RUSCHEWEYH DERIVATIVE AND STRONGLY STARLIKE FUNCTIONS

By
Liu JINLIN

Abstract. Let $A$ denote the class of analytic functions $f(z)$ defined in the unit disc satisfying the condition $f(0) = f'(0) - 1 = 0$. Let $S^*(\beta, \gamma)$ be the class of strongly starlike functions of order $\beta$ and type $\gamma$, and let $C(\beta, \gamma)$ denote the class of strongly convex functions of order $\beta$ and type $\gamma$. Certain new classes $S^*_a(\beta, \gamma)$ and $C_\gamma(\beta, \gamma)$ are introduced by virtue of Ruscheweyh derivative and some properties of $S^*_2(\beta, \gamma)$ and $C_\gamma(\beta, \gamma)$ are discussed.

1. Introduction

Let $A$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $E = \{ z : |z| < 1 \}$. A function $f(z)$ belonging to $A$ is said to be starlike of order $\gamma$ if it satisfies

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \gamma \quad (z \in E)$$

for some $\gamma$ ($0 \leq \gamma < 1$). We denote by $S^*(\gamma)$ the subclass of $A$ consisting of functions which are starlike of order $\gamma$ in $E$. Also, a function $f(z)$ in $A$ is said to be convex of order $\gamma$ if it satisfies $zf''(z) \in S^*(\gamma)$, or

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \gamma \quad (z \in E)$$

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for some $\gamma$ $(0 \leq \gamma < 1)$. We denote by $C(\gamma)$ the subclass of $A$ consisting of all functions which are convex of order $\gamma$ in $E$.

If $f(z) \in A$ satisfies

\begin{equation}
\left| \arg \left( \frac{zf'(z)}{f(z)} - \gamma \right) \right| < \frac{\pi}{2} \beta \quad (z \in E)
\end{equation}

for some $\gamma$ $(0 \leq \gamma < 1)$ and $\beta$ $(0 < \beta \leq 1)$, then $f(z)$ is said to be strongly starlike of order $\beta$ and type $\gamma$ in $E$, and denoted by $f(z) \in S^*(\beta, \gamma)$. If $f(z) \in A$ satisfies

\begin{equation}
\left| \arg \left( \frac{zf''(z)}{f'(z)} - \gamma \right) \right| < \frac{\pi}{2} \beta \quad (z \in E)
\end{equation}

for some $\gamma$ $(0 \leq \gamma < 1)$ and $\beta$ $(0 < \beta \leq 1)$, then we say that $f(z)$ is strongly convex of order $\beta$ and type $\gamma$ in $E$, and we denote by $C(\beta, \gamma)$ the class of all such functions. It is obvious that $f(z) \in A$ belongs to $C(\beta, \gamma)$ if and only if $zf''(z) \in \overline{S}^*(\beta, \gamma)$. Also, we note that $\overline{S}^*(1, \gamma) = S^*(\gamma)$ and $C(1, \gamma) = C(\gamma)$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$, then the Hadamard product (or convolution product) $(f \ast g)(z)$ of $f(z)$ and $g(z)$ is defined by

\begin{equation}
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.
\end{equation}

By the Hadamard product, we define

\begin{equation}
D^\alpha f(z) = \frac{z}{(1-z)^{1+\alpha}} \ast f(z) \quad (\alpha \geq -1)
\end{equation}

for $f(z) \in A$. $D^\alpha f(z)$ is called the Ruscheweyh derivative and was introduced by Ruscheweyh in [1].

We now introduce the following classes:

\[ S^*_{\alpha}(\beta, \gamma) = \left\{ f(z) \in A : D^\alpha f(z) \in S^*(\beta, \gamma), \alpha \geq -1 \text{ and } \frac{z(D^2 f(z))'}{D^\alpha f(z)} \neq \gamma \text{ for } z \in E \right\} \]

and

\[ \overline{C}_{\alpha}(\beta, \gamma) = \left\{ f(z) \in A : D^\alpha f(z) \in \overline{C}(\beta, \gamma), \alpha \geq -1 \text{ and } 1 + \frac{z(D^2 f(z))'}{(D^\alpha f(z))'} \neq \gamma \text{ for } z \in E \right\} \]

In this note, we shall investigate some properties of $S^*_{\alpha}(\beta, \gamma)$ and $\overline{C}_{\alpha}(\beta, \gamma)$.  

2. Main Results

We shall need the following lemma.

**Lemma.** (see [2] [3]). Let a function \( p(z) = 1 + b_1 z + \cdots \) be analytic in \( E \) and \( p(z) \neq 0 \) (\( z \in E \)). If there exists a point \( z_0 \in E \) such that

\[
|\arg(p(z))| < \frac{\pi}{2} \beta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| = \frac{\pi}{2} \beta \quad (0 < \beta \leq 1),
\]

then we have

\[
\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,
\]

where

\[
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \left( \text{when } \arg(p(z_0)) = \frac{\pi}{2} \beta, \right)
\]

\[
k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \left( \text{when } \arg(p(z_0)) = -\frac{\pi}{2} \beta, \right)
\]

and \( (p(z_0))^{1/\beta} = \pm ia \quad (a > 0) \).

**Theorem 1.** \( S^*_{x+1}(\beta, \gamma) \subset \tilde{S}^*_{x}(\beta, \gamma) \) for \( x \geq -\gamma \) and \( 0 \leq \gamma < 1 \).

**Proof.** Let \( f(z) \in \tilde{S}^*_{x+1}(\beta, \gamma) \). Then we set

\[
(2.1) \quad \frac{z(D^x f(z))'}{D^x f(z)} = \gamma + (1 - \gamma)p(z),
\]

where \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is analytic in \( E \) and \( p(z) \neq 0 \) for all \( z \in E \). According to the well known identity (see [1] [4])

\[
(2.2) \quad z(D^x f(z))' = (x + 1)D^{x+1} f(z) - xD^x f(z),
\]

we have

\[
(2.3) \quad \frac{D^{x+1} f(z)}{D^x f(z)} = \frac{1}{x + 1} \left[ \frac{z(D^x f(z))'}{D^x f(z)} + x \right] = \frac{1}{x + 1} [(1 - \gamma)p(z) + \gamma + x].
\]

Differentiating both sides of (2.3) logarithmically, it follows that
Suppose that there exists a point $z_0 \in E$ such that $|\arg(p(z_0))| < \frac{\pi}{2} \beta$ ($|z| < |z_0|$) and $|\arg(p(z_0))| = \frac{\pi}{2} \beta$.

Then, applying the Lemma, we can write that $z_0 p'(z_0) = ik \beta$ and $(p(z_0))^{1/\beta} = \pm ia \ (a > 0)$.

Therefore, if $\arg(p(z_0)) = \frac{\pi}{2} \beta$, then

$$\frac{z_0 (D^{x+1} f(z_0))'}{D^{x+1} f(z_0)} - \gamma = (1 - \gamma)p(z_0) \left[ 1 + \frac{z_0 p'(z_0)}{(1 - \gamma)p(z_0) + \gamma + \alpha} \right]$$

$$= (1 - \gamma)a^\beta e^{i\beta/2} \left[ 1 + \frac{ik \beta}{(1 - \gamma)a^\beta e^{i\beta/2} + \gamma + \alpha} \right].$$

This implies that

$$\arg\left\{ \frac{z_0 (D^{x+1} f(z_0))'}{D^{x+1} f(z_0)} - \gamma \right\}$$

$$= \frac{\pi}{2} \beta + \arg\left\{ 1 + \frac{ik \beta}{(1 - \gamma)a^\beta e^{i\beta/2} + \gamma + \alpha} \right\}$$

$$= \frac{\pi}{2} \beta + \tan^{-1}$$

$$\times \left\{ \frac{k \beta \left( \gamma + \alpha + (1 - \gamma)a^\beta \cos\left(\frac{\pi}{2} \beta \right) \right)}{(\gamma + \alpha)^2 + 2(\gamma + \alpha)(1 - \gamma)a^\beta \cos((\pi/2) \beta) + (1 - \gamma)^2 a^{2\beta} + k \beta (1 - \gamma)a^\beta \sin((\pi/2) \beta)} \right\}$$

$$\geq \frac{\pi}{2} \beta. \quad \left( \text{where } k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) > 1 \right),$$

which contradicts the hypothesis that $f(z) \in \mathbb{S}_{x+1}(\beta, \gamma)$. 

\[ z(D^{x+1} f(z))' = \frac{z(D^x f(z))'}{D^{x+1} f(z)} + \frac{(1 - \gamma)p'(z)}{(1 - \gamma)p(z) + \gamma + \alpha} \]

$$= (1 - \gamma)p(z) + \gamma + \frac{(1 - \gamma)p'(z)}{(1 - \gamma)p(z) + \gamma + \alpha} \]
Similarly, if \( \arg(p(z_0)) = -(\pi/2)\beta \), then we obtain that
\[
\arg\left\{ \frac{z_0(D^{x+1}f(z_0))'}{D^{x+1}f(z_0)} - \gamma \right\} \leq -\frac{\pi}{2} \beta,
\]
which also contradicts the hypothesis that \( f(z) \in S_{x+1}^*(\beta, \gamma) \).

Thus the function \( p(z) \) has to satisfy \( |\arg(p(z))| < \frac{\pi}{2} \beta \) (\( z \in E \)). This shows that
\[
\left| \arg\left\{ \frac{z(D^2f(z))'}{D^2f(z)} - \gamma \right\} \right| < \frac{\pi}{2} \beta \quad (z \in E),
\]
or \( f(z) \in \bar{S}_x^*(\beta, \gamma) \).

**Theorem 2.** Let \( \alpha \geq -\gamma \) and \( 0 \leq \gamma < 1 \), then \( \bar{C}_{x+1}(\beta, \gamma) \subseteq \bar{C}_x(\beta, \gamma) \).

**Proof.** \( f(z) \in \bar{C}_{x+1}(\beta, \gamma) \iff D^{x+1}f(z) \in \bar{C}(\beta, \gamma) \iff z(D^{x+1}f(z))' \in \bar{S}^*(\beta, \gamma) \)
\[
\iff D^{x+1}(zf'(z)) \in \bar{S}^*(\beta, \gamma) \iff zf'(z) \in \bar{S}_{x+1}^*(\beta, \gamma) \]
\[
\iff zf'(z) \in \bar{S}_x^*(\beta, \gamma) \iff D^x(zf'(z)) \in \bar{S}^*(\beta, \gamma) \]
\[
\iff z(D^2f(z))' \in \bar{S}^*(\beta, \gamma) \iff D^2f(z) \in \bar{C}(\beta, \gamma) \]
\[
\iff f(z) \in \bar{C}_x(\beta, \gamma) \).

For \( c > -1 \), and \( f(z) \in \mathcal{A} \), we define the integral operator \( L_c(f) \) as
\[
L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1}f(t) \, dt.
\]
The operator \( L_c(f) \) when \( c \in \mathbb{N} = \{1, 2, 3, \ldots\} \) was studied by Bernardi [6]. For \( c = 1, L_1(f) \) was investigated by Libera [5].

**Theorem 3.** Let \( c > -\gamma \) and \( 0 \leq \gamma < 1 \). If \( f(z) \in \bar{S}^*_x(\beta, \gamma) \) with \( z(D^2(L_c(f)))'/(D^2(L_c(f))) \neq \gamma \) for all \( z \in E \), then we have \( L_c(f) \in \bar{S}^*_x(\beta, \gamma) \).

**Proof.** Set
\[
\frac{z(D^2(L_c(f)))'}{D^2(L_c(f))} = \gamma + (1 - \gamma)p(z),
\]
where \( p(z) \) is analytic in \( E \), \( p(0) = 1 \) and \( p(z) \neq 0 \) (\( z \in E \)). From (2.5), we have
Using \((2.6)\) and \((2.7)\), we get

\[
(c + 1) \frac{D^2 f}{D^2(L_c(f))} = c + \gamma + (1 - \gamma)p(z).
\]

Differentiating \((2.8)\) logarithmically, we obtain

\[
\frac{z(D^2 f(z))'}{D^2 f(z)} - \gamma = (1 - \gamma)p(z) + \frac{(1 - \gamma)z'p'(z)}{c + \gamma + (1 - \gamma)p(z)}.
\]

Suppose that there exists a point \(z_0 \in E\) such that

\[
|\arg(p(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| = \frac{\pi}{2}\beta.
\]

Then, applying the Lemma, we can write that \(z_0p'(z_0)/p(z_0) = ik\beta\) and

\[
(p(z_0))^{1/\beta} = \pm ia \quad (a > 0).
\]

If \(\arg(p(z_0)) = -(\pi/2)\beta\), then

\[
\frac{z_0(D^2 f(z_0))'}{D^2 f(z_0)} - \gamma = (1 - \gamma)p(z_0) \left[ 1 + \frac{z_0p'(z_0)/p(z_0)}{1 + (1 - \gamma)p(z_0)} \right] + \frac{ik\beta}{c + \gamma + (1 - \gamma)a^\beta e^{-i\pi/2}}.
\]

This shows that

\[
\arg\left\{ \frac{z_0(D^2 f(z_0))'}{D^2 f(z_0)} - \gamma \right\}
= -\frac{\pi}{2}\beta + \arg\left\{ 1 + \frac{ik\beta}{c + \gamma + (1 - \gamma)a^\beta e^{-i\pi/2}} \right\}
= -\frac{\pi}{2}\beta + \tan^{-1}
\times \left\{ \frac{k\beta \left( c + \gamma + (1 - \gamma)a^\beta \cos\left(\frac{\pi}{2}\beta\right) \right)}{(c + \gamma)^2 + 2(c + \gamma)(1 - \gamma)a^\beta \cos((\pi/2)\beta) + (1 - \gamma)^2 a^{2\beta} - k\beta(1 - \gamma)a^\beta \sin((\pi/2)\beta)} \right\}
\leq -\frac{\pi}{2}\beta \quad \text{(where} \quad k \leq -\frac{1}{2}\left( a + \frac{1}{a} \right) < -1),
\]

which contradicts the condition \(f(z) \in S^*_a(\beta, \gamma)\).
Similarly, we can prove the case \( \arg(p(z_0)) = (\pi/2)\beta \). Thus we conclude that the function \( p(z) \) has to satisfy \( |\arg(p(z))| < (\pi/2)\beta \) for all \( z \in E \). This gives that

\[
\left| \arg \left( \frac{z(D^\gamma(L_c(f)))'}{D^\gamma(L_c(f))} - \gamma \right) \right| < \frac{\pi}{2} \beta \quad (z \in E),
\]

or \( L_c(f) \in \overline{S}_\beta^\gamma(\beta, \gamma) \).

**THEOREM 4.** Let \( c > -\gamma \) and \( 0 \leq \gamma < 1 \). If \( f(z) \in \overline{C}_\beta(\beta, \gamma) \) and

\[
1 + z(D^\gamma(L_c(f)))''/D^\gamma(L_c(f)))' \neq \gamma \quad \text{for all} \quad z \in E,
\]

then we have \( L_c(f) \in \overline{C}_\beta(\beta, \gamma) \).

**Proof.** \( f \in \overline{C}_\beta(\beta, \gamma) \Leftrightarrow zf' \in \overline{S}_\beta^\gamma(\beta, \gamma) \Leftrightarrow L_c(zf') \in \overline{S}_\beta^\gamma(\beta, \gamma) \)

\[
\Leftrightarrow z(L_c(f))' \in \overline{S}_\beta^\gamma(\beta, \gamma) \Leftrightarrow L_c(f) \in \overline{C}_\beta(\beta, \gamma).
\]

**References**


Department of Mathematics
Yangzhou University
Yangzhou 225002
People’s Republic of China