For the labeling the intersection \( L \cap L^+_p \), we take the following two ways.

1. Label the intersection \( \ell \cap L^+_p \) and \( \ell' \cap L^-_p \) with \( \ell, \ell' \in L_u \) as \( D_u(X) \).
2. Label the intersection \( \ell \cap L^+_p \) with \( \ell \in L_u \) by \( D_u(X) \), and the intersection \( \ell' \cap L^-_p \) with \( \ell' \in L_u \) by \( F_u \circ D_u(X) \).

**type 2:** \( \#(L \cap \mathbb{Z}^3) = \infty \).

Pick up two (possibly equal) elements \( X^+, X^- \) of \( S_3 \). Label the points of the intersection \( L^+_p \cap \mathbb{Z}^3 \) by \( X^+ \) and \( L^-_p \cap \mathbb{Z}^3 \) by \( X^- \). Then label \( L^+_p \cap \ell \) with \( \ell \in L_u \) by \( D_u(X^+) \) and \( L^-_p \cap \ell' \) with \( \ell' \in L_u \) by \( D_u(X^-) \).

These give one or more bi-infinite sequences with symbols \( A, B, C \). Such sequences are called the **3D cutting sequences** obtained from \( L \).

**Remark 2.1.** The function \( D_u \) is naturally extended to a function \( D_u : \Sigma \to \Sigma \) of the set \( \Sigma \) of all infinite sequences with symbols \( A, B, C \).

If \( S \) is a 3D cutting sequence associated with a line \( L \), then \( D_u(S) \) is a 2D cutting sequence associated with the line \( P_{uv}(L) \), where \( \{u, v\} \subset \{x, y, z\} \). In this way, 2D cutting sequences are obtained from 3D cutting sequences.

### 3 Three-Distance Sequence

In this section, we define the notion of three-distance sequences with three symbols. The following definitions are the natural extensions of those for two-distance sequences with two symbols \( A, B \) [1].

Let \( S \) be a bi-infinite sequence with three symbols \( A, B, C \).

**Definition 3.1.** A word \( w \) in \( S \) is a finite string of consecutive symbols from \( S \).
Definition 3.2. The length $|w|$ of a word $w$ is the total number of symbols which are contained in $w$.

Definition 3.3. The $i$-weight $|w|i$ of a word $w$ ($i \in \{A, B, C\}$) is the number of the symbol $i$ in the word $w$. Notice that $|w| = |w|A + |w|B + |w|C$.

Definition 3.4. A sequence $S$ is called a three-distance sequence, if, for each $l \in \mathbb{Z}_+$ and for each $i \in \{A, B, C\}$, we have the inequality

$$\#\{|w|i : w \text{ is a word of } S \text{ and } |w| = l\} \leq 3.$$  

Similarly we define $m$-distance sequences for infinite sequences with $n$ symbols ($n \geq 2$).

Definition 3.5. An infinite sequence $S$ with $n$ symbols $x_1, x_2, \ldots, x_n$ is called an $m$-distance sequence if, for each $l \in \mathbb{Z}_+$ and for each $x_z$ ($1 \leq z \leq n$), we have the inequality

$$\#\{|w|_{x_z} : |w| = l\} \leq m.$$ 

By the definition, every $(m - 1)$-distance sequence is an $m$-distance sequence.

Lemma 3.1. Let $S$ be an infinite sequence with $n$ symbols $x_1, x_2, \ldots, x_n$.

1. If $S$ is $m$-distance, then, for each $l \in \mathbb{Z}_+$ and for each $x_z$ ($1 \leq z \leq n$), there exist $\mu \in \mathbb{Z}_+$ and $m'$ with $0 \leq m' \leq m - 1$ such that

$$\{|w|_{x_z} : |w| = l\} = \{|w|_{x_z} : |w| = l\}.$$  

2. If $S$ is not $m$-distance, then there exist an $l \in \mathbb{Z}_+$ an $z \in \{1, \ldots, n\}$ and two words $w_1, w_2$ in $S$ of length $l$, such that $|w_2|_{x_z} - |w_1|_{x_z} = m$.

Proof. Fix an arbitrary $l \in \mathbb{Z}_+$ and $z \in \{1, \ldots, n\}$. We put $\mu = \min\{|w|_{x_z} : |w| = l\}$ and $M = \max\{|w|_{x_z} : |w| = l\}$. Then for each word $w$ such that $|w| = l$, $\mu \leq |w|_{x_z} \leq M$. When $M - \mu \leq 1$, there is nothing to prove. In what follows, we consider the case $M - \mu \geq 2$. The sequence $S$ is written as

$$S = \cdots w_{-1}w_0w_1 \cdots w_lw_{l+1}w_{l+2} \cdots$$

Take two words $w_1, w_1^+$ in $S$, such that $|w_1|_{x_z} = \mu$, $|w_1^+|_{x_z} = M$. We assume, without loss of generality, that $w_1 = w_1w_2 \cdots w_{l-1}w_l$, $w_1^+ = w_1w_2 \cdots w_{l+1}w_{l+2} \cdots w_{l+1+d}w_{l+d}$, $d > 0$. We define

$$\chi(w_1) = w_2 \cdots w_{l+1},$$
and

$$\chi^c(w_1) = \chi^c(\chi^{c-1}(w_1)) = w_{1+c} \cdots w_{l+c}, \quad (c \in \mathbb{Z}_+)$$

If $$|\chi^{c}(w_1)|_{x_1} = |w_1|_{x_1}$$, for each $$c \geq 0$$, then $$S$$ is three-distance. If it is not the case, let

$$c_1 = \max\{c : |\chi^{c}(w_1)|_{x_1} = |w_1|_{x_1}\}.$$ 

By the definition, it follows that

$$|\chi^{c_1+1}(w_1)|_{x_1} = |w_1|_{x_1} + 1.$$ 

If $$|\chi^{c}(w_1)|_{x_1} \leq |w_1|_{x_1} + 1$$, for each $$c \geq c_1$$, then $$S$$ is three-distance. If it is not the case, we put

$$c_2 = \max\{c : |\chi^{c}(w_1)|_{x_1} \leq |w_1|_{x_1} + 1, c \geq c_1\}.$$

Then

$$|\chi^{c_2+1}(w_1)|_{x_1} = |w_1|_{x_1} + 2.$$ 

If $$|\chi^{c}(w_1)|_{x_1} \leq |w_1|_{x_1} + 2$$, for each $$c \geq c_2$$, then $$S$$ is three-distance. If it is not the case, let

$$c_3 = \max\{c : |\chi^{c}(w_1)|_{x_1} \leq |w_1|_{x_1} + 2, c \geq c_2\}.$$ 

Then

$$|\chi^{c_3+1}(w_1)|_{x_1} = |w_1|_{x_1} + 3.$$ 

We repeat this process up to $$M - \mu$$ steps. If $$S$$ is $$m$$-distance, then $$M - \mu < m$$. Then $$\mu$$ and $$m' := M - \mu$$ satisfy the conclusion of (1). If $$S$$ is not $$m$$-distance, then there exist an $$l \in \mathbb{Z}_+$$ and an $$x$$ such that $$\#\{w_{x_1} : |w| = l\} > m$$. Arguing as above, we may find two words $$w_1, w_2$$ in $$S$$ of length $$l$$, such that $$|w_2|_{x_1} - |w_1|_{x_1} = m$$.

This completes the proof.

Some examples of three-distance sequences with three symbols will be given in the next section.

4 3D Cutting Sequences and Three-Distance Sequences

**Example 4.1.** The line in $$\mathbb{R}^3$$ defined by the equation “$$x = y = z$$” yields a periodic 3D cutting sequence

$$(ABC)^\infty = \cdots ABCABCABC \cdots ABCABCABCABC \cdots.$$ 

It is easy to see that the above sequence is two-distance.
Table 1 is a list of the words in the above sequence of length up to 5, and their weights.

<table>
<thead>
<tr>
<th>Length</th>
<th>Words</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>w</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>A, B, C</td>
<td>0, 1</td>
</tr>
<tr>
<td>2</td>
<td>AB, BC, CA</td>
<td>0, 1</td>
</tr>
<tr>
<td>3</td>
<td>ABC, BCA, CAB</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>ABCA, BCAB, CABC</td>
<td>1, 2</td>
</tr>
<tr>
<td>5</td>
<td>ABCAB, BCABC, CABCA</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Table 2 is a list of the words in the above sequence of length up to 4, and their weights.

<table>
<thead>
<tr>
<th>Length</th>
<th>Words</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>w</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>A, B, C</td>
<td>0, 1</td>
</tr>
<tr>
<td>2</td>
<td>AB, BA, BB, AC, CB, CA, BC</td>
<td>0, 1</td>
</tr>
<tr>
<td>3</td>
<td>ABC, CBB, BAB, BBA, BCB, CBC, BAC, CAB, CBA, BBC, BCA, ACB, ABB</td>
<td>0, 1</td>
</tr>
<tr>
<td>4</td>
<td>ACBB, ABCB, ACBC, ABCB, ABCB, BBBC, BBCA, BCAB, BCBB, BBAC, BCBA, BABC, BBBC, CBC, CAB, CBCA, CBBA, CBAB, CBAC, CBBA, CBBC</td>
<td>0, 1</td>
</tr>
</tbody>
</table>

We show that each 3D cutting sequence is three-distance.

The orthogonal projection on the u-axis \((u \in \{x, y, z\})\) in \(\mathbb{R}^3\) is denoted by \(P_u\). Let \(S\) be a 3D cutting sequence associated with a line \(L\) in \(\mathbb{R}^3\). Take an arbitrary word \(w = w_1 \cdots w_l\) in \(S\), \(\{w_1, \ldots, w_l\} \subset \{A, B, C\}\). And take the points...
Therefore, we have

\[ m \text{, } m' \text{ which correspond to } w_1 \text{ and } w_l \text{ respectively, as the point of the intersection } L \cap H_i \text{, } H_i \in \mathcal{H}_l, l \in \{A, B, C\} \text{, or } L \cap \ell \text{, } \ell \in \mathcal{L} \text{, or } L \cap \mathbb{Z}^3. \text{ Let } M \text{ be the segment of } L \text{ whose end-points are } m \text{ and } m'. \text{ The length of the projection of } M \text{ on the } u\text{-axis is denoted by } P_u(M) \text{. Then we obtain the following inequalities.}

\[
\begin{align*}
|w|_A - 1 &\leq P_u(M) \leq |w|_A + 1 \\
|w|_B - 1 &\leq P_v(M) \leq |w|_B + 1 \\
|w|_C - 1 &\leq P_v(M) \leq |w|_C + 1
\end{align*}
\]

(4.0)

The symbols A, B, C correspond to x, y, z, respectively via the above inequality.

**Theorem 4.1.** Each 3D cutting sequence is three-distance.

**Proof.** Let S be a 3D cutting sequence associated with a line L in \( \mathbb{R}^3 \). We assume that there exist an \( i \in \{A, B, C\} \) and two words \( w_1, w_2 \) in S, such that \(|w_1| = |w_2| \) and \(|w_1| + 2 < |w_2|\). Then we obtain

\[ 0 < |w_1| + 1 < |w_2| - 1. \]

(4.1)

Let \( u \) be the coordinate corresponding to \( i \) via (4.0). And let \( M_1, M_2 \) be the segments of L whose end-points are the points corresponding to the first and last symbols of \( w_1, w_2 \) respectively. Then the slope of \( P_{uv}(L) \) is

\[ \frac{P_v(M_1)}{P_u(M_1)} = \frac{P_v(M_2)}{P_u(M_2)}. \]

Let \( k \) be a symbol, \( k \in \{A, B, C\} \setminus \{i\} \) and v the coordinate corresponding to \( k \), \( v \in \{x, y, z\} \setminus \{u\} \). By using the inequalities (4.0) and (4.1), it follows that

\[
\frac{|w_1|_k - 1}{|w_1| + 1} \leq \frac{P_v(M_1)}{P_u(M_1)} = \frac{P_v(M_2)}{P_u(M_2)} \leq \frac{|w_2|_k + 1}{|w_2| - 1}.
\]

Therefore, we have

\[ \frac{|w_1|_k - 1}{|w_1| + 1} \leq \frac{|w_2|_k + 1}{|w_2| - 1}. \]

(4.2)

From (4.1) and (4.2), we obtain

\[ |w_1|_k - 1 < |w_2|_k + 1. \]

(4.3)

Let \( j \) be the symbol other then \( i, k \). Namely \( \{i, j, k\} = \{A, B, C\} \). Then,

\[
|w_1| = |w_1|_i + |w_1|_j + |w_1|_k = |w_2|_i + |w_2|_j + |w_2|_k
\]

\[
< |w_2|_i - 2 + |w_1|_j + |w_2|_j + 2 = |w_2|_i + |w_1|_j + |w_2|_k.
\]
Hence
\[ |w_2|_j < |w_1|_j. \] (4.4)

By the symmetric argument, from (4.2), we have
\[ \frac{|w_1|_j - 1}{|w_1|_j + 1} \leq \frac{|w_2|_j + 1}{|w_2|_j - 1}, \] (4.5)
and thus
\[ |w_1|_j - 1 < |w_2|_j + 1. \] (4.6)

The inequalities (4.4) and (4.6) imply \(|w_1|_j - 1 < |w_2|_j + 1 < |w_1|_j + 1\). Hence, we have
\[ |w_2|_j + 1 = |w_1|_j. \] (4.7)

Then, \(|w_1|_j + |w_1|_j = |w_1|_j + |w_2|_j + 1 < |w_2|_j + |w_2|_j - 1\). Therefore, we obtain
\[ |w_1|_k > |w_2|_k. \] (4.8)

The inequalities (4.8) and (4.3) imply \(|w_1|_k - 1 < |w_2|_k + 1 < |w_1|_k + 1\). Hence we have
\[ |w_2|_k + 1 = |w_1|_k. \] (4.9)

From (4.7) and (4.9),
\[
|w_1| = |w_1|_i + |w_1|_j + |w_1|_k \\
= |w_1|_i + |w_2|_j + |w_2|_k + 2 < |w_2|_i + |w_2|_j + |w_2|_k = |w_2|.
\]

This is the contradiction. Hence for each \(i \in \{A, B, C\}\), there exist no words \(w_1, w_2\) such that \(|w_2|_i - |w_1|_i| > 2\). So \(S\) is a three-distance sequence. Q.E.D

There exists a three-distance sequence which is not a 3D cutting sequence. We give such an example.

**Example 4.3.** A periodic infinite sequence which repeats the word AACABCAB

\[ S = \cdots CABAACABCABAACAB \cdots = (AACABCAB)^\infty \]

is three-distance. We show that \(S\) is not a 3D cutting sequence. If \(S\) is a 3D cutting sequence associated with a line \(L\) in \(\mathbb{R}^3\), then by Remark 2.1, for each \(u\), \(\mathcal{D}_u(S)\) is a 2D cutting sequence associated with \(P_{uv}(L)\) (\(\{u, v\} \subset \{x, y, z\}\)). Here by [1, Theorem 1], \(\mathcal{D}_u(S)\) is a two-distance sequence. However,

\[ \mathcal{D}_y(S) = \cdots CAAACACAAACA \cdots = (CAAACA)^\infty \]
is not two-distance with two symbols \(A, C\), since the \(C\)-weight of the words \(AAA, ACA, CAC\) of length 3 in \(D_{\gamma}(S)\) is 0, 1, 2 respectively. Thus \(D_{\gamma}(S)\) cannot be a 2D cutting sequence. Accordingly, \(S\) is a three-distance sequence which is not a 3D cutting sequence.

5 Three-Distance Sequences which are not 3D Cutting Sequences

In this section, we show that there exist infinitely many three-distance sequences which are not 3D cutting sequences. Let \(x_1, \ldots, x_n\) be the \(n\) symbols. We fix a bijection

\[ f_n : \{1, 2, \ldots, n!\} \rightarrow S_n, \]

where

\[ S_n = \{x_{\sigma_1} \cdots x_{\sigma_n} : \{\sigma_1, \ldots, \sigma_n\} = \{1, \ldots, n\}\}. \]

Note that \(\#(S_n) = n!\). For each bi-infinite sequence \(R_n = \cdots \rho_{-1}\rho_0\rho_1\rho_2 \cdots\) with \(\rho_v \in \{1, 2, \ldots, n!\}\) \((v \in \mathbb{Z})\), we define a bi-infinite sequence with \(n\) symbols \(x_1, \ldots, x_n\) as follows.

\[ f_n(R_n) = \cdots f_n(\rho_{-1})f_n(\rho_0)f_n(\rho_1)f_n(\rho_2) \cdots. \]

The set of all such sequences is denoted by \(\Sigma_{t_n}\).

**Proposition 5.1.**

1. If \(n \leq 3\), then each sequence of \(\Sigma_{t_n}\) is three-distance.
2. If \(n \geq 4\), then each sequence of \(\Sigma_{t_n}\) is four-distance.

**Proof.** When \(n = 1\), there is nothing to prove. Assume \(n \geq 2\). Let \(S\) be an element of \(\Sigma_{t_n}\). Fix an arbitrary \(l \in \mathbb{Z}_+\). We put \(l = nt + r\) with \(t \in \mathbb{Z}_+\) and \(0 \leq r < n\). Let \(w\) be a word of \(S\) such that \(|w| = l\). When \(l = |w| < n\), we obtain \(|w|_{x_a} \leq 2 \quad (x_a \in \{x_1, \ldots, x_n\})\). Now suppose \(l \geq n\). We write \(w\) as \(w = w_1\overline{w}w_2\), where \(\overline{w} = f_n(\rho_v) \cdots f_n(\rho_{v+h}), \quad v \in \mathbb{Z}, \quad h \in \mathbb{Z}_+\), and \(w_1, w_2\) are the words of \(S\) such that \(w_1\) is a proper subword of \(f_n(\rho_{v-1})\) and \(w_2\) is a proper subword of \(f_n(\rho_{v+h+1})\). If \(|w_1| = |w_2| = 0\), then \(|w| = |\overline{w}| = nt\). If \(|w_a| \neq 0\) and \(|w_b| = 0\) \((a, b \in \{1, 2\})\), then \(|\overline{w}| = nt\) and \(1 \leq |w_a| = r < n\). If \(|w_1| \neq 0\) and \(|w_2| \neq 0\), then \(2 \leq |w_1| + |w_2| \leq 2n - 2\). Thus we have

\[ nt + r - 2 \leq |\overline{w}| \leq nt + r - 2n + 2. \]

Since \(0 \leq r < n\), we obtain

\[ nt - 2 \leq nt + r - 2 \leq |\overline{w}| \leq nt + r - 2n + 2 < nt - n + 2 = n(t - 1) + 2. \]
Namely

\[ n(t-1) \leq nt - 2 \leq |\overline{w}| < n(t-1) + 2. \]

Therefore \( |\overline{w}| = n(t-1) \). First, we consider the case \( |\overline{w}| = nt \). Then \( |w_1| + |w_2| = r \) and \( |\overline{w}|_{x_a} = t, \ 0 \leq |w_1|_{x_a} + |w_2|_{x_a} \leq 2 \). Since \( |w|_{x_a} = |w_1|_{x_a} + |\overline{w}|_{x_a} + |w_2|_{x_a} \), we have

\[ t \leq |w|_{x_a} \leq t + 2. \quad (5.10) \]

Next, we consider the case \( |\overline{w}| = n(t-1) \). Then \( |w_1| + |w_2| = n + r \) and \( 0 \leq |w_1|_{x_a} + |w_2|_{x_a} \leq 2 \), and \( |\overline{w}|_{x_a} = t - 1 \). Thus we have

\[ t - 1 \leq |w|_{x_a} \leq t + 1. \quad (5.11) \]

By inequalities (5.10) and (5.11), we obtain \( t - 1 \leq |w|_{x_a} \leq t + 2 \). Therefore \( S \) is at most four-distance. Furthermore, if \( n \geq 4 \), it is easy to create a four-distance sequence. Next, we consider the following case: \( n \leq 3 \).

**Case 1:** When \( n = 2 \), an arbitrary \( l \) is written as \( l = 2t \) or \( l = 2t + 1 \).

First, we assume \( l = |w| = 2t \). If \( |\overline{w}| = 2t \), then \( |w|_{x_a} = |\overline{w}|_{x_a} = t \). If \( |\overline{w}| = 2(t-1) \), then \( t - 1 \leq |w|_{x_a} \leq t + 1 \). Hence, we obtain \( t - 1 \leq |w|_{x_a} \leq t + 1 \).

Next, we assume \( l = |w| = 2t + 1 \). If \( |\overline{w}| = 2t \), then \( t \leq |w|_{x_a} \leq t + 1 \). We note that \( |\overline{w}| = 2(t-1) \) does not hold in this case. Because, if \( |\overline{w}| = 2(t-1) \), then we obtain \( |w_1| + |w_2| = 3 \). Hence \( |w_1| = 1 \) and \( |w_2| = 2 \), or \( |w_1| = 2 \) and \( |w_2| = 1 \). This is contrary to our assumption that \( w_1 \) and \( w_2 \) are proper subwords of \( f_n(p_{r-1}) \) and \( f_n(p_{r+h+1}) \), respectively.

Therefore, if \( n = 2 \), then \( S \) is three-distance.

**Case 2:** When \( n = 3 \), an arbitrary \( l \) is written as \( l = 3t \) or \( l = 3t + 1 \) or \( l = 3t + 2 \).

First, we assume \( l = |w| = 3t \). If \( |\overline{w}| = 3t \), then \( |w|_{x_a} = |\overline{w}|_{x_a} = t \). If \( |\overline{w}| = 3(t-1) \), then \( t - 1 \leq |w|_{x_a} \leq t + 1 \). Hence, we obtain \( t - 1 \leq |w|_{x_a} \leq t + 1 \).

Next, we assume \( l = |w| = 3t + 1 \). If \( |\overline{w}| = 3t \), then \( t \leq |w|_{x_a} \leq t + 1 \). If \( |\overline{w}| = 3(t-1) \), then \( t - 1 \leq |w|_{x_a} \leq t + 1 \). Hence, we have \( t - 1 \leq |w|_{x_a} \leq t + 1 \).

Assume \( l = |w| = 3t + 2 \). If \( |\overline{w}| = 3t \), then \( t \leq |w|_{x_a} \leq t + 2 \). We note that \( |\overline{w}| = 3(t-1) \) does not hold in this case. Because, if \( |\overline{w}| = 3(t-1) \), then we obtain \( |w_1| + |w_2| = 5 \). Hence \( |w_1| = 1 \) and \( |w_2| = 4 \), or \( |w_1| = 4 \) and \( |w_2| = 1 \), or \( |w_1| = 2 \) and \( |w_2| = 3 \), or \( |w_1| = 3 \) and \( |w_2| = 2 \). This is contrary to our assumption that \( w_1 \) and \( w_2 \) are proper subwords of \( f_n(p_{r-1}) \) and \( f_n(p_{r+h+1}) \), respectively.
Therefore, if $n = 3$, then $S$ is three-distance. This completes the proof.

**Example 5.1.** When $n = 3$, $\#(S_3) = 6$. We put $\{x_1, x_2, x_3\} = \{A, B, C\}$. Let $f_3 : \{1, 2, \ldots, 6\} \rightarrow S_3$ be a bijection given by:

$1 \mapsto ABC, \quad 2 \mapsto ACB, \quad 3 \mapsto BAC, \quad 4 \mapsto BCA, \quad 5 \mapsto CAB, \quad 6 \mapsto CBA.$

By Proposition 5.1, an infinite sequence

$$R_3 = \cdots 52435364564311432253522451353624626625316243341334622466243235$$

$$543456625426166216231525522166544 \cdots,$$

produces a three-distance sequence $S \in \Sigma_{f_3}$,

$$S = \cdots CABACBBBCABACCABBCABCABCBABCABACABCA$$

$$BCBCA \cdots.$$

However,

$$D_x(S) = \cdots CBCBCBCCCBBCCBB \cdots$$

and

$$D_y(S) = \cdots CAACCAACCAACCAAC \cdots,$$

$$D_z(S) = \cdots ABABBABAABBABABAABBABAB \cdots$$

are not two-distances with two symbols BC, CA, AB respectively. Namely, there does not exist a line in $\mathbb{R}^2$ which has $D_u(S)$ as its 2D cutting sequence. Therefore $S$ is a three-distance sequence which is not a 3D cutting sequence. From the above construction, it is easy to see that there are infinitely many such sequences.

The set of the elements of $\Sigma_{f_3}$ which are not 3D cutting sequences is denoted by $\Sigma_{f_3}^\ast$.

**Corollary 5.2.** $\text{card } \Sigma_{f_3}^\ast = \text{card } \Sigma_{f_3} = \text{card } \mathbb{R}$.

**Proof.** The set of bi-infinite sequences with symbols 1, 2, \ldots, 6 is denoted by $\mathcal{R}_3$. For a sequence $R_3 = \cdots r_{-1}r_0r_1r_2 \cdots \in \mathcal{R}_3$ with $r_v \in \{1, 2, \ldots, 6\}$ ($v \in \mathbb{Z}$), we define the infinite sequence $R_3^\ast = \cdots r_{-1}135r_0r_1r_2 \cdots$. We put

$$\mathcal{R}_3^\ast = \{R_3^\ast : R_3 \in \mathcal{R}_3\}.$$
Then we have \( \text{card } R_3^+ = \text{card } R_3 = \text{card } \mathbb{R} \). Note that

\[
D_z \circ f_3(135) = D_z(f_3(1)f_3(3)f_3(5)) = D_z(\text{ABCBACCAB}) = \text{ABBAAB}.
\]

Hence, for any element \( R_3^+ \) of \( R_3^+ \), \( D_z \circ f_3(R_3^+) \) is not two-distance with two symbols \( A, B \). Thus \( D_z \circ f_3(R_3^+) \) cannot be a 2D cutting sequence. From Remark 2.1, we see \( f_3(R_3^+) \in \Sigma_{f_3} \). We put

\[
\Sigma_{f_3}(135) = \{ f_3(R_3^+) : R_3^+ \in R_3^+ \}.
\]

Note that \( \Sigma_{f_3}(135) \subset \Sigma_{f_3} \). Since there exists an injection:

\[
R_3^+ \rightarrow \Sigma_{f_3}(135), \quad R_3^+ \mapsto f_3(R_3^+),
\]

we have \( \text{card } \mathbb{R} \leq \text{card } \Sigma_{f_3}(135) \). Hence \( \text{card } \mathbb{R} \leq \text{card } \Sigma_{f_3} \). Therefore we obtain

\[
\text{card } \mathbb{R} \leq \text{card } \Sigma_{f_3} \leq \text{card } \Sigma \leq \text{card } \mathbb{R},
\]

and

\[
\text{card } \Sigma_{f_3} = \text{card } \Sigma = \text{card } \mathbb{R}.
\]

Q.E.D

References