AN ULTRAPOWER WHICH DOES NOT PRESERVE THE TRUTH OF A $\Pi_2$ SENTENCE

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Abstract. We construct a 'counterexample' to Łoś' theorem in the ordered Mostowski model for set theory ZFA.

The proof of the fundamental theorem of ultraproducts, as is well known, uses AC (the axiom of choice). Howard [2] showed that it is necessary even if for proving its special case: ultrapowers. In fact, he showed how to construct an ultrapower, which does not preserve some $\Pi_2$ sentence, in a model for BPI (the Boolean Prime Ideal Theorem) $\neg AC$. In this paper, we give another such ultrapower in the ordered Mostowski model for ZFA (Zermelo-Fraenkel set theory with atoms, see Jech [1]).

Let $I$ be a non-empty set, let $U$ be an ultrafilter on $I$ and let $\mathfrak{A}$ be a model for the first order language $\mathcal{L}$. Let $\mathbb{A}$ be the universe set of $\mathfrak{A}$. Consider the equivalence relation $\equiv$ over the set $A^I$ defined by:

$$f \equiv g \iff \{i \in I \mid f(i) = g(i)\} \in U \quad \text{for } f, g \in A^I.$$

If $f \in A^I$, let $[f]$ denote the equivalence class of $f$ ([f] = {g ∈ A^I | f ≡ g}). The ultrapower $\mathfrak{A}^I/U$ is the model for $\mathcal{L}$ described as follows:

(i) The universe of $\mathfrak{A}^I/U$ is $A^I/U = \{[f] \mid f ∈ A^I\}$

(ii) Let $P$ be an $n$-placed predicate symbol of $\mathcal{L}$. The interpretation of $P$ in $\mathfrak{A}^I/U$ is the relation $R$ such that $R([f_1], [f_2], \ldots, [f_n])$ iff

$$\{i \in I \mid \mathfrak{A} \models P(f_1(i), f_2(i), \ldots, f_n(i))\} \in U \quad (f_1, f_2, \ldots, f_n \in A^I).$$

Then Łoś' Theorem reads (see [3]):

For each formula $\phi$ of $\mathcal{L}$, and for each $f_1, f_2, \ldots, f_n \in A^I$

$$\mathfrak{A}^I \models \phi([f_1], [f_2], \ldots, [f_n]) \quad \text{iff} \quad \{i \in I \mid \mathfrak{A} \models \phi(f_1(i), f_2(i), \ldots, f_n(i))\} \in U.$$ 

This theorem is proved by using $AC$. We can prove without $AC$ easily the following:

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PROPOSITION. Let $\sigma$ be a $\Sigma_2$ sentence. If $\sigma$ is true in a model $\mathfrak{A}$, then $\sigma$ is true in every ultrapower of $\mathfrak{A}$. \hfill $\square$

So, the least possible hierarchy of sentences whose truth is not preserved is $\Pi_2$. In fact, Howard [2] showed that

If every ultrapower preserves the truth of every $\Pi_2$ sentence and if BPI holds, then the axiom of choice holds.

In this paper, we give another ultrapower which does not preserve the truth of a $\Pi_2$ sentence in the ordered Mostowski model for $\text{ZFA}$. For the $\text{ZF}$ model which is translated by P. J. Cohen (see Jech [1], 5.5.), we can obtain the same result.

Recall the ordered Mostowski model $M$ for $\text{ZFA}$. Let $N$ be a model for $\text{ZFA} + \text{AC}$ with countable atoms. Since the set of atoms $A$ of $N$ is countable, we can endow dense linear ordering to $A$ by an isomorphism: $\langle Q, < \rangle \to \langle A, <_A \rangle$. Consider the automorphism group $\mathcal{G}$ of $\langle A, <_A \rangle$. Each automorphism $\pi \in \mathcal{G}$ can be extended to an automorphism of $N$ by the recursion: $\pi(0) = 0, \pi(x) = \{\pi(y) \mid y \in x\}$. For $x \in N$,

$x$ is symmetric if there is a finite subset $E$ of $A$ such that

$$\forall \pi \in \mathcal{G}[\forall e \in E(\pi(e) = e) \Rightarrow \pi(x) = x]$$

(such an $E$ is called a support of $x$).

Let $M$ be the class of all the hereditarily symmetric elements of $N$. Then $M$ is a model for $\text{ZFA}$, which contains all the elements of $A \cup \{A\} \cup \{<_A\} \cup \{\langle A, <_A \rangle\} \cup N_0$, where $N_0$ is the class of hereditarily atomless elements of $N$. In $M$, $\langle A, <_A \rangle$ is a dense linearly ordered set without endpoints, and $A$ cannot be well-ordered, a fortiori, $A$ has no countably infinite subset (Jech [1], p. 50 and p. 52).

**Lemma.** (1) In $M$, every subset of $A$ is a finite union of intervals of $A$ of the form $(\leftarrow, a) \cup \{a\} \cup (a, b) \cup (a, \rightarrow)$ where $a, b \in A$, and where $(\leftarrow, a) = \{x \in A \mid x <_A a\}$, similarly for others.

(2) In $M$, only the non-principal ultrafilters on $A$ are

$$\{x \in A \mid \exists a \in A (a, \rightarrow) \subset x\} \text{ and } \{x \in A \mid \exists a \in A (\leftarrow, a) \subset x\}.$$

**Proof.** (1) Trivial. (2) Let $U_0 = \{x \in A \mid \exists a \in A (a, \rightarrow) \subset x\}$ and $U_1 = \{x \in A \mid \exists a \in A (\leftarrow, a) \subset x\}$. First we prove $U_0$ is a non-principal ultrafilter in $M$. As $\langle A, <_A \rangle$ is a linearly ordered set without largest element in $M$, $U_0$ is a non-principal filter in $M$. If $x \in M$ and $x \subset A$, then there is an $a \in A$ such that $(a, \rightarrow) \subset x$ or $(a, \leftarrow) \subset A - x$ by (1), so exactly one of $x$ and $A - x$ is in $U_0$. So
$U_0$ is an ultrafilter in $M$. Similarly for $U_1$. Next we consider in $N$, to determine non-principal ultrafilters in $M \cap \mathcal{P}(A)$. Let $U$ be a non-principal ultrafilter in $M \cap \mathcal{P}(A)$ ($U$ may be not in $M$). First assume that non of bounded intervals of $A$ belong to $U$. Then by (1), it is clear that $U$ is of either form given in (2). So in the following, we assume $U$ contain a bounded interval as an element, and lead to a contradiction. Since $U$ is a filter, $U$ contains a bounded closed interval. Let $\psi : \langle \mathcal{Q}, \prec \rangle \rightarrow \langle A, \prec_A \rangle$ be the isomorphism which endows the dense linear ordering to $A$. Fix a bounded closed interval $I_0 = [a_0, b_0] \in U$. Using Lemma (1), by induction on $n < \omega$, we can make $I_n = [a_n, b_n]$ in such a way that the following conditions hold:

(i) $I_n \in U$,

(ii) the sequence $\{I_n\}$ is strictly descending,

(iii) $\lim_{n \to \omega} |\psi^{-1}(I_n)| = 0$, where $| \cdot |$ represents the length of interval.

Hence, there is a real $\alpha$ such that $\bigcap_{n \in \omega} \psi^{-1}(I_n) = \{\alpha\}$. Then

$$U = \{ x \in A \mid \exists a, b \in A (\psi^{-1}(a) < \alpha < \psi^{-1}(b) \wedge (a, b) \subset x) \}$$

If $\alpha$ is a rational, then $U$ is a principal filter, contradicting our assumption. So $\alpha$ is an irrational. Now, assume that $U$ is in $M$, and fix a support $S$ of $U$. Since $\alpha$ is an irrational and $S$ is finite, by (ii) and (iii), there is an $m < \omega$ such that $I_m \cap S = 0$. Take an $n$ such that $m < n$ and $a_m <_A a_n$. Let $\pi$ be an order automorphism such that if either $x \leq_A a_m$ or $b_m \leq_A x$, $\pi(x) = x$ and such that $\alpha < \psi^{-1}(\pi(a_m)) < \psi^{-1}(\pi(b_m))$. Then $\pi(U) = U$, for every member of a support $S$ of $U$ is preserved by $\pi$, and any member of $\psi^{-1}(\pi(I_n))$ is larger than $\alpha$, and so $-\pi(I_n) \in U$, which is a contradiction.

Now, we state our theorem. (Note that the statement "an ordered set has no end points" is $\Pi_2$.)

**Theorem.** In $M$, let $U$ be a non-principal ultrafilter on $A$. Then $\langle A, \prec_A \rangle^A / U$ is a dense linearly ordered set with an end point. So, $\langle A, \prec_A \rangle$ and $\langle A, \prec_A \rangle^A / U$ are not elementarily equivalent.

**Proof.** Claim. $A^A / U = \{ [c_a] | a \in A \} \cup \{ [i_A] \}$, where $c_a$ is the constant function with the value $a$ and $i_A$ is the identity function on $A$.

Firstly, assuming the Claim, we prove our theorem: That $\langle A, \prec_A \rangle^A / U$ is a dense linearly ordered set is obvious. Now, if $U$ is $\{ x \in A \mid \exists a \in A (a, \rightarrow) \subset x \}$, then $[i_A]$ is the largest element of $\langle A, \prec_A \rangle^A / U$. If $U$ is $\{ x \in A \mid \exists a \in A (\leftarrow, a) \subset x \}$,
then \([i_A]\) is the least element of \(\langle A, <_A \rangle / U\). Whereas \(\langle A, <_A \rangle\) has no end points.

**Proof of the Claim.** We consider only the case where \(U = \{x \in A \mid \exists a \in A (a, \rightarrow) \subset x\}\), another case is proved similarly. Let \(f : A \rightarrow A\) be in \(M\). First we assume \([f] < [i_A]\), i.e. \(\{x \in A \mid f(x) <_A x\} \in U\) and prove \([f] = [c_a]\) for some \(a \in A\). From the choice of \(U\), there is an \(a_0\) such that

\[(a_0, \rightarrow) \subset \{x \in A \mid f(x) <_A x\}.

Fix a support of \(f\) whose maximum element \(a^*\) is larger than \(a_0\). Fix \(a_1\) with \(a^* <_A a_1\). Then \(f(a_1) <_A a_1\). It suffices to show that if \(a_1 <_A a\), then \(f(a) = f(a_1)\), for letting \(f(a_1) = b\), we have \([f] = [c_b]\). To show this, fix an arbitral \(a\) with \(a_1 <_A a\). As \(f(a_1) <_A a_1\) and \(a^* <_A a_1\) we can take an order automorphism \(\pi\) of \(A\) such that if \(x \subseteq_A f(a_1)\) or \(x \subseteq_A a^*\) then \(\pi(x) = x\), and \(\pi(a_1) = a\). Since \(\pi\) preserves the support of \(f\), \(\pi f = f\), so

\[f(a) = (\pi f)(a) = (\pi f)(\pi(a_1)) = \pi(f(a_1)) = f(a_1).

Next, assume that \([i_A] < [f]\), i.e. \(\{x \in A \mid x <_A f(x)\} \in U\). Again we prove that \([f] = [c_a]\) for some \(a \in A\). From the choice of \(U\), there is an \(a_0\) such that

\[(a_0, \rightarrow) \subset \{x \in A \mid x <_A f(x)\}.

Fix a support of \(f\) whose maximum element \(a^*\) is larger than \(a_0\). Fix \(a_1\) with \(a^* <_A a_1\). Then \(a_1 <_A f(a_1)\). It suffices to show that if \(a_1 <_A a\), then \(f(a) = f(a_1)\). To show this, fix an \(a\) with \(f(a_1) <_A a\). As \(a_1 <_A f(a_1)\) and \(a_1 <_A f(a)\), we can take an order automorphism \(\pi\) of \(A\) such that if \(x \subseteq_A a_1\), then \(\pi(x) = x\) and \(\pi(f(a_1)) = f(a)\). Since \(\pi\) preserves the support of \(f\), \(\pi f = f\) and so

\[f(a) = \pi(f(a_1)) = (\pi f)(\pi(a_1)) = f(a_1).

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**References**

