SPACELIKE MINIMAL SURFACES IN 4-DIMENSIONAL
LORENTZIAN SPACE FORMS

By

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Abstract. We give a necessary and sufficient condition for the existence of spacelike minimal surfaces in 4-dimensional Lorentzian space forms, which is a generalization of the Ricci condition for minimal surfaces in 3-dimensional Riemannian space forms.

1. Introduction

Let $N^n(c)$ and $N^4_1(c)$ denote the $n$-dimensional simply connected Riemannian space form and Lorentzian space form of constant curvature $c$, respectively. Every minimal surface in $N^3(c)$ may be seen as a minimal surface in $N^4(c)$. We note that $N^3(c)$ is naturally included in $N^4_1(c)$, and every minimal surface in $N^3(c)$ may be seen also as a spacelike minimal surface in $N^4_1(c)$. So minimal surfaces in $N^3(c)$ can be generalized into two ways, that is, minimal surfaces in $N^4(c)$ and spacelike minimal surfaces in $N^4_1(c)$. Then it seems interesting to compare the geometry of (spacelike) minimal surfaces in $N^4(c)$ and $N^4_1(c)$.

Let $M$ be a minimal surface in $N^3(c)$ with induced metric $ds^2$ and Gaussian curvature $K$. Then $M$ satisfies the Ricci condition, that is, the metric $d\tilde{s}^2 = \sqrt{c - K} \, ds^2$ is flat at points where $K < c$. Conversely, let $M$ be a 2-dimensional simply connected Riemannian manifold with metric $ds^2$ and Gaussian curvature $K (< c)$. If $M$ satisfies the Ricci condition, then there exists an isometric minimal immersion of $M$ into $N^3(c)$ (cf. [4]). Hence, the Ricci condition is a necessary and sufficient condition for the existence of minimal surfaces in $N^3(c)$.

In [2, Th. 1], the Ricci condition is generalized for minimal surfaces in $N^4(c)$. In this paper, we give another generalization for spacelike minimal surfaces in $N^4_1(c)$.

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THEOREM. (i) Let $M$ be a spacelike minimal surface in $N^4_1(c)$. We denote by $K$, $K_v$ and $\Delta$ the Gaussian curvature, the normal curvature and the Laplacian of $M$, respectively. Then

\[ \Delta \log \{(c - K)^2 + K_v^2\} = 8K \]

at points where $(c - K)^2 + K_v^2 > 0$, and

\[ \Delta \tan^{-1}\left(\frac{K_v}{c - K}\right) = -2K_v \]

at points where $K \neq c$.

(ii) Conversely, let $M$ be a 2-dimensional simply connected Riemannian manifold with Gaussian curvature $K (\neq c)$ and Laplacian $\Delta$. If $K_v$ is a function on $M$ satisfying (1) and (2), then there exists an isometric minimal immersion of $M$ into $N^4_1(c)$ with normal curvature $K_v$.

REMARK. The condition (1) is equivalent to that the metric

\[ ds^2 = \{(c - K)^2 + K_v^2\}^{1/4} ds^2 \]

is flat at points where $(c - K)^2 + K_v^2 > 0$. Here $ds^2$ is the induced metric on $M$.

Using the divergence theorem for (1) and (2), we get the following corollaries.

COROLLARY 1. Let $M$ be a compact spacelike minimal surface in $N^4_1(c)$ with Gaussian curvature $K$ and normal curvature $K_v$.

(i) If $(c - K)^2 + K_v^2 > 0$ on $M$, then $M$ is of genus 1.

(ii) If $K$ is constant, then $K = c$ or $K = 0$.

COROLLARY 2. Let $M$ be a compact spacelike minimal surface in $N^4_1(c)$ with Gaussian curvature $K$ and normal curvature $K_v$.

(i) If $K \neq c$ on $M$, then $\int_M K_v \, dM = 0$.

(ii) If $K \neq c$ and $K_v$ does not change sign on $M$, then $K_v = 0$.

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2. Preliminaries

In this section, we recall the method of moving frames for spacelike surfaces in $N^4_1(c)$. Unless otherwise stated, we shall use the following convention on the
ranges of indices:

\[ 1 \leq A, B, \cdots \leq 4, \quad 1 \leq i, j, \cdots \leq 2, \quad 3 \leq \alpha, \beta, \cdots \leq 4. \]

Let \( \{ e_A \} \) be an oriented local orthonormal frame field in \( N_f^4(c) \), and \( \{ \omega^A \} \) be the dual coframe. Here the Lorentzian metric of \( N_f^4(c) \) is given by

\[ ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2. \]

We can define the connection forms \( \{ \omega_B^A \} \) by

\[ de_B = \sum_A \omega_B^A e_A. \]

Then

\[ \omega_B^A + \omega_A^B = 0, \quad \omega_4^A = \omega_A^4, \quad \text{where } 1 \leq A, B \leq 3. \]

The structure equations are given by

\[ d\omega^A = -\sum_B \omega_B^A \wedge \omega^B, \]

\[ d\omega_B^A = -\sum_C \omega_C^A \wedge \omega_B^C + \frac{1}{2} \sum_{C,D} R_{BCD}^A \omega_C^D \wedge \omega_D, \]

\[ R_{BCD}^A = c e_B (\delta_C^A \delta_{BD} - \delta_D^A \delta_{BC}), \]

where \( e_B = 1 \) for \( 1 \leq B \leq 3 \) and \( e_4 = -1 \).

Let \( M \) be an oriented spacelike surface in \( N_f^4(c) \), that is, the induced metric on \( M \) is Riemannian. We choose the frame \( \{ e_A \} \) so that \( \{ e_i \} \) are tangent to \( M \). Then \( \omega^i = 0 \) on \( M \). In the following, our argument will be restricted on \( M \). By (4)

\[ 0 = -\sum_i \omega_i^x \wedge \omega^i. \]

So there is a symmetric tensor \( h_{ij}^x \) such that

\[ \omega_i^x = \sum_j h_{ij}^x \omega^j, \]

where \( h_{ij}^x \) are the components of the second fundamental form of \( M \).

The Gaussian curvature \( K \) and the normal curvature \( K_n \) of \( M \) are given by

\[ d\omega_2^1 = K \omega^1 \wedge \omega^2, \quad d\omega_3^1 = K \omega^1 \wedge \omega^2. \]

Then by (3), (5), (6) and (7) we have

\[ K = c + h_{11}^3 h_{22}^3 - (h_{12}^3)^2 - h_{11}^4 h_{22}^4 + (h_{12}^4)^2, \]
The mean curvature vector $H$ of $M$ is given by
\[ H = \frac{1}{2} \sum_{i,x} h^2 e_x. \]

The surface $M$ is called minimal if $H = 0$ on $M$.

3. Proof of Theorem

(i) As $M$ is a spacelike minimal surface in $N^4(c)$, using the notations in Section 2, we may write
\[ \omega_1^3 = so^1 + t\omega^2, \quad \omega_2^3 = t\omega^1 - so^2, \quad \omega_4^1 = u\omega^1 + v\omega^2, \quad \omega_4^2 = v\omega^1 - u\omega^2. \]

By (9) and (10)
\[ K = c - s^2 - t^2 + u^2 + v^2, \quad K_v = -2(sv - tu). \]

Using (4), (5), (6) and (11) we have
\[
\begin{align*}
    d\omega_1^3 &= ds \wedge \omega^1 - so^1 \wedge \omega^2 + dt \wedge \omega^2 - t\omega_1^2 \wedge \omega^1 \\
    &= -\omega_3^1 \wedge \omega_2^2 - \omega_4^1 \wedge \omega_4^4 \\
    &= -(t\omega^1 - so^2) \wedge \omega_1^1 - \omega_4^3 \wedge (u\omega^1 + v\omega^2).
\end{align*}
\]

So, using the notation like
\[
\begin{align*}
    ds &= s_1 \omega^1 + s_2 \omega^2, \\
    dt &= t_1 \omega^1 + t_2 \omega^2, \\
    \omega_1^1 &= (\omega_2^1)\omega^1 + (\omega_2^1)\omega^2 = -\omega_2^2, \\
    \omega_3^4 &= (\omega_4^3)\omega^1 + (\omega_4^3)\omega^2 = \omega_3^4,
\end{align*}
\]
we get
\[ 2s(\omega_2^1) + 2t(\omega_2^1) - v(\omega_4^3) + u(\omega_4^3) = -s_2 + t_1. \]

Similarly, from the exterior derivative of $\omega_2^3$, $\omega_4^4$ and $\omega_2^4$,
\[
\begin{align*}
    2s(\omega_2^1) - 2t(\omega_2^1) - v(\omega_4^3) + u(\omega_4^3) &= s_1 + t_2, \\
    2u(\omega_4^3) + 2v(\omega_4^3) - t(\omega_4^3) + s(\omega_4^3) &= -u_2 + v_1, \\
    2u(\omega_4^3) - 2v(\omega_4^3) - t(\omega_4^3) + s(\omega_4^3) &= u_1 + v_2.
\end{align*}
\]
Therefore we have

\[
\begin{pmatrix}
  s & -t & v & u \\
  t & s & -u & v \\
  u & -v & t & s \\
  v & u & -s & t
\end{pmatrix}
\begin{pmatrix}
  2\omega_2^1 \\
  2(\ast \omega_2^1) \\
  -\omega_4^1 \\
  -\ast \omega_4^1
\end{pmatrix} =
\begin{pmatrix}
  *ds + dt \\
  *dt - ds \\
  *du + dv \\
  *dv - du
\end{pmatrix},
\]

where \( \ast \) denotes the Hodge star operator on \( M \).

Set

\[
A = \begin{pmatrix}
  s & -t & v & u \\
  t & s & -u & v \\
  u & -v & t & s \\
  v & u & -s & t
\end{pmatrix},
\]

and

\[
X = s^2 - t^2 - u^2 + v^2, \quad Y = 2(st - uv),
\]

\[
Z = s^2 + t^2 + u^2 + v^2, \quad W = 2(su + tv).
\]

Let \( A_{ij} \) (\( 1 \leq i, j \leq 4 \)) denote the cofactors of \( A \). Then, noting (12) and (14), we can see that

\[
A_{11} = -A_{44} = s(c - K) - vK_v = sX + tY = sZ - uW,
\]

\[
A_{21} = A_{33} = t(c - K) + uK_v = -tX + sY = tZ - vW,
\]

\[
A_{31} = -A_{23} = -u(c - K) + tK_v = -uX - vY = uZ - sW,
\]

\[
A_{41} = A_{13} = -v(c - K) - sK_v = vX - uY = vZ - tW,
\]

and

\[
\det A = (c - K)^2 + K_v^2 = X^2 + Y^2 = Z^2 - W^2.
\]

By (12)-(19), at points where \( (c - K)^2 + K_v^2 > 0 \),

\[
2\omega_2^1 = \frac{1}{\det A} \left\{ A_{11}(\ast ds + dt) + A_{21}(\ast dt - ds) + A_{31}(\ast du + dv) + A_{41}(\ast dv - du) \right\}
\]

\[
= \frac{*d\{(c - K)^2 + K_v^2\}}{4\{(c - K)^2 + K_v^2\}} + \frac{X\,dY - Y\,dX}{2(X^2 + Y^2)}
\]

\[
= \frac{1}{4} * d\log\{(c - K)^2 + K_v^2\} + \frac{X\,dY - Y\,dX}{2(X^2 + Y^2)}.
\]
Hence, by the exterior derivative of this equation, together with (8), we get the equation (1).

Similarly, by (12)–(19), at points where \( K \neq c \),

\[
-\omega_4^3 = \frac{1}{\det A} \left\{ A_{13}(\ast ds + dt) + A_{23}(\ast dt - ds) + A_{33}(\ast du + dv) + A_{43}(\ast dv - du) \right\}
\]

\[
= \frac{(c - K)(\ast dK_v) - K_v\{\ast d(c - K)\}}{2\{(c - K)^2 + K_v^2\}} + \frac{Z dW - W dZ}{2(Z^2 - W^2)}
\]

\[
= \frac{1}{2} \ast d \tan^{-1}\left( \frac{K_v}{c - K} \right) + \frac{Z dW - W dZ}{2(Z^2 - W^2)}.
\]

Hence, by the exterior derivative of this equation, together with (8), we get the equation (2).

(ii) We may assume that \( M \) is a small neighborhood. Let \( ds^2 \) be the metric on \( M \). As noted in the remark in the introduction, the condition (1) implies that the metric

\[
ds^2 = ((c - K)^2 + K_v^2)^{1/4} ds^2
\]

is flat. So there exists a coordinate system \((x^1, x^2)\) such that

\[
ds^2 = ((c - K)^2 + K_v^2)^{-1/4} \{ (dx^1)^2 + (dx^2)^2 \}.
\]

Set

\[
\omega^i = ((c - K)^2 + K_v^2)^{-1/8} dx^i,
\]

so that \{\( \omega^i \)\} is an orthonormal coframe field with dual frame \{\( e_i \)\}. By

\[
d\omega^1 = -\omega^1_2 \wedge \omega^2, \quad d\omega^2 = -\omega^2_1 \wedge \omega^1,
\]

we can find that the connection form \( \omega^1_2 = -\omega^2_1 \) is given by

\[
\omega^1_2 = -\omega^2_1 = \frac{1}{8} \ast d \log((c - K)^2 + K_v^2).
\]

As \( K \neq c \), we may choose smooth functions \( s \) and \( v \) so that

\[
s^2 - v^2 = c - K, \quad sv = -\frac{1}{2} K_v.
\]

Let \( E \) be a 2-plane bundle over \( M \) with metric \( \langle , \rangle \) and orthonormal sections \{\( e_3 \)\} such that

\[
\langle e_3, e_3 \rangle = 1, \quad \langle e_3, e_4 \rangle = 0, \quad \langle e_4, e_4 \rangle = -1.
\]
Let $h$ be a symmetric section of $\text{Hom}(TM \times TM, E)$ such that
\[
(h^3_{ij}) = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix},
\]
and set
\[
\omega_1^3 = -\omega_1^1 = -\omega_2^1 = -s\omega_2^1, \quad \omega_2^3 = -\omega_2^2 = -s\omega_2^2,
\]
\[
\omega_1^4 = \omega_4^1 = v\omega_1^1, \quad \omega_2^4 = \omega_2^2 = v\omega_2^1.
\]
We define a compatible connection $\nabla$ of $E$ so that
\[
\nabla e_3 = \omega_3^2 e_4, \quad \nabla e_4 = \omega_4^3 e_3,
\]
where
\[
\omega_4^3 = \omega_3^2 = -\frac{1}{2} \ast d \tan^{-1} \left( \frac{K_v}{c - K} \right).
\]

Now, almost reversing the argument in (i) with $t = u = 0$, we can find that $\{\omega_B^4\}$ satisfy the structure equations:
\[
d\omega_2^1 = -\omega_2^1 \wedge \omega_3^1 - \omega_4^1 \wedge \omega_4^2 + c\omega_1^1 \wedge \omega_2^2,
\]
\[
d\omega_1^3 = -\omega_2^3 \wedge \omega_1^3 - \omega_4^3 \wedge \omega_4^4 + c\omega_1^3 \wedge \omega_2^4,
\]
\[
d\omega_1^4 = -\omega_2^4 \wedge \omega_1^3 - \omega_4^4 \wedge \omega_4^1 + c\omega_1^4 \wedge \omega_2^2,
\]
\[
d\omega_3^3 = -\omega_1^3 \wedge \omega_4^3 - \omega_2^3 \wedge \omega_2^4,
\]
which are the integrability conditions. Therefore, by the fundamental theorem, there exists an isometric immersion of $M$ into $N_4^4(c)$, which is minimal and has normal curvature $K_v$.

4. Some Problems

Refering to our results and the case of minimal surfaces in $N_4^4(c)$, it should be natural to consider the following problems (cf. [3], [1], [5], [6] and their references).

**Problem 1.** Classify spacelike minimal surfaces with constant Gaussian curvature in $N_4^4(c)$.

**Problem 2.** Classify spacelike minimal surfaces with constant normal curvature in $N_4^4(c)$.
Problem 3. Classify spacelike minimal surfaces in $N^4_1(c)$ which are locally isometric to minimal surfaces in $N^3(c)$, or spacelike minimal surfaces in $N^3_1(c)$.

Of course, we may consider the higher codimensional problems. Here we should note that a spacelike minimal surface with constant Gaussian curvature $c$ in $N^4_1(c)$ may not be totally geodesic. These problems will be discussed elsewhere.

References


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