SPACES OF UPPER SEMI-CONTINUOUS MULTI-VALUED FUNCTIONS ON SEPARABLE METRIC SPACES

By

Katsuro Sakai and Shigenori Uehara

Abstract. Let \( X = (X, d) \) be a metric space. By \( \operatorname{USCC}(X, I) \), we denote the space of upper semi-continuous multi-valued functions \( \varphi : X \to I = [0, 1] \) such that each \( \varphi(x) \) is a closed interval. Each \( \varphi \in \operatorname{USCC}(X, I) \) can be identified with its graph, which is a closed subset of \( X \times I \). The space \( \operatorname{USCC}(X, I) \) admits the Hausdorff metric induced by the product metric on \( X \times I \). In this paper, by proving the converse of Fedorchuk's result, we show that \( \operatorname{USCC}(X, I) \) is homeomorphic to the Hilbert cube \( Q = [-1, 1]^\omega \) if and only if \( X \) is infinite, locally connected and compact. In case \( X \) is a dense subset of a locally connected metric space \( Y \) such that \( Y \setminus X \) is locally non-separating in \( Y \), \( \operatorname{USCC}(X, I) \) can be regarded as a subspace of \( \operatorname{USCC}(Y, I) \). It is also proved that the pair \( (\operatorname{USCC}(Y, I), \operatorname{USCC}(X, I)) \) is homeomorphic to \( (Q, s) \) if and only if \( X \neq Y \), \( X \) is \( G_\delta \) in \( Y \), and \( Y \) is compact, where \( s = (-1, 1)^\omega \subset Q \).

Introduction

Let \( X = (X, d) \) be a metric space. By \( (2^X)_m \), we denote the hyperspace of non-empty bounded closed subsets of \( X \) with the Hausdorff metric \( d_H \) defined by \( d \) (cf. [Ku, p. 214]). Let \( 2^X \) be the totality of non-empty closed subsets of \( X \). In case \( X \) is unbounded, \( 2^X \neq (2^X)_m \) and \( d_H \) is not a metric on the whole \( 2^X \) (e.g., \( d_H({\{x}\}}, X) = \infty \) for any \( x \in X \)) but \( d_H \) induces a topology on \( 2^X \). This topology depends on the metric \( d \) (cf. [SU_2, §1]).

1991 Mathematics Subject Classification. 54C60, 57N20, 58C06, 58D17.

Key words and phrases. The space of upper semi-continuous multi-valued functions, the hyperspace of non-empty closed sets, the Hausdorff metric, the Hilbert cube, the pseudo-interior, locally non-separating, Property \( S \).

Received November 24, 1998

Revised June 29, 1999
We endow the product space $X \times \mathbb{R}$ with the metric
\[
\rho((x, t), (x', t')) = \max\{d(x, x'), |t - t'|\}.
\]

Let $\varphi : X \to \mathbb{R}$ be a multi-valued function such that each $\varphi(x)$ is compact. Then, $\varphi$ is upper semi-continuous (u.s.c.) if and only if the graph of $\varphi$ is closed in $X \times \mathbb{R}$, whence we can regard $\varphi \in 2^{X \times \mathbb{R}}$. By $\text{USC}_B(X)$, we denote the space of bounded u.s.c. multi-valued functions $\varphi : X \to \mathbb{R}$ such that each $\varphi(x)$ is non-empty and compact, where $\varphi : X \to \mathbb{R}$ is bounded means that the image $\varphi(x) = \bigcup_{x \in X} \varphi(x)$ is bounded. The space $\text{USC}_B(X)$ is now regarded as a subspace of $2^{X \times \mathbb{R}}$. One should note that $\text{USC}_B(X) \neq (2^{X \times \mathbb{R}})_m$ in general, but $\rho_H(\varphi, \psi) < \infty$ can be defined for each $\varphi, \psi \in \text{USC}_B(X)$ because $\varphi$ and $\psi$ are bounded. Let $\text{USC}(X, I)$ be the subspace of $\text{USC}_B(X)$ consisting of all $\varphi \in \text{USC}_B(X)$ with the image $\varphi(X) \subseteq I$. By $\text{USCC}_B(X)$, we denote the subspace of $\text{USC}_B(X)$ consisting of all $\varphi \in \text{USCC}_B(X)$ such that each $\varphi(x)$ is connected (i.e., a closed interval). Let $\text{USCC}(X, I) = \text{USCC}_B(X) \cap \text{USC}(X, I)$.

In case $X$ is compact, every u.s.c. multi-valued function $\varphi : X \to \mathbb{R}$ is bounded, so we denote $\text{USC}_B(X) = \text{USC}(X)$ and $\text{USCC}_B(X) = \text{USCC}(X)$. In this case, every admissible metric for $X$ induces the same topology for $\text{USC}_B(X)$, that is, the topology for $\text{USC}_B(X)$ does not depend on the metric $d$. In case $X$ is non-compact, it depends on the metric $d$ (see the end of Introduction).

Fedorchuk [Fe1, 2] proved that if $X$ is an infinite locally connected compact metric space then $\text{USC}(X, I)$ is homeomorphic to $(\approx)$ the Hilbert cube $Q = [-1, 1]^\omega$ and $\text{USCC}(X) \approx Q \setminus \{0\} \approx Q \times [0, 1]$ (cf. [SU1, Appendix]). In this paper, by showing the converse of this result, we have the following:

**Theorem 1.** For a metric space $X$, the following are equivalent:
(a) $\text{USCC}(X, I) \approx Q$;
(b) $\text{USCC}_B(X) \approx Q \setminus \{0\} \approx Q \times [0, 1]$;
(c) $X$ is infinite, locally connected and compact.

In case $X$ is a dense subset of a metric space $Y$, we have the natural isometric embedding $e_Y : \text{USC}_B(X) \to \text{USC}_B(Y)$ defined by $e_Y(\varphi) = \text{cl}_{X \times \mathbb{R}} \varphi$. Then $e_Y(\text{USC}(X, I)) \subset \text{USC}(Y, I)$. But, in general,

$e_Y(\text{USC}_B(X)) \neq \text{USC}_B(Y)$ nor $e_Y(\text{USCC}(X, I)) \neq \text{USCC}(Y, I)$.

For example, let $Y = S^1$ be the unit circle of Euclidean plane $\mathbb{R}^2$ with the usual metric, $X = S^1 \setminus \{(1, 0)\}$, and $f : X \to \mathbb{R}$ be the map defined by $f(x, y) = y$ if $x \leq 0$ and $f(x, y) = y/|y|$ if $x > 0$. Then $e_Y(f)(1, 0) = \{-1, 1\}$ is not connected.
In case $Y$ is locally connected, it will be shown that

$$e_Y(\text{USCC}_B(X)) \subset \text{USCC}_B(Y) \quad \text{and/or} \quad e_Y(\text{USCC}(X,I)) \subset \text{USCC}(Y,I)$$

if and only if the complement $Y \setminus X$ is \textit{locally non-separating} in $Y$, that is, $U \cap X \neq \emptyset$ is connected for each non-empty connected open set $U$ in $Y$ (Proposition 2). Let $s = (-1,1)^{\circ}$ be the pseudo-interior of $Q$, which is homeomorphic to the separable Hilbert space $\ell_2$. We generalize Theorem 1 to pairs as follows:

**Theorem 2.** Let $X$ be a dense subset of a locally connected metric space $Y$ with the locally non-separating complement in $Y$. Then the following are equivalent:

(a) $\langle \text{USCC}(Y,I), e_Y(\text{USCC}(X,I)) \rangle \approx (Q,s)$;
(b) $\langle \text{USCC}_B(Y), e_Y(\text{USCC}_B(X)) \rangle \approx (Q \times [0,1), s \times [0,1))$;
(c) $X \neq Y$, $X$ is $G_\delta$ in $Y$ and $Y$ is compact.

In the above, it should be observed that if $Y$ is locally connected and $Y \setminus X$ is locally non-separating in $Y$ then $X$ is dense in $Y$.

A metric space $X = (X,d)$ (or a metric $d$) has \textit{Property S} if $X$ is covered by finitely many connected sets with arbitrarily small diameters. It should be remarked that a metric space with Property S is totally bounded, hence \textit{a complete metric space with Property S is compact}. The subspace of $2^X$ consisting of compacta is denoted by $\exp(X)$. In case $X$ is compact, $\exp(X) = 2^X$. In [Cu], Curtis proved that $X$ admits a Peano compactification $\tilde{X}$ such that $(\exp(\tilde{X}), \exp(X)) \approx (Q,s)$ if and only if $X$ is connected, locally connected, completely metrizable, nowhere locally compact and admits a metric $d$ with Property S. We have the following version of this Curtis' result:

**Theorem 3.** A metrizable space $X$ has a metrizable compactification $\tilde{X}$ such that

$$(\text{USCC}((\tilde{X},I), e_{\tilde{X}}(\text{USCC}(X,I))) \approx (Q,s)$$

if and only if $X$ is completely metrizable, non-compact and admits a metric with Property S.

One should note that some admissible metric $d$ for $X$ cannot be extended to $\tilde{X}$ even if $d$ has Property S. For example, let $X = (0,1)$ and $\tilde{X} = [0,1]$. Then, $X \approx S^1 \setminus \{(1,0)\}$. The metric on $X$ inherited from $S^1$ has Property S but cannot be extended to $\tilde{X}$. The following is a direct consequence of Theorems 2 and 3:
Corollary 1. Let $X$ be completely metrizable, non-compact and admits a metric with Property S. Then $X$ admits a metric which induces the topology on $\text{USCC}_B(X)$ such that $\text{USCC}(X, I) \approx \text{USCC}_B(X) \approx \ell_2$.

In the above, the topology of $\text{USCC}(X, I)$ is not defined by using a complete metric on $X$. In [SU2], it is proved that the spaces $\text{USCC}_B(X)$ and $\text{USCC}(X, I)$ are homeomorphic to a non-separable Hilbert space for a uniformly locally connected, non-compact and complete metric space $X$ (even if $X$ is separable). One should observe that $\text{USCC}_B(\mathbb{R})$ is non-separable but $\text{USCC}_B((0,1))$ is separable, where $\mathbb{R}$ and $(0,1)$ have the usual metrics.

Proofs of Theorems

We start with the following:

Proposition 1. For a locally compact metric space $X$, $\text{USCC}(X, I)$ is closed in $2^{x \times I}$ if and only if $X$ is locally connected.

Proof. The "if" part is Proposition 1.1 in [SU2], where the local compactness of $X$ need not be assumed.

To see the "only if" part, assume that $X$ is not locally connected. Then some $x_0 \in X$ has a compact neighborhood $B_0$ such that any neighborhood of $x_0$ contained in $B_0$ is not connected. Let $\delta = d(x_0, X \setminus B_0) > 0$. Then we have disjoint non-empty closed sets $A_1$ and $B_1$ in $X$ such that $B_0 = A_1 \cup B_1$, $d(x_0, A_1) < 2^{-1}\delta$ and $x_0 \in B_1$. In fact, since $B_0$ is compact, the intersection of clopen sets in $B_0$ containing $x_0$ is the component of $B_0$, which is not a neighborhood of $x_0$. Then we have a clopen set $B_1$ in $B_0$ and $x_1 \in B_0 \setminus B_1$ with $d(x_0, x_1) < 2^{-2}\delta$, whence $A_1 = B_0 \setminus B_1$ and $B_1$ satisfy the condition. Using the same argument inductively, we have disjoint non-empty closed sets $A_n$ and $B_n$ in $X$, $n \in \mathbb{N}$, such that $B_{n-1} = A_n \cup B_n, d(x_0, A_n) < 2^{-n}\delta$ and $x_0 \in B_n$. For each $n \in \mathbb{N}$, let

$$\varphi_n = \bigcup_{i=1}^n A_i \times \{0\} \cup B_n \times \{1\} \cup (X \setminus \text{int}_X B_0) \times I \in \text{USCC}(X, I).$$

Note that $\varphi_n(\text{int}_X B_0) = \{0, 1\}$. Since $2^{B_0 \times I} = \exp(B_0 \times I)$ is compact, $(\varphi_n|B_0)_{n \in \mathbb{N}}$ has a subsequence $(\varphi_{n_i}|B_0)_{i \in \mathbb{N}}$ converging to some $\varphi' \in 2^{B_0 \times I}$. Then $(\varphi_{n_i})_{i \in \mathbb{N}}$ converges to $\varphi = \varphi' \cup (X \setminus \text{int}_X B_0) \times I$ in $2^{x \times I}$. Since $(x_0, 0) \in \varphi_n$ for all $n \in \mathbb{N}$, we have $(x_0, 0) \in \varphi$. For each $n \in \mathbb{N}$, choose $x_n \in A_n$ so that $d(x_n, x_0) < 2^{-n}\delta$.
Spaces of upper semi-continuous multi-valued functions

Since \( \rho((x_0, 1), (x_n, 1)) < 2^{-n}\delta \) and \((x_n, 1) \in \varphi_n\), we have \((x_0, 1) \in \varphi\). However \((x_0, 1/2) \notin \varphi\) because \( \text{int} X B_0 \times (0, 1) \cap \varphi_n = \emptyset \) for any \( n \in \mathbb{N} \). This means that \( \varphi \cap \{x_0\} \times I \) (i.e., \( \varphi(x_0) \)) is not connected, hence \( \varphi \notin \text{USCC}(X, I) \). This is a contradiction. \( \square \)

For a metric space \( X \), there exists the natural closed embedding \( i_X : X \to \text{USCC}(X, I) \) defined as follows:

\[
i_X(x) = X \times \{0\} \cup \{x\} \times I \subset X \times I \quad \text{for each } x \in X,
\]

whence each \( i_X(x) \in \text{USCC}(X, I) \) is defined by

\[
i_X(x)(y) = \begin{cases} \{0\} & \text{if } y \neq x, \\ I & \text{if } y = x. \end{cases}
\]

Observe that \( \rho_H(i_X(x), i_X(x')) = d(x, x') \) if \( d(x, x') < 1 \), hence \( i_X \) is locally isometric. It is easy to see that \( i_X(X) \) is closed in \( \text{USCC}(X, I) \).

**Proof of Theorem 1.** The implications \((c) \Rightarrow (a)\) and \((c) \Rightarrow (b)\) are Fedorchuk's results [Fe1, Fe2] (cf. [SU1, Appendix]).

\((a) \Rightarrow (c)\): By using the embedding \( i_X \) above, \( X \) can be embedded in \( \text{USCC}(X, I) \) as a closed set, hence \( X \) is compact. By Proposition 1, \( X \) is locally connected. If \( X \) is a singleton, the space \( \text{USCC}(X, I) \) is homeomorphic to the hyperspace of subcontinua (i.e., closed subintervals) of \( I \), so \( \text{USCC}(X, I) \approx I^2 \) (cf. [Du, §3]). Hence, if \( X \) is finite then \( \text{USCC}(X, I) \approx I^{2n} \), where \( n \) is the number of points of \( X \). Therefore, \( X \) must be infinite.

\((b) \Rightarrow (c)\): Since \( \text{USCC}_B(X) \) is locally compact, \( \varphi_0 = X \times \{0\} \in \text{USCC}_B(X) \) has a compact neighborhood \( N \) in \( \text{USCC}_B(X) \). Choose \( \delta > 0 \) so that every \( \varphi \in \text{USCC}_B(X) \) with \( \rho_H(\varphi, \varphi_0) < \delta \) belongs to \( N \). Then, \( \text{USCC}(X, [0, \delta]) \subset N \) and \( \text{USCC}(X, [0, \delta]) \) is closed in \( \text{USCC}_B(X) \). Hence, \( \text{USCC}(X, I) \approx \text{USCC}(X, [0, \delta]) \) is compact. As seen in the above, it follows that \( X \) is compact and locally connected. Since

\[
\text{USCC}_B(X) = \text{USCC}(X) \approx \text{USCC}(X, (0, 1)) \subset \text{USCC}(X, I),
\]

\( \text{USCC}(X, I) \) is infinite-dimensional, which implies that \( X \) is infinite. \( \square \)

By \( C_B(X) \), we denote the Banach space of bounded continuous real-valued functions of \( X \) with the sup-norm and let \( C(X, I) = \{ f \in C_B(X) | f(X) \subset I \} \). Although \( C_B(X) \subset \text{USCC}_B(X) \) as sets, the Banach space \( C_B(X) \) is not a subspace of \( \text{USCC}_B(X) \) in case \( X \) is non-compact (cf. [FK, Remark 3.6] and Supplement).
In [SU2, Corollary 1.5], it is also shown that if \( X \) is locally connected and has no isolated points then the closures of \( C(X,I) \) and \( C_B(X) \) in \( 2^{X \times I} \) are USCC\(_X(I) \) and USCC\(_B(X) \), respectively. In case \( X \) is locally compact, the converse also holds by Proposition 1.

**Corollary 2.** For a locally compact metric space \( X \),
\[
\operatorname{cl}_{2^{X \times I}} C(X,I) = \text{USCC}(X,I) \text{ and/or } \operatorname{cl}_{2^{X \times I}} C_B(X) = \text{USCC}_B(X)
\]
if and only if \( X \) is locally connected and has no isolated point.

Next, we show the following:

**Proposition 2.** Let \( X \) be a dense subset of a locally connected metric space \( Y \). Then, the following are equivalent:

(a) \( e_Y(\text{USCC}_B(X,I)) \subset \text{USCC}(Y,I) \);

(b) \( e_Y(\text{USCC}_B(X)) \subset \text{USCC}_B(Y) \);

(c) \( Y \setminus X \) is locally non-separating in \( Y \).

**Proof.** (c) \( \Rightarrow \) (b): Suppose \( e_Y(\text{USCC}_B(X)) \not\subset \text{USCC}_B(Y) \), that is, there exists \( \varphi \in \text{USCC}_B(X) \) such that \( e_Y(\varphi) \not\in \text{USCC}_B(Y) \). Then \( e_Y(\varphi)(y) \) is not connected for some \( y \in Y \setminus X \), whence we have \( t_1 < t < t_2 \) such that \( t_1, t_2 \in e_Y(\varphi)(y) \) but \( t \notin e_Y(\varphi) \). Since \( e_Y(\varphi) \) is closed in \( Y \times I \) and \( Y \) is locally connected, we have a connected open neighborhood \( U \) in \( y \) in \( Y \) and \( \delta > 0 \) such that
\[
U \times (t - \delta, t + \delta) \cap e_Y(\varphi) = \emptyset,
\]
whence \( t \notin \varphi(x) \) for all \( x \in U \cap X \), \( t_1 < t - \delta \) and \( t_2 > t + \delta \). By the definition of \( e_Y(\varphi) \), we have \( x_i \in U \cap X \) and \( s_i \in \varphi(x_i) \), \( i = 1, 2 \), such that \( |s_i - t_i| < \delta \), whence \( t \notin \varphi(x_i) \) and \( s_1 < t < s_2 \). Since \( \varphi(x_i) \) is connected, \( \varphi(x_1) \subset (-\infty, t) \) and \( \varphi(x_2) \subset (t, \infty) \). Since \( \varphi \) is u.s.c.,
\[
U_1 = \{ x \in U \mid \varphi(x) \subset (-\infty, t) \} \text{ and } U_2 = \{ x \in U \mid \varphi(x) \subset (t, \infty) \}
\]
are open in \( U \). It follows that \( U = U_1 \cup U_2 \), \( U_1 \cap U_2 = \emptyset \) and \( x_i \in U_i \cap X \), \( i = 1, 2 \). Hence, \( U \cap X \) is not connected, which means that \( Y \setminus X \) is not locally non-separating in \( Y \).

(b) \( \Rightarrow \) (a): This is observed as follows:
\[
e_Y(\text{USCC}(X,I)) = e_Y(\text{USCC}_B(X)) \cap \text{USC}(Y,I)
\subset \text{USCC}_B(Y) \cap \text{USC}(Y,I) = \text{USCC}(Y,I).
\]
Spaces of upper semi-continuous multi-valued functions

(a) ⇒ (c): First, note that $X$ is dense in $Y$. Otherwise, $e_Y(\varphi)(y) = \emptyset$ for each $\varphi \in \text{USCC}(X, I)$ and $y \in Y \setminus \text{cl} X$. Now, suppose that $Y \setminus X$ is not locally non-separating in $Y$, that is, there exists a connected open set $U$ in $Y$ such that $U \cap X$ is not connected. (Note that $U \cap X \neq \emptyset$ because $X$ is dense in $Y$.) Let $U \cap X = U_1 \cup U_2$, where $U_1$ and $U_2$ are disjoint non-empty open sets in $X$. Note that $\text{cl}_Y U_1 \cup \text{cl}_X U_2 \supset U$. Let

$$\varphi = (X \setminus U) \times I \cup U_1 \times \{0\} \cup U_2 \times \{1\} \in \text{USCC}(X, I).$$

Since $U$ is connected, we have $y \in U \cap \text{cl}_Y U_1 \cap \text{cl}_Y U_2 \subset U \setminus X$ because $X$ is dense in $Y$. It follows that $e_Y(\varphi)(y) = \{0, 1\}$. Thus $e_Y(\varphi) \notin \text{USCC}(Y, I)$, which contradicts to $e_Y(\text{USCC}(X, I)) \subset \text{USCC}(Y, I)$. Therefore, $Y \setminus X$ is locally non-separating in $Y$.

\textbf{Proposition 3.} Let $X$ be a dense subset of a locally connected compact metric space $Y$ with the locally non-separating complement $Y \setminus X$ in $Y$. Then, $e_Y(\text{USCC}_B(X))$ is $G_\delta$ in $\text{USCC}(Y)$ if and only if $X$ is $G_\delta$ in $Y$.

\textbf{Proof.} The "only if" part follows from

$$i_Y(X) = i_Y(Y) \cap e_Y(\text{USCC}_B(X)),$$

where $i_Y : Y \to \text{USCC}(Y, I) \subset \text{USCC}_B(Y)$ is the natural closed embedding.

To see the "if" part, let $X = \bigcap_{n \in \mathbb{N}} U_n$, where each $U_n$ is open in $Y$. For each $m, n \in \mathbb{N}$, let

$$G_{m, n} = \{\varphi \in \text{USCC}_B(Y) | \rho_H(\varphi, e_Y(\varphi|U_n)) < 1/m\}.$$

Since $e_Y(\text{USCC}_B(X)) = \bigcap_{m, n \in \mathbb{N}} G_{m, n}$, it suffices to show that each $G_{m, n}$ is open in $\text{USCC}_B(Y)$, or each $F_{m, n} = \text{USCC}_B(Y) \setminus G_{m, n}$ is closed in $\text{USCC}_B(Y)$.

Assume that a sequence $\varphi_i \in F_{m, n}$, $i \in \mathbb{N}$, converges to $\varphi \in \text{USCC}_B(Y)$. Since $\varphi$ is bounded, $\varphi \subset Y \times [-a, a]$ for some $a > 0$. Then, we may assume that $\varphi_i \subset Y \times [-a, a]$ for all $i \in \mathbb{N}$. Since each $\varphi_i$ is compact, we can choose $(x_i, t_i) \in \varphi_i$ so that

$$\rho((x_i, t_i), e_Y(\varphi_i|U_n)) = \rho_H(\varphi_i, e_Y(\varphi_i|U_n)) \geq 1/m.$$

Since $Y \times [-a, a]$ is compact, we may assume that $(x_i, t_i)$ converges to $(x_0, t_0) \in Y \times [-a, a]$, whence $(x_0, t_0) \in \varphi$. We show that $\rho((x_0, t_0), e_Y(\varphi|U_n)) \geq 1/m$, which means that $\varphi \in F_{m, n}$. Then, $F_{m, n}$ would be closed in $\text{USCC}(Y, [-a, a])$.

Now, assume that $\rho((x_0, t_0), e_Y(\varphi|U_n)) < 1/m$. Then, we have $(y_0, s_0) \in \varphi|U_n$ such that $\rho((x_0, t_0), (y_0, s_0)) < 1/m$. Let

$$\delta = \min\{d(y_0, Y \setminus U_n), \frac{1}{2} (1/m - \rho((x_0, t_0), (y_0, s_0)))\} > 0.$$
Choose \(i\) so that \(\rho_{y_i}((x_i, t_i), (x_0, t_0)) < \delta\) and \(\rho((x_i, t_i), (x_0, t_0)) < \delta\). Then, we have \((y_i, s_i) \in \varphi_i\) such that \(\rho((y_0, s_0), (y_i, s_i)) < \delta\). Since \(d(y_0, y_i) < d(y_0, Y \setminus U_n)\), it follows that \(y_i \in U_n\), hence \((y_i, s_i) \in \varphi_i | U_n\). Therefore,

\[
\rho((x_i, t_i), (y_i, s_i)) \geq \rho((x_i, t_i), e_Y(\varphi_i | U_n)) \geq 1/m.
\]

On the other hand,

\[
\rho((x_i, t_i), (y_i, s_i)) \leq \rho((x_i, t_i), (x_0, t_0)) + \rho((x_0, t_0), (y_0, s_0)) + \rho((y_0, s_0), (y_i, s_i)) < 2\delta + \rho((x_0, t_0), (y_0, s_0)) < 1/m,
\]

which is a contradiction. The proof is completed.

Now, we prove Theorems 2 and 3.

**Proof of Theorem 2.** \((a) \Rightarrow (b)\): As saw in the proof of [Fe2, Proposition 2.4], \(D = \text{USCC}(Y, I) \setminus \text{USCC}(Y, (0, 1))\) is a contractible \(Z\)-set in \(\text{USCC}(Y, I)\) and then

\[
\text{USCC}(Y, (0, 1)) \approx \text{USCC}(Y, I) \setminus D \approx Q \times [0, 1).
\]

It follows from [Ch, Theorem 6.6] that

\[
(\text{USCC}(Y, (0, 1)), e_Y(\text{USCC}(X, I)) \setminus D) \approx (Q \times [0, 1), s \times [0, 1)),
\]

where it should be noted that \(e_Y(\text{USCC}(X, I)) \setminus D \neq e_Y(\text{USCC}(X, (0, 1)))\) but

\[
e_Y(\text{USCC}(X, I)) \setminus D = \{e_Y(\varphi) | \varphi \in \text{USCC}(X, (a, b))\} \text{ for some } 0 < a < b < 1\}.
\]

By Theorem 1, \(Y\) is compact, whence \(\text{USCC}_B(Y) = \text{USCC}(Y)\) and there exists a homeomorphism \(h : \text{USCC}(Y) \rightarrow \text{USCC}(Y, (0, 1))\) such that

\[
h(e_Y(\text{USCC}_B(X))) = \{e_Y(\varphi) | \varphi \in \text{USCC}(X, (a, b))\} \text{ for some } 0 < a < b < 1\}.
\]

Consequently, we have

\[
(\text{USCC}_B(Y), e_Y(\text{USCC}_B(X))) \approx (\text{USCC}(Y, (0, 1)), e_Y(\text{USCC}(X, I)) \setminus D)
\]

\[
\approx (Q \times [0, 1), s \times [0, 1)).
\]

\((b) \Rightarrow (c)\): By Theorem 1, the condition \((b)\) implies that \(X \neq Y\) and \(Y\) is compact and locally connected. Moreover, \(Y \setminus X\) is locally non-separating in \(Y\) by Proposition 2, and \(X\) is \(G_\delta\) in \(Y\) by Proposition 3.

\((c) \Rightarrow (a)\): We first consider the case that \(Y\) is connected, hence it is a Peano continuum. In this case, \(\text{USCC}(Y, I)\) is the closure of \(C(Y, I)\) in \(\exp(Y \times I) =\)
Spaces of upper semi-continuous multi-valued functions

2^{Y \times I} [Fe_2, Theorem 1.10]. Since \( (\text{USCC}(Y, I), C(Y, I)) \approx (Q, s) \) [SU_1, Corollary 1'], the complement \( \text{USCC}(Y, I) \setminus C(Y, I) \) is a \( Z_\sigma \)-set in \( \text{USCC}(Y, I) \). By Proposition 3, \( e_Y(\text{USCC}_B(X)) \) is \( G_\delta \) in \( \text{USCC}_B(Y) \), whence

\[ e_Y(\text{USCC}(X, I)) = e_Y(\text{USCC}_B(X)) \cap \text{USCC}(Y, I) \]

is also \( G_\delta \) in \( \text{USCC}(Y, I) \). Then, the complement

\[ M = \text{USCC}(Y, I) \setminus e_Y(\text{USCC}(X, I)) \]

is \( F_\sigma \) in \( \text{USCC}(Y, I) \) and \( M \subset \text{USCC}(Y, I) \setminus C(Y, I) \), hence \( M \) is a \( Z_\sigma \)-set in \( \text{USCC}(Y, I) \). Let \( (A, B) \) be a pair of compacta in \( \text{USCC}(Y, I) \) such that \( B \subset M \) and \( \varepsilon > 0 \). By all the same way as the proof of Main Theorem of [SU_1], but using a point \( x_0 \in Y \setminus X \), we can define an embedding \( h : A \to M \) such that \( h|B = \text{id} \) and \( h \) is \( \varepsilon \)-close to \( \text{id} \). Applying the characterization of \( B(Q) = Q \setminus s \) [An] (cf. [Ch, Lemma 8.1]), we have \( (\text{USCC}(Y, I), M) \approx (Q, B(Q)) \), hence

\[ (\text{USCC}(Y, I), e_Y(\text{USCC}(X, I))) \approx (Q, s). \]

In the general case, we write \( Y = \bigcup_{i=1}^n Y_i \), where each \( Y_i \) is a component of \( Y \), which is closed and open in \( Y \) because of locally connectedness of \( Y \). Since \( Y \setminus X \) is locally non-separating in \( Y \), each \( X_i = X \cap Y_i \) is a component of \( X \). Then

\[ (\text{USCC}(Y, I), e_Y(\text{USCC}(X, I))) \approx \left( \prod_{i=1}^n \text{USCC}(Y_i, I), \prod_{i=1}^n e_{Y_i}(\text{USCC}(X_i, I)) \right). \]

In case \( Y_i \) is a singleton, \( X_i = Y_i \) and \( \text{USCC}(Y_i, I) \) is homeomorphic to the hyperspace of subcontinua of \( I \), hence \( \text{USCC}(Y_i, I) \approx \mathbb{I}^2 \) (cf. [Du, §3]). Hence the general case can be obtained the connected case.

**Proof of Theorem 3.** First, assume that \( X \) is completely metrizable and has an admissible metric with Property \( S \). Then, \( X \) has only finitely many components, which are closed and open in \( X \). Replacing the metric, we may assume that the distance between any two components of \( X \) is positive. Thus, as in the proof of Theorem 2, it suffices to treat the case \( X \) is connected. In this case, \( X \) has a Peano compactification \( \hat{X} \) with a locally non-separating remainder \( \hat{X} \setminus X \) by [Cu, Proposition 2.4]. By complete metrizability, \( X \) is \( G_\delta \) in \( \hat{X} \). Then, the "if" part follows from Theorem 2.

Conversely, assume that \( X \) has a compactification \( \hat{X} \) such that

\[ (\text{USCC}(\hat{X}, I), e_{\hat{X}}(\text{USCC}(X, I))) \approx (Q, s). \]

By Theorem 2, \( X \neq \hat{X} \), \( X \) is \( G_\delta \) in \( \hat{X} \), \( \hat{X} \) is locally connected and the remainder \( \hat{X} \setminus X \) is locally non-separating in \( \hat{X} \). Then \( X \) is completely metrizable and, as is
easily observed, each component of $\tilde{X}$ is a Peano compactification of a component of $X$ with locally non-separating remainder. By [Cu, Proposition 2.4], $X$ admits an admissible metric $d$ with Property S. Thus we have the “only if” part.

Supplement

As mentioned before Corollary 2, the Banach space $C_0(X)$ is not a subspace of USCC$C_0(X)$ in case $X$ is non-compact (cf. [FK, Remark 3.6]). Here we show the following:

**Proposition 4.** In the following cases, the topology for $C(X, I)$ induced by the sup-norm is different from the one induced by the Hausdorff metric $\rho_H$:

1. $X$ has a non-complete component;
2. $X$ has a non-totally bounded component;
3. $X$ has infinitely many components $X_i$, $i \in \mathbb{N}$, such that $\inf_{i \in \mathbb{N}} \diam X_i > 0$ and $\inf_{i \neq j} \dist(X_i, X_j) > 0$.

**Proof.** (1) Let $X_0$ be a non-complete component of $X$. Then $X_0$ has a non-convergent Cauchy sequence $(x_i)_{i \in \mathbb{N}}$. For each $n \in \mathbb{N}$, we have $m > n$ such that $d(x_i, x_j) < (1/3)d(x_n, x_m)$ for all $i, j \geq m$. In fact, $x_n$ is not an accumulation point of $(x_i)_{i \in \mathbb{N}}$, whence there is some $\delta > 0$ such that $d(x_n, x_i) > \delta$ for almost all $i \in \mathbb{N}$. Since $(x_i)_{i \in \mathbb{N}}$ is a Cauchy sequence, we can choose $m > n$ such that $d(x_n, x_m) > \delta$ and $d(x_i, x_j) < (1/3)\delta$ if $i, j \geq m$, whence $d(x_i, x_j) < (1/3)d(x_n, x_m)$ for all $i, j \geq m$. Therefore, by taking a subsequence, we can assume that $d(x_{i_n}, x_{j_n}) < (1/3)d(x_n, x_{n+1})$ for every $n \in \mathbb{N}$ and $i_n, j_n > n$. For each $n \in \mathbb{N}$, let $\varepsilon_n = (1/3)d(x_n, x_{n+1})$. Then, the collection $\{B(x_n, \varepsilon_n) : n \in \mathbb{N}\}$ is discrete in $X$ and

$$(*) \quad \bigcup_{i > n} B(x_i, \varepsilon_i) \subseteq B(x_{n+1}, 2\varepsilon_n) \subseteq X \setminus \bigcup_{j \leq n} B(x_j, \varepsilon_j).$$

Moreover, since $X_0$ is connected, it follows that

$$(\dagger) \quad [0, \varepsilon_n] \subseteq [0, 2\varepsilon_1] \subseteq \{d(x_n, y) : y \in X_0\} \quad \text{for every } n \in \mathbb{N}.$$

We define a map $f \in C(X, I)$ as follows:

$$f(x) = \begin{cases} 1 - \varepsilon_i^{-1}d(x, x_i) & \text{if } x \in B(x_i, \varepsilon_i), \ i \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

One should note that any map $g \in C(X, I)$ with $\sup_{x \in X} |f(x) - g(x)| = \gamma < 1/2$ is not uniformly continuous. In fact, by $(\dagger)$, we have $y_i \in X_0$, $i \in \mathbb{N}$,
Spaces of upper semi-continuous multi-valued functions

such that \( d(x_i, y_j) = \varepsilon_i \), whence \( \lim_{i \to \infty} d(x_i, y_j) = 0 \) but

\[
|g(x_i) - g(y_i)| \geq |f(x_i) - f(y_j)| - |f(x_i) - g(x_i)| - |f(y_j) - g(y_i)|
\]

\[
\geq 1 - \gamma - \gamma = 1 - 2\gamma > 0.
\]

However, for each \( \varepsilon > 0 \), there exists a uniformly continuous map \( h \in C(X, I) \) with \( \rho_H(f, h) < \varepsilon \). In fact, choose \( n \in \mathbb{N} \) so that \( 2\varepsilon_n < \varepsilon \), and define a map \( h \in C(X, I) \) as follows:

\[
h(x) = \begin{cases} 
1 - 2^{-1} \varepsilon_n^{-1} d(x, x_{n+1}) & \text{if } x \in B(x_{n+1}, 2\varepsilon_n), \\
\phantom{1 - 2^{-1} \varepsilon_n^{-1} d(x, x_{n+1})} f(x) & \text{otherwise.}
\end{cases}
\]

It follows from (\ref{eq1}) that \( f(\text{cl} B(x_i, \varepsilon_i)) = h(\text{cl} B(x_{n+1}, 2\varepsilon_n)) = I \) for every \( i > n \). Then, by (\ref{eq1}), it can be easily seen that \( \rho_H(f, h) < 2\varepsilon_n < \varepsilon \).

(2) Let \( X_0 \) be a non-totally bounded component of \( X \). Then, we have \( \delta > 0 \) and \( x_i \in X_0 \), \( i \in \mathbb{N} \), such that \( d(x_i, x_j) > \delta \) if \( i \neq j \). Observe that

(\ref{eq2}) \[ [0, \delta] \subset \{ d(x_i, y) \mid y \in X_0 \} \text{ for every } i \in \mathbb{N}. \]

For each \( i \in \mathbb{N} \), let \( \delta_i = \min \{ i^{-1}, 1/3 \delta \} > 0 \). Now, we define a map \( f \in C(X, I) \) as follows:

\[
f(x) = \begin{cases} 
1 - \delta_i^{-1} d(x, x_i) & \text{if } x \in B(x_i, \delta_i), \ i \in \mathbb{N}, \\
\phantom{1 - \delta_i^{-1} d(x, x_i)} f(x) & \text{otherwise.}
\end{cases}
\]

By the same reason as the case (1), any map \( g \in C(X, I) \) with \( \sup_{x \in X} |f(x) - g(x)| < 1/2 \) is not uniformly continuous. However, for each \( \varepsilon > 0 \), choose \( n \in \mathbb{N} \) so that \( n^{-1} < \varepsilon \), and define a uniformly continuous map \( h \in C(X, I) \) defined by

\[
h(x) = \begin{cases} 
1 - \min \{ \varepsilon, \delta \}^{-1} d(x, x_i) & \text{if } x \in B(x_i, \min \{ \varepsilon, \delta \}), \ i \geq n, \\
\phantom{1 - \min \{ \varepsilon, \delta \}^{-1} d(x, x_i)} f(x) & \text{otherwise.}
\end{cases}
\]

From (\ref{eq2}), it follows that

\[
f(\text{cl} B(x_i, \delta_i)) = h(\text{cl} B(x_{n+1}, \min \{ \varepsilon, \delta \})) = I \text{ for every } i \geq n,
\]

Then, we have \( \rho_H(f, h) < \varepsilon \).

(3) For each \( i \in \mathbb{N} \), take \( x_i \in X_i \). Choose \( 2\delta > 0 \) so that \( \delta < \inf_{i \in \mathbb{N}} \text{diam } X_i \) and \( \delta < \inf_{i \neq j} \text{dist}(X_i, X_j) \). Since \( \sup_{x \in X_i} d(x, x_i) > \delta \), it follows that

(\ref{eq3}) \[ [0, \delta] \subset \{ d(x_i, y) \mid y \in X_i \} \text{ for every } i \in \mathbb{N}. \]

Then, by replacing \( X_0 \) by \( X_i \)'s in the proof of the case (2), we have the proof of this case.

\( \square \)
Acknowledgments

The authors would like to thank the referee for his helpful comments.

References


Katsuro Sakai: Institute of Mathematics,
University of Tsukuba, Tsukuba, 305-8571
Japan
E-mail address: sakaiktr@sakura.cc.tsukuba.ac.jp

Shigenori Uehara: Takamatsu National College of Technology, Takamatsu, 761-8058
Japan
E-mail address: uehara@takamatsu-nct.ac.jp