Introduction

The notion of separable functor was introduced in [5], where some applications in the framework of group-graded rings where done. This notion fits satisfactorily to the classical notion of separable algebra over a commutative ring. The concept of coseparable coalgebra over a field appears in [1] to prove a result of Sullivan [7]. A more complete study of the separability of coalgebras was performed in [2]. In this last paper, an analysis of the relationship between coseparability and the cohomology theory for coalgebras is developed.

Our aim is to study the separability, in the sense of [5], of some canonical functors stemming from a morphism of coalgebras.

In Section 1 we fix some notation and we prove a preliminary characterization of the bicomodules.

The Section 2 contains the theoretical body of the paper. For a morphism of coalgebras $\phi : C \rightarrow D$, we characterize the separability of the corestriction functor $(-)_{\phi}$ (Theorem 2.4) and of the coinduction functor $(-)^{\phi}$ (Theorem 2.7). The reader can find the definitions of these functors in Section 1. For the particular case of the coalgebra morphism $\varepsilon : C \rightarrow k$ given by the counit of the $k$-coalgebra $C$, the separability of the corestriction functor gives precisely the notion of coseparable coalgebra. We finish the section with Theorem 2.9, that entails that a coseparable coalgebra need not to be necessarily of finite dimension (Theorem 3.4).

Section 3 is devoted to study the relationship between coseparability and cosemisimplicity for coalgebras. As a consequence, we obtain that a $k$-coalgebra
C is coseparable if and only if the coalgebra induced by any field extension of \( k \) is co-semi-simple.

1. Notation and Preliminaries

Let \( k \) be a commutative field. Any tensor product \( \otimes_k \) over \( k \) will be simply denoted by \( \otimes \). The identity map on a set \( X \) will be denoted by \( 1_X \) or even by \( 1 \).

A coalgebra over \( k \) is a \( k \)-vector space \( C \) together with two \( k \)-linear maps \( \Delta_C : C \to C \otimes C \) and \( \varepsilon_C : C \to k \) such that \((1 \otimes \Delta_C) \circ \Delta_C = (\Delta_C \otimes 1) \circ \Delta_C \) and \((\varepsilon_C \otimes 1) \circ \Delta_C = (1 \otimes \varepsilon_C) \circ \Delta_C = 1 \). We shall refer to [8] for details. The dual space \( C^* = \text{Hom}_k(C,k) \) can be canonically endowed with structure of \( k \)-algebra. A right \( C \)-comodule is a \( k \)-vector space \( M \) together with a structure \( k \)-linear map \( \rho_M : M \to M \otimes C \) such that \((1_M \otimes \varepsilon_C) \circ \rho_M = 1_M \) and \((\rho_M \otimes 1_C) \circ \rho_M = (1_M \otimes \Delta_C) \circ \rho_M \). The coalgebra \( C \) can be considered as a right \( C \)-comodule with structure map \( \rho_C = \Delta_C \). A \( k \)-linear map \( f : M \to N \) between right \( C \)-comodules is said to be \( C \)-colinear or a morphism of right \( C \)-comodules if \((f \otimes 1) \circ \rho_M = \rho_N \circ f \). The right \( C \)-comodules together with the \( C \)-colinear maps between them form a Grothendieck category \( M^C \). In fact, \( M^C \) is isomorphic to a closed subcategory of the category \( C^* \text{-} \text{Mod} \) of all left modules over \( C^* \). In particular, the \( C \)-colinear maps between \( C \)-comodules are precisely the \( C^* \)-linear maps between them. For the notion of closed subcategory we shall refer to [3, p. 395]. The notation \( \text{Com}_C(M,N) \) stands for the \( k \)-vector space of all the \( C \)-colinear maps between two \( C \)-comodules \( M,N \). The category of left \( C \)-comodules will be denoted by \( ^C M \). We will use Sweedler's \( \Sigma \)-notation. For example, if \( M \) is a right \( C \)-comodule, then \( \rho_M(m) = \sum_m m_0 \otimes m_1 \in M \otimes C \) for \( m \in M \). The structure of left \( C^* \)-module is given by \( fm = \sum_m m_0 f(m_1) \), for \( f \in C^* \).

It is not difficult to see that if \( W \) is a \( k \)-vector space and \( X \) is a right \( C \)-comodule, then \( W \otimes X \) is a right \( C \)-comodule with structure map \( 1_W \otimes \rho_X : W \otimes X \to W \otimes X \otimes C \). Moreover, if \( W \) is a right \( C \)-comodule, then the structure map \( \rho_W : W \to W \otimes C \) becomes \( C \)-colinear. Consider coalgebras \( C \) and \( D \). Following [9], a \( C-D \)-bicomodule \( M \) is a left \( C \)-comodule and a right \( D \)-comodule such that the \( C \)-comodule structure map \( \rho_M^C : M \to C \otimes M \) is \( D \)-colinear or, equivalently, that the \( D \)-comodule structure map \( \rho_M^D : M \to M \otimes D \) is \( C \)-colinear. Equivalently, if \( \rho_M^D(m) = \sum_{(m)} m_{-1} \otimes m_0 \) and \( \rho_M^C(m) = \sum_{(m)} m_0 \otimes m_1 \), then

\[
\sum_{(m)} m_{-1} \otimes (m_0)_0 \otimes (m_0)_1 = \sum_{(m)} (m_0)_{-1} \otimes (m_0)_0 \otimes m_1.
\]

Furthermore, given a \( k \)-coalgebra \( D \) and a \( k \)-algebra \( R \) we can consider the
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category \( M^D_R \) consisting of the right \( D \)-comodules and right \( R \)-modules \( M \) satisfying the compatibility condition

\[
\sum_{(mr)} (mr)_0 \otimes (mr)_1 = \sum_{(m)} m_0 r \otimes m_1
\]

for every \( m \in M \) and \( r \in R \) or, equivalently, that the homothety \( h_r : M \to M, \ m \mapsto mr \) is a \( D \)-comodule map on \( M \) for every \( r \in R \). The morphisms in this category are the right \( D \)-colinear and right \( R \)-linear maps.

The following characterization of the bicomodules will be useful in this paper.

**Proposition 1.1.** Let \( C, D \) be two coalgebras and consider a \( k \)-vector space \( M \) such that \( M \) is a left \( C \)-comodule and a right \( D \)-comodule. The following statements are equivalent

(i) \( M \) is a \( C \)-\( D \)-bicomodule.

(ii) \( M \) is a \( D^* \)-\( C^* \)-bimodule.

(iii) \( M \in M^D_C \).

(iv) \( M \in C^D \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( f \in C^*, \ g \in D^* \) and \( m \in M \).

\[
(gm)f = \left( \sum_{(m)} m_0 g(m_1) \right) f = \left( \sum_{(m)} m_0 f \right) g(m_1) = \sum_{(m)} f((m_0)_0)(m_0)_0 g(m_1)
\]

Moreover

\[
g(mf) = g \left( \sum_{(m)} f(m_0)(m_0)_0 \right) = \sum_{(m)} f(m_0)(m_0)_0 = \sum_{(m)} f((m_0)_0)(m_0)_0 g((m_0)_0)
\]

Since

\[
\sum_{(m)} m_{-1} \otimes (m_0)_0 \otimes (m_0)_1 = \sum_{(m)} (m_0)_{-1} \otimes (m_0)_0 \otimes m_1
\]

we conclude that \( (gm)f = g(mf) \).

(ii) \( \Rightarrow \) (i) Let \( m \in M \). The \( k \)-subspace \( mC^* \) of \( M \) is finite-dimensional and, thus, \( D^*(mC^*) \) is finite-dimensional. Since \( M \) is a bimodule, \( D^*(mC^*) = (D^*m)C^* \). Let
\{e_1, \ldots, e_n\} be a \(k\)-basis of this vector space. We will prove that
\[
(1 \otimes \rho_M^+) \circ \rho_M^- (e_i) = (\rho_M^- \otimes 1) \circ \rho_M^+ (e_i)
\]  
for every \(i = 1, \ldots, n\). Put
\[
\rho_M^+ (e_i) = \sum_j c_{ij}^+ \otimes e_j \quad \rho_M^- (e_i) = \sum_k c_{ik}^- \otimes e_k
\]
for \(d_i \in D\) and \(c_i \in C\). Choose a \(k\)-basis \(\{c_1, \ldots, c_r\}\) of the \(k\)-vector subspace of \(C\) spanned by the \(c_i^k\)'s, for \(i, k = 1, \ldots, n\). Analogously, let \(\{d_1, \ldots, d_r\}\) be a \(k\)-basis of the \(k\)-vector subspace of \(D\) spanned by the \(d_i^j\)'s. After some computations, we obtain
\[
(\rho_M^- \otimes 1) \circ \rho_M^+ (e_i) = \sum_{h,l} c_h \otimes m_{h,l} \otimes d_l \\
(1 \otimes \rho_M^+) \circ \rho_M^- (e_i) = \sum_{h,l} c_h \otimes m_{h,l}' \otimes d_l
\]
where \(m_{h,l}, m_{h,l}' \in D^* C^*\). Moreover, for \(f \in C^*\) and \(g \in D^*\), we can check that
\[
(g e_i)f = \sum_{h,l} f(c_h) m_{h,l} g(d_l) \\
g(e_i f) = \sum_{h,l} f(c_h) m_{h,l}' g(d_l)
\]
It is evident that certain particular choices of \(f \in C^*\), \(g \in D^*\) give rise to \(m_{h,l} = m_{h,l}'\) for every \(h = 1, \ldots, s\), \(l = 1, \ldots, r\). Thus, the identity (I) holds.

(ii) \(\Rightarrow\) (iii). Let \(f \in C^*\). We have to prove that the homothety \(h_f : M \rightarrow M\) is a morphism of right \(D\)-comodules, i.e., that \(h_f\) is a morphism of left \(D^*\)-modules. But this is true because \(M\) is a \(D^* - C^*\)-bimodule.

(iii) \(\Rightarrow\) (ii) Since \(h_f : M \rightarrow M\) is a \(D\)-comodule map we have
\[
\sum_{(m)} f((m_0)_{-1})(m_0)_0 \otimes m_1 = \sum_{(m)} f(m_{-1})(m_0)_0 \otimes (m_0)_1
\]
Therefore, for every \(g \in D^*\), the equality
\[
\sum_{(m)} f((m_0)_{-1})(m_0)_0 g(m_1) = \sum_{(m)} f(m_{-1})(m_0)_0 g((m_0)_1)
\]
holds, that is, \((gm)f = g(mf)\).

The proof of (ii) \(\iff\) (iv) is similar to that of (ii) \(\iff\) (iii).
We shall recall the concept of cotensor product from [9, 2]. If \( M \) is a right \( C \)-comodule and \( N \) is a left \( C \)-comodule, then the cotensor product \( M \boxtimes_C N \) is the kernel of the \( k \)-linear map

\[
\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \to M \otimes C \otimes N
\]

Let \( C \) and \( D \) be two coalgebras. A morphism of coalgebras is a \( k \)-linear map \( \varphi : C \to D \) such that \( \Delta_D \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_C \). The morphism of coalgebras \( \varphi \) induces a morphism of \( k \)-algebras \( \varphi^* : D^* \to C^* \). Let \( \varphi : C \to D \) be a morphism of \( k \)-coalgebras. Every right \( C \)-comodule \( M \) with structure map \( \rho_M : M \to M \otimes C \) can be considered as a right \( D \)-comodule with structure map

\[
M \xrightarrow{\rho_M} M \otimes C \xrightarrow{1_M \otimes \varphi} M \otimes D
\]

This gives an exact functor \((-)_\varphi : M^C \to M^D \) called co-restriction functor. In particular, \( C \) can be viewed as \( D \)-bicomodule and we can also consider the coinduction functor \((-)^\rho : M^D \to M^C \) where \( N^\rho = N \boxtimes_D C \) for every right \( D \)-comodule \( N \). In fact, \( N \boxtimes_D C \) is a \( D \)-subcomodule of the right \( D \)-comodule \( N \otimes C \) whose structure map is

\[
N \otimes C \xrightarrow{1_N \otimes \Delta_C} N \otimes C \otimes C \xrightarrow{1_N \otimes 1_C \otimes \varphi} N \otimes C \otimes D
\]

It is proved in [4] that if \( M \) is a right \( C \)-comodule then the structure map \( \rho_M \) induces a \( C \)-collinear map \( \overline{\rho}_M : M \to M \rho \boxtimes D C \) such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\rho_M} & M \otimes C \\
& \searrow_{\overline{\rho}_M} & \downarrow_i \\
& & M \rho \boxtimes D C
\end{array}
\]

is commutative. Taking \( M = C \) we obtain a \( C \)-bicomodule map \( \overline{\Delta}_C : C \to C \boxtimes_D C \).

2. Separability of functors over comodules

Definition 2.1. Consider abelian categories \( \mathcal{C} \) and \( \mathcal{D} \). A covariant functor \( F : \mathcal{C} \to \mathcal{D} \) is said to be a separable functor (see [5]) if for all objects \( M, N \in \mathcal{C} \) there are maps

\[
v_{M,N}^F : \text{Hom}_\mathcal{D}(F(M), F(N)) \to \text{Hom}_\mathcal{C}(M, N)
\]

satisfying the following separability conditions.
1. For every \( \alpha \in \text{Hom}_\mathfrak{a}(M, N) \) we have \( v_{M,N}^F(F(\alpha)) = \alpha \).

2. For \( M', N' \in \mathcal{C} \), \( f \in \text{Hom}_\mathfrak{a}(F(M), F(N)) \), \( g \in \text{Hom}_\mathfrak{a}(F(M'), F(N')) \), \( \alpha \in \text{Hom}_\mathfrak{a}(M, M') \) and \( \beta \in \text{Hom}_\mathfrak{a}(N, N') \), such that the following diagram is commutative

\[
\begin{array}{ccc}
F(M) & \xrightarrow{f} & F(N) \\
\downarrow F(\alpha) & & \downarrow F(\beta) \\
F(M') & \xrightarrow{g} & F(N')
\end{array}
\]

then the following diagram is also commutative

\[
\begin{array}{ccc}
M & \xrightarrow{v_{M,N}^F(f)} & N \\
\downarrow \alpha & & \downarrow \beta \\
M' & \xrightarrow{v_{M',N'}^F(g)} & N'
\end{array}
\]

In this section we will characterize the separability of the corestriction and coinduction functors defined by a morphism of coalgebras.

Let \( C, D \) be \( k \)-coalgebras. Let \( F : C \rightarrow D \) be a \( k \)-functor, i.e., the induced map \( \text{Com}_C(M, N) \rightarrow \text{Com}_D(F(M), F(N)) \) is assumed to be a \( k \)-linear map. If \( M \) is a \( C \)-bicomodule, then \( M \in M^C_C \), by Proposition 1.1. For \( z \in F(M) \) and \( f \in C^* \), define \( z \cdot f = F(h_f)(z) \), where \( h_f : M \rightarrow M, m \mapsto h_f(m) = mf \) is a morphism of \( C \)-comodules. This implies that \( F(h_f) \) is a morphism of \( D \)-comodules. Thus, \( F(M) \in M^D_C \).

**Proposition 2.2.** Let \( F : C \rightarrow D \) be a separable \( k \)-functor. Assume that \( M, N \) are \( C \)-bicomodules and let

\[
v_{M,N} : \text{Com}_D(F(M), F(N)) \rightarrow \text{Com}_C(M, N)
\]

be the map given by the separability conditions. If \( \alpha \in \text{Com}_D(F(M), F(N)) \) is also a morphism of right \( C^* \)-modules, then \( v_{M,N}(\alpha) \) is a morphism of \( C \)-bicomodules.

**Proof.** Since \( \alpha \) is \( C^* \)-linear, the following square is commutative

\[
\begin{array}{ccc}
F(M) & \xrightarrow{\alpha} & F(N) \\
\downarrow F(h_f) & & \downarrow F(h_f) \\
F(M) & \xrightarrow{\alpha} & F(N)
\end{array}
\]
The separability of $F$ implies that the following diagram is commutative

$$
\begin{array}{ccc}
M & \xrightarrow{\nu_{M,N}(\alpha)} & N \\
\downarrow{h_f} & & \downarrow{h_f} \\
M & \xrightarrow{\nu_{M,N}(\alpha)} & N
\end{array}
$$

This means that $\nu_{N,M}(\alpha)$ is a morphism of right $C^*$-modules, that is, it is a morphism of left $C$-comodules and, thus, of $C$-bicomodules. \qed

We will denote by $f - M^C$ the full subcategory of $M^C$ consisting of the comodules of finite dimension.

**Proposition 2.3.** Let $F : M^C \to M^D$ be a left exact $k$-functor that commutes with direct limits. Assume that $F(f - M^C) \subseteq f - M^D$. The functor $F$ is separable if and only if its restriction $F' : f - M^C \to f - M^D$ is separable.

**Proof.** It is clear that if $F$ is separable then its restriction $F' : f - M^C \to f - M^D$ is separable. Conversely, assume that this last functor is separable. Take $M, N \in M^C$ and $\alpha \in \text{Com}_D(F(M), F(N))$. Write $M = \bigcup_{i \in I} M_i$, $N = \bigcup_{j \in J} N_j$, as direct unions of finite-dimensional subcomodules. It is clear that $F(M) = \bigcup_{i \in I} F(M_i) = \bigcup_{i \in I} F'(M_i)$ and analogously $F(N) = \bigcup_{j \in J} F(N_j) = \bigcup_{j \in J} F'(N_j)$. For every $i \in I$, there is $j \in J$ such that $\alpha(F(M_i)) \subseteq F(N_j)$. Put $\alpha_i = \alpha_{i,F(M_i)}$. Since $F'$ is separable, there is a map

$$\nu_{M_i,N_j} : \text{Com}_D(F'(M_i), F'(N_j)) \to \text{Com}_C(M_i, N_j)$$

which satisfies the separability conditions. Put $\beta_i = \nu_{M_i,N_j}(\alpha_i) : M_i \to N_j$. Consider $i \leq i'$ with $i, i' \in I$ and let $\iota_{i,i'}$ denote the inclusion $M_i \leq M_{i'}$. There are $j, j' \in J$ with $j \leq j'$ such that $\alpha(F(M_i)) \subseteq F(N_j)$ and $\alpha(F(M_{i'})) \subseteq F(N_{j'})$. In other words, the diagram

$$
\begin{array}{ccc}
F'(M_i) & \xrightarrow{\beta_i} & F'(N_j) \\
\downarrow{F'(\iota_{i,i'})} & & \downarrow{F'(\iota_{j,j'})} \\
F'(M_{i'}) & \xrightarrow{\beta_{i'}} & F'(N_{j'})
\end{array}
$$

is commutative, where $\iota_{j,j'}$ denotes the inclusion $N_j \leq N_{j'}$. Since $F'$ is separable,
we have that the following diagram is commutative

\[ \begin{array}{ccc}
M_i & \xrightarrow{\beta_i} & N_j \\
\downarrow{\iota_{i'}} & & \downarrow{\iota_{j'}} \\
M_i & \xrightarrow{\beta_i} & N_j,
\end{array} \]

Therefore, we can define \( v_{M,N}(x) = \lim \beta_i \). Thus, we have defined a map \( v_{M,N} : \text{Com}_D(F(M), F(N)) \to \text{Com}_C(M, N) \). Now it is a routine matter to check that these maps satisfy the separability conditions, i.e., \( F \) is separable. \( \square \)

Let \( r : X \to Y, s : Y \to X \) be morphisms of bicomodules such that \( r \circ s = 1_Y \). We will say that \( s \) is a \textit{splitting monomorphism} of bicomodules and that \( r \) is a \textit{splitting epimorphism} of bicomodules. The proof of the following Theorem was performed after [5, Proposition 1.3.(1)].

**Theorem 2.4.** Let \( \varphi : C \to D \) be a morphism of coalgebras. The functor \( (-)_{\varphi} : M^C \to M^D \) is separable if and only if the canonical morphism \( \overline{\Delta}_C : C \to C \square_D C \) is a splitting monomorphism of \( C \)-bicomodules.

**Proof.** Assume that \( (-)_{\varphi} \) is separable and consider the map \( p : C \square_D C \to C \) defined as the restriction of the map \( C \otimes C \to C, c_1 \otimes c_2 \mapsto c_1 \varphi(c_2) \). This \( p \) is a morphism of right \( D \)-comodules and of right \( C^* \)-modules. Let \( \phi = v_{C \square_D C, C}(p) \), where the map

\( v_{C \square_D C, C} : \text{Com}_D((C \square_D C)_{\varphi}, C_{\varphi}) \to \text{Com}_C(C \square_D C, C) \)

is given by the separability of \( (-)_{\varphi} \). By Proposition 2.2, \( \phi \) is a morphism of \( C \)-bicomodules. Write \( \overline{\Delta} = \overline{\Delta}_C \). Now, the diagram

\[ \begin{array}{ccc}
(C \square_D C)_{\varphi} & \xrightarrow{p} & C_{\varphi} \\
\overline{\Delta}_{\varphi} \downarrow & & 1 \\
\overline{\Delta} & \xrightarrow{1} & C
\end{array} \]

is commutative. Since \( (-)_{\varphi} \) is separable, the diagram

\[ \begin{array}{ccc}
C \square_D C_{\varphi} & \xrightarrow{\phi} & C \\
\overline{\Delta} \downarrow & & 1 \\
C & \xrightarrow{1} & C
\end{array} \]
is commutative. Thus, $\tilde{\Delta}$ is a splitting monomorphism of $C$-bicomodules. Assume that there is a morphism of $C$-bicomodules $\phi : C \square_D C \rightarrow C$ such that $\phi \circ \tilde{\Delta} = 1_C$. Let $M, N \in M^C$ and $f \in \text{Com}_D(M_{\phi}, N_{\phi})$. Define $\tilde{f}$ by the following commutative diagram of right $C$-comodule maps:

\[
\begin{array}{ccc}
M \square_D C & \xrightarrow{\phi \square 1} & N \square_D C \\
\cong & & \\
M \square_D C & \xrightarrow{\gamma \square 1} & N \square_D C \\
\downarrow & & \downarrow \\
M \square C & \xrightarrow{1 \square D \phi} & N \square C \\
\cong & & \\
M & \xrightarrow{\tilde{f}} & N
\end{array}
\]

where $\gamma$ denotes the isomorphism $N \cong N \square_C C$. Let $u_M : M \rightarrow M \square_D C$, $v_N : N \square_D C \rightarrow N$ be the compositions of the vertical maps on the left and on the right in the diagram, respectively. From the condition $\phi \circ \tilde{\Delta} = 1_C$ it follows easily that $v_M \circ u_M = 1_M$. Given $C$-comodules $M'$, $N'$ and $\alpha \in \text{Com}_C(M, M')$, $\beta \in \text{Com}_C(N, N')$ and $g \in \text{Com}_D(M_{\phi}, N_{\phi})$, consider the diagram

\[
\begin{array}{ccc}
M \square_D C & \xrightarrow{\phi \square 1} & N \square_D C \\
\uparrow & & \downarrow \\
M & \xrightarrow{\tilde{f}} & N \\
\downarrow & & \downarrow \\
M' \square_D C & \xrightarrow{g \square 1} & N' \square_D C
\end{array}
\]

Now, it is easy to see that if the outer square is commutative then the inner square is commutative. Moreover, if $f$ is a morphism of right $C$-comodules, then
\( f = f \). Therefore, if we define
\[
v_{M,N} : \text{Com}_D(M_\varphi, N_\varphi) \to \text{Com}_C(M, N)
\]
by \( v_{M,N}(f) = \tilde{f} \), then the functor \((-)_\varphi\) is separable. \( \square \)

Following [4], we will say that a morphism of coalgebras \( \varphi \) is a monomorphism of coalgebras provided that \( \varphi \circ u = \varphi \circ v \), for \( u, v \) morphisms of coalgebras, it follows that \( u = v \). Although every injective morphism of coalgebras is a monomorphism of coalgebras, both notions are not equivalent.

**Corollary 2.5.** If \( \varphi : C \to D \) is a monomorphism of coalgebras, then the functor \((-)_\varphi : M^C \to M^D\) is separable.

**Proof.** By [4, Theorem 3.5], the map \( \overline{\Delta_C} : C \to C \square_D C \) is an isomorphism. By Theorem 2.4, \((-)_\varphi\) is a separable functor. \( \square \)

**Corollary 2.6.** Let \( \varphi : C \to D \) be a morphism of coalgebras. If the functor \((-)_\varphi : M^C \to M^D\) is separable and \( D \) is a co-semi-simple coalgebra, then \( C \) is a co-semi-simple coalgebra.

**Proof.** Let \( M \) be any right \( C \)-comodule. Since \( D \) is co-semi-simple, \( M_\varphi \) is completely reducible. By [5, Proposition 1.2.(2)], \( M \) is completely reducible and, thus, \( C \) is co-semi-simple. \( \square \)

The proof of the following Theorem was performed after [5, Proposition 1.3.(2)].

**Theorem 2.7.** Let \( \varphi : C \to D \) be a morphism of \( k \)-coalgebras. The functor \((-)_\varphi = - \square_D C : M^D \to M^C\) is separable if and only if \( \varphi \) is a splitting epimorphism of \( D \)-bicomodules.

**Proof.** Assume that \(- \square_D C\) is separable. For \( M, N \in M^D \), there exists the map
\[
v_{M,N} : \text{Com}_C(M \square_D C, N \square_D C) \to \text{Com}_D(M, N)
\]
satisfying the separability conditions. Taking \( M = D \) and \( N = C \), we have \( v_{D,C} : \text{Com}_C(D \square_D C, C \square_D C) \to \text{Com}_D(D, C) \). Now, consider the canonical \( C \)-bicolinear map \( \Delta : C \to C \square_D C \) and define \( \tilde{\Delta}' : D \square_D C \to C \square_D C \) as \( \tilde{\Delta}' = \tilde{\delta} \circ (\delta_C)^{-1} \), where
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\[ \delta_C : C \to D \square D C \] is the canonical isomorphism. Put \( \psi = v_{D,C}(\tilde{\alpha}') \). Since \( \tilde{\alpha}' \) is a morphism of right \( D \)-comodules and of right \( D^* \)-comodules, we can apply Proposition 2.2 to obtain that \( \psi \) is a morphism of \( D \)-bicomodules. On the other hand, the following triangle is commutative

\[
\begin{array}{ccc}
C \square_D C & \xrightarrow{\varphi \square_D 1} & D \square_D C \\
\alpha & & 1 \\
D \square_D C & \xrightarrow{\psi} & \\
\end{array}
\]

By the separability conditions, we deduce that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi} & D \\
\psi & & 1 \\
D & \xrightarrow{} & \\
\end{array}
\]

is commutative. Hence, \( \varphi \) is a splitting epimorphism of \( D \)-bicomodules. Conversely, assume that there is a \( D \)-bilinear map \( \theta : D \to C \) such that \( \varphi \circ \theta = 1_D \). If \( f \in \text{Com}_C(M \square_D C, N \square_D C) \), then we define them map \( v_{M,N} : \text{Com}_C(M \square_D C, N \square_D C) \to \text{Com}_D(M,N) \) by putting \( v_{M,N}(f) = \tilde{f} \), where \( \tilde{f} \) makes the following diagram commutative.

\[
\begin{array}{ccc}
M \square_D C & \xrightarrow{f} & N \square_D C \\
1_M \square_D \theta & & 1_N \square_D \varphi \\
M \square_D D & \xrightarrow{\cong} & N \square_D D \\
\cong & & \cong \\
M & \xrightarrow{\tilde{f}} & N \\
\end{array}
\]

A verification shows (as in Theorem 2.4) that the functor \(- \square_D C\) is then separable.

\[ \Box \]

**Lemma 2.8.** Let \( A, B \) be subcoalgebras of \( C \). If \( \varphi : C \to D \) is a morphism of coalgebras such that \( \varphi(A) \cap \varphi(B) = 0 \), then \( A \square_D B = 0 \).

**Proof.** We have the following equalizer

\[ A \square_D B \to A \otimes B \xrightarrow{\rho_A \otimes 1_B} A \otimes D \otimes B \xrightarrow{1_A \otimes \rho_B} A \otimes D \otimes B \]
where \( \rho_A = (1_A \otimes \varphi) \circ \Delta_A \) and \( \rho_B = (\varphi \otimes 1_B) \circ \Delta_B \). If \( z \in A \square D_B \), then \((\rho_A \otimes 1_B)(z) = (1_A \otimes \rho_B)(z) \). Since \( \text{Im}(\rho_A \otimes 1_B) \subseteq A \otimes \varphi(A) \otimes B \) and \( \text{Im}(1_A \otimes \rho_B) \subseteq A \otimes \varphi(B) \otimes B \), we have that \((\rho_A \otimes 1_B)(z) = (1_A \otimes \rho_B)(z) \subseteq (A \otimes \varphi(A) \otimes B) \cap (A \otimes \varphi(B) \otimes B) = 0 \). Thus, \((\rho_A \otimes 1_B)(z) = 0 \) and, since \( \rho_A \otimes 1_B \) is a monomorphism, \( z = 0 \).

Let \( \{ C_i : i \in I \} \) be a set of coalgebras with structure maps \( \Delta_i, \varepsilon_i \). The vector space \( \oplus C_i \) can be canonically endowed with structure of coalgebra (see e.g. [8, page 50]). Moreover, from a set of morphisms of coalgebras \( \{ \varphi_i : C_i \rightarrow D_i : i \in I \} \), we obtain the coalgebra morphism \( \oplus \varphi_i : \oplus C_i \rightarrow \oplus D_i \).

**Theorem 2.9.** Let \( \varphi_i : C_i \rightarrow D_i \) be morphisms of \( k \)-coalgebras, \( i \in I \).

1. If the functor \( (\_)_i : M^{C_i} \rightarrow M^{D_i} \) is separable for every \( i \in I \), then the functor \( (\_)_i : M^{\oplus C_i} \rightarrow M^{\oplus D_i} \) is separable.
2. If the functor \( (\_)_i : M^{D_i} \rightarrow M^{C_i} \) is separable for every \( i \in I \), then the functor \( (\_)_i : M^{\oplus D_i} \rightarrow M^{\oplus C_i} \) is separable.

**Proof.** (1) By Theorem 2.4, for every \( i \in I \), there is a morphism of \( C_i \)-bicomodules \( \phi_i : C_i \square D_i C_i \rightarrow C_i \) such that \( \phi \circ \Delta_i = 1_{C_i} \). The map \( \oplus \phi_i : \oplus C_i \square D_i C_i \rightarrow \oplus C_i \) is a morphism of \( \oplus C_i \)-bicomodules and \( \oplus \phi_i \circ \oplus \Delta_i = 1_{\oplus C_i} \). If we prove that there is an isomorphism of \( \oplus C_i \)-bicomodules

\[
(\oplus C_i) \square D_i (\oplus C_i) \cong \oplus (C_i \square D_i C_i),
\]

then we can deduce from Theorem 2.4 that the functor \( (\_)_i \) is separable. The Lemma 2.8 assures that \( C_i \square \oplus D_i C_j = 0 \) if \( i \neq j \). Therefore,

\[
(\oplus C_i) \square D_i (\oplus C_i) \cong \bigoplus_{i,j} (C_i \square \oplus D_i C_j) \cong \bigoplus_{i,j} (C_i \square D_i C_j)
\]

(2) By Theorem 2.7, for every \( i \in I \) there is a morphism of \( D_i \)-bicomodules \( \psi_i : D_i \rightarrow C_i \) such that \( \varphi_i \circ \psi_i = 1_{D_i} \). It is clear that \( \oplus \varphi_i \circ \oplus \psi_i = 1_{\oplus D_i} \). Moreover, it is not difficult to see that \( \oplus \psi_i \) is a morphism of \( \oplus D_i \)-bicomodules. By Theorem 2.7, the functor \( (\_)_i \) is separable.

**Remark 2.10.** Theorem 2.9 can be used to construct coalgebra morphisms between infinite-dimensional coalgebras such that the corestriction and the coinduction functors are separable.
3. Applications

Recall that a $k$-algebra $A$ is said to be separable if the canonical map $A \otimes A \to A$ is a splitting epimorphism of $A$-bimodules. By [5, Proposition 1.3] this is equivalent to say that the restriction functor $A - \text{Mod} \to k - \text{Mod}$ is separable. In this section we investigate the coseparable coalgebras.

A morphism of $k$-coalgebras $\varphi : C \to D$ induces a morphism of $k$-algebras $\varphi^* : D^* \to C^*$. Let us denote by $(-)_{\varphi} : C^* - \text{Mod} \to D^* - \text{Mod}$ the functor restriction of scalars. Recall that if $C$ is a finite-dimensional coalgebra, then there is an isomorphism of categories $M^C \cong C^* - \text{Mod}$.

**Proposition 3.1.** Let $\varphi : C \to D$ be a morphism of coalgebras. Assume that $C$ and $D$ are finite-dimensional. The following statements are equivalent.

(i) The functor $(-)_{\varphi} : M^C \to M^D$ is separable.

(ii) The functor $(-)_{\varphi^*} : C^* - \text{Mod} \to D^* - \text{Mod}$ is separable.

**Proof.** (i) $\Leftrightarrow$ (ii) The functorial diagram

\[
\begin{array}{ccc}
M^C & \xrightarrow{(-)_{\varphi}} & M^D \\
\downarrow & & \downarrow \\
C^* - \text{Mod} & \xrightarrow{(-)_{\varphi^*}} & D^* - \text{Mod}
\end{array}
\]

where the vertical arrows represent canonical isomorphisms of categories, commutes. This entails that $(-)_{\varphi}$ is separable if and only if $(-)_{\varphi^*}$ is separable. \(\square\)

**Proposition 3.2.** Let $\varphi : C \to D$ be a morphism of coalgebras.

1. The functor $(-)_{\varphi} : M^C \to M^D$ is separable if and only if the restriction $(-)_{\varphi} : f - M^C \to f - M^D$ is separable.

2. Let $C' \leq C$, $D' \leq D$ be subcoalgebras such that $\varphi(C') \leq D'$, and let us denote by $\varphi' : C' \to D'$ the induced coalgebra map. If the functor $(-)_{\varphi}$ is separable then the functor $(-)_{\varphi'} : M^{C'} \to M^{D'}$ is separable.

3. If $C$ is cosemisimple then the functor $(-)_{\varphi}$ is separable if and only if for any finite-dimensional subcoalgebras $C' \leq C$ and $D' \leq D$ such that $\varphi(C') \leq D'$, the functor $(-)_{\varphi'}$ is separable.

**Proof.** (1) This follows from Proposition 2.3.
(2) Let \( C' \) be a subcoalgebra of \( C \) and let us denote by \( i : C' \to C \) the inclusion coalgebra map. By [4, Theorem 3.5] the functor \( (-)_i : M^C \to M^C \) is separable. For any subcoalgebra \( D' \) of \( D \) with \( \varphi(C') \subseteq D' \) we can consider the commutative diagram

\[
\begin{array}{ccc}
M^C & \xrightarrow{(-)_i} & M^D \\
\downarrow & & \downarrow \\
M^{C'} & \xrightarrow{(-)_{i'}} & M^{D'}
\end{array}
\]

where \( j : D' \to D \) is the inclusion map. If \( (-)_{i'} \) is separable, then \( (-)_{i'} \circ (-)_i = (-)_i \circ (-)_{i'} \) is separable. By [5, Lemma 1.1.(3)], \( (-)_{i'} \) is a separable functor.

(3) Assume that \( C \) is co-semi-simple. Then \( C = \bigoplus C_i \), where the \( C_i \)'s are simple subcoalgebras. If we put \( D_i = \varphi(C_i) \), and we denote by \( \varphi_i : C_i \to D_i \) the induced coalgebra morphism, we have by hypothesis that \( (-)_{\varphi_i} \) is a separable functor. For every \( i \), there is a \( D_i \)-bicomodule morphism \( \psi_i : C_i \square_D C_i \to C_i \) such that \( \psi_i \circ \Delta_{C_i} = 1_{C_i} \). We have that \( C_i \square_D C_i = C_i \square_D C_i \) and

\[
C \square_D C = \bigoplus_i (C_i \square_D C_i) \oplus \bigoplus_{i \neq j} (C_i \square_D C_j)
\]

Then the maps \( \{\psi_i\} \) give a bicomodule map \( \psi : C \square_D C \to C \) if we put \( \psi = \bigoplus \psi_i \) on \( \bigoplus_i (C_i \square_D C_i) \) and zero on \( \bigoplus_{i \neq j} (C_i \square_D C_j) \). Clearly, \( \psi \circ \Delta_C = 1_C \). By Theorem 2.4, \( (-)_{\varphi} \) is a separable functor.

A \( k \)-coalgebra \( C \) is co-separable (see [2]) if there exists \( k \)-linear map \( \tau : C \otimes C \to k \) such that \( (I \otimes \tau)(\Delta \otimes I) = (\tau \otimes I)(I \otimes \Delta) \) and \( \tau I = \varepsilon \). As it was observed in [2, p. 41], \( C \) is co-separable if and only if there exists a \( C \)-bicomodule map \( \pi : C \otimes C \to C \) such that \( \pi \Delta = I \). Now, the counit \( \varepsilon : C \to K \) is a morphism of coalgebras and, in this case, \( C \square_k C = C \otimes C \). It follows from Theorem 2.4 that \( C \) is co-separable if and only if the corestriction functor \( (-)_k : M^C \to k - \text{Mod} \) is separable. From Corollary 2.6 every co-separable coalgebra is co-semi-simple. This result is also given in [2]. Moreover, from Proposition 3.2.3, if \( C \) is co-semi-simple then \( C \) is co-separable if and only if any finite-dimensional subcoalgebra of \( C \) is co-separable.

If \( C \) is a \( k \)-coalgebra and \( k \subseteq K \) is any field extension, then we can define the \( K \)-coalgebra \( C \otimes_k K = C \otimes_k K \), with comultiplication given by

\[
\Delta_{C \otimes K} = \Delta_C \otimes_k 1_K : C \otimes K \to (C \otimes C) \otimes_k K \cong (C \otimes K) \otimes_k (C \otimes K)
\]
and counit given by

$$\varepsilon_{C \otimes K} = \varepsilon \otimes 1_K : C \otimes K \to k \otimes K \cong K$$

With this notation, we can prove the following result.

**Proposition 3.3.** Let $\varphi : C \to D$ be a morphism of $k$-coalgebras. The following statements are equivalent.

(i) The functor $(-)_\varphi : M^C \to M^D$ is separable.

(ii) For any field extension $k \subseteq K$ the functor $(-)^{\otimes 1_K} : M^{C \otimes K} \to M^{D \otimes K}$ is separable.

**Proof.** (i) $\Rightarrow$ (ii) By Theorem 2.4, $\Delta_C$ is a splitting monomorphism of $C$-bicomodules. It is not difficult to see that $(C \otimes K) \square_D K (C \otimes K) \cong (C \square_D C) \otimes K$. Therefore, $\overline{\Delta_{C \otimes K}} = \overline{\Delta_C} \otimes 1_K$ is a splitting monomorphism of $C \otimes K$-bicomodules.

(ii) $\Rightarrow$ (i) This is clear. $\square$

The following theorem gives a "classical" interpretation of the notion of coseparable coalgebra.

**Theorem 3.4.** A $k$-coalgebra $C$ is coseparable if and only if $C \otimes K$ is cosemi-simple for every field extension $k \subseteq K$.

**Proof.** Assume that $C$ is coseparable. By Proposition 3.3, $C \otimes K$ is a $K$-coalgebra coseparable for every field extension $k \subseteq K$. By Corollary 2.6, $C \otimes K$ is a co-semi-simple coalgebra.

Conversely, assume that $C \otimes K$ is co-semi-simple for every field extension $k \subseteq K$. By Proposition 3.2 we have only to prove that every finite-dimensional subcoalgebra of $C$ is coseparable. Indeed, if $E \subseteq C$ is a finite-dimensional subcoalgebra then $E \otimes K$ is co-semi-simple by Corollary 2.6. Therefore,

$$\text{Hom}_K(E \otimes K, K) = (E \otimes K)^* \cong E^* \otimes K$$

is a semisimple $K$-algebra for every field extension $k \subseteq K$. This entails that $E^*$ is a separable $k$-algebra. By Proposition 3.1, $E$ is a coseparable $k$-coalgebra. $\square$

**Remark 3.5.** If $H$ is a Hopf $k$-algebra then $H$ is coseparable as $k$-coalgebra if and only if $H$ is co-semi-simple. This fact follows from Theorem 3.4 and [1, Theorem 3.3.2].
References


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