

AREA-MINIMIZING OF THE CONE OVER SYMMETRIC R -SPACES

By

Daigo HIROHASHI, Takahiro KANNO and Hiroyuki TASAKI

Introduction

Let B denote a submanifold of the unit sphere in \mathbf{R}^n and C_B the cone over B , which is the union of rays starting from the origin and passing through B .

A cone is called area-minimizing if the truncated cone C_B^1 inside the unit ball is area-minimizing among all surfaces with boundary B . The surfaces we will use are integral currents. A tangent cone to surface S at a point $p \in S$ can be thought of as the union of rays starting from p and tangent to S at p . This is the generalization of the notion of tangent plane. If the tangent cone at p is not a plane, then p is a singular point of S . If S is area-minimizing, then each tangent cone to S is area-minimizing. Thus in order to study area-minimizing surface with singularities, we need to know which cones are area-minimizing.

G. R. Lawlor proposed a criterion for area-minimization in [5]. His principal idea is to construct an area-nonincreasing retraction $\Pi : \mathbf{R}^n \rightarrow C$. If S is another surface which has the same boundary as C_B^1 , it will follow that

$$\text{vol}(S) \geq \text{vol}(\Pi(S)) \geq \text{vol}(C_B^1)$$

since $\Pi(S)$ must cover all of C_B^1 . Using this method, he gave a complete classification of area-minimizing cones C over products of spheres and the first example of minimizing cone over a nonorientable manifold. In order to construct the retraction he solved a differential equation with numerical analysis.

In this paper, we consider the canonical imbeddings of symmetric R -spaces which are linear isotropy orbits of symmetric pairs. Using root systems, we construct area-nonincreasing retractions concretely.

In section 1 we prepare some notation and terminology, and prove an essential theorem (Theorem 1.6) for construction of the retractions. In section 2 we describe the canonical imbeddings of symmetric R -spaces, and construct

retractions onto the cones over them. In section 3 we apply the result of section 2 to symmetric R -spaces associated with symmetric pairs of type B_I .

Concerning the cones over symmetric R -spaces, B. N. Cheng [1] proved the cone over $U(n)/O(n)$ and $U(n)$ are area-minimizing in \mathbf{R}^{n^2+n} for $n \geq 7$ and \mathbf{R}^{2n^2} , respectively, by calibration. G. R. Lawlor [5] proved the cone over $SO(n)$ are area-minimizing in \mathbf{R}^{n^2} . Using the criterion of Lawlor in [5], M. Kerckhove [4] proved the cone over an isolated orbit of the action of $SU(n)$ on the unit sphere in the vector space of traceless n -by- n Hermitian symmetric matrices is area-minimizing for $n > 2$ and the cone over an isolated orbit of the adjoint action of $SO(n)$ is area-minimizing for $n > 3$.

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1. Preliminaries

Let G be a compact connected Lie group and θ an involutive automorphism of G . We denote by G_θ the closed subgroup of all fixed points of θ in G . For a closed subgroup K of G which lies between G_θ and the identity component of G_θ , (G, K) is a Riemannian symmetric pair. Let \mathfrak{g} and \mathfrak{k} be the Lie algebras of G and K respectively. The involutive automorphism θ of G induces an involutive automorphism of \mathfrak{g} , which is also denoted by θ . Since K lies between G_θ and the identity component of G_θ , we have

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}.$$

An inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} which is invariant under the actions of $\text{Ad}(G)$ and θ induces a bi-invariant Riemannian metric on G and G -invariant Riemannian metric on the homogeneous space $M = G/K$, which are also denoted by the same symbol $\langle \cdot, \cdot \rangle$. Then M is a compact Riemannian symmetric space with respect to $\langle \cdot, \cdot \rangle$. Conversely any compact symmetric space is constructed in this way. Put

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}.$$

Since θ is involutive, we have an orthogonal direct sum decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

This decomposition is called a canonical decomposition of the orthogonal symmetric Lie algebra (\mathfrak{g}, θ) .

Take and fix a maximal Abelian subspace \mathfrak{a} in \mathfrak{m} and a maximal Abelian subalgebra \mathfrak{t} in \mathfrak{g} including \mathfrak{a} . Let \mathfrak{c} be the center of \mathfrak{g} and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. We have an

orthogonal direct sum decomposition:

$$\mathfrak{g} = \mathfrak{c} + \mathfrak{g}'.$$

We set

$$\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{g}', \quad \mathfrak{c}_m = \mathfrak{c} \cap \mathfrak{m}.$$

We have an orthogonal direct sum decomposition:

$$\mathfrak{a} = \mathfrak{c}_m + \mathfrak{a}'.$$

Put

$$\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}.$$

Since \mathfrak{t} is θ -invariant we get an orthogonal direct sum decomposition of \mathfrak{t} :

$$\mathfrak{t} = \mathfrak{b} + \mathfrak{a}.$$

For $\alpha \in \mathfrak{t}$ we put

$$\tilde{\mathfrak{g}}_\alpha = \{X \in \mathfrak{g}^C \mid [H, X] = \sqrt{-1}\langle \alpha, H \rangle X (H \in \mathfrak{t})\}$$

and define the root system $\tilde{R}(\mathfrak{g})$ of \mathfrak{g} by

$$\tilde{R}(\mathfrak{g}) = \{\alpha \in \mathfrak{t} - \{0\} \mid \tilde{\mathfrak{g}}_\alpha \neq \{0\}\}.$$

We also denote \tilde{R} instead of $\tilde{R}(\mathfrak{g})$. For $\alpha \in \mathfrak{a}$ we put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^C \mid [H, X] = \sqrt{-1}\langle \alpha, H \rangle X (H \in \mathfrak{a})\}$$

and define the root system $R(\mathfrak{g}, \mathfrak{k})$ of $(\mathfrak{g}, \mathfrak{k})$ by

$$R(\mathfrak{g}, \mathfrak{k}) = \{\alpha \in \mathfrak{a} - \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}.$$

We also denote R instead of $R(\mathfrak{g}, \mathfrak{k})$. Put

$$\tilde{R}_0(\mathfrak{g}) = \tilde{R}(\mathfrak{g}) \cap \mathfrak{b}$$

and denote the orthogonal projection from \mathfrak{t} to \mathfrak{a} by $H \mapsto \bar{H}$. Then we have

$$R(\mathfrak{g}, \mathfrak{k}) = \{\bar{\alpha} \mid \alpha \in \tilde{R}(\mathfrak{g}) - \tilde{R}_0(\mathfrak{g})\}.$$

We extend a basis of \mathfrak{a} to that of \mathfrak{t} and define the lexicographic orderings $>$ on \mathfrak{a} and \mathfrak{t} with respect to these bases. Then for $H \in \mathfrak{t}$, $\bar{H} > 0$ implies $H > 0$. We denote by $\tilde{F}(\mathfrak{g})$ the fundamental system of $\tilde{R}(\mathfrak{g})$ with respect to the ordering $>$. We also denote \tilde{F} instead of $\tilde{F}(\mathfrak{g})$. Put

$$\tilde{F}_0(\mathfrak{g}) = \tilde{F}(\mathfrak{g}) \cap \tilde{R}_0(\mathfrak{g}).$$

Then the fundamental system $F(\mathfrak{g}, \mathfrak{k})$ of $R(\mathfrak{g}, \mathfrak{k})$ with respect to the ordering $>$ is given by

$$F(\mathfrak{g}, \mathfrak{k}) = \{\bar{\alpha} | \alpha \in \tilde{F}(\mathfrak{g}) - \tilde{F}_0(\mathfrak{g})\}.$$

We define positive root systems by

$$\tilde{R}_+(\mathfrak{g}) = \{\alpha \in \tilde{R}(\mathfrak{g}) | \alpha > 0\}$$

$$R_+(\mathfrak{g}, \mathfrak{k}) = \{\alpha \in R(\mathfrak{g}, \mathfrak{k}) | \alpha > 0\}.$$

We also denote \tilde{R}_+ and R_+ instead of $\tilde{R}_+(\mathfrak{g})$ and $R_+(\mathfrak{g}, \mathfrak{k})$. Then

$$R_+(\mathfrak{g}, \mathfrak{k}) = \{\bar{\alpha} | \alpha \in \tilde{R}_+(\mathfrak{g}) - \tilde{R}_0(\mathfrak{g})\}$$

holds. We set

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} | [X, H] = 0 (H \in \mathfrak{a})\}$$

and

$$\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

$$\mathfrak{m}_\alpha = \mathfrak{m} \cap (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$$

for $\alpha \in R_+(\mathfrak{g}, \mathfrak{k})$. We have the following lemma ([2]).

LEMMA 1.1. (1) *We have orthogonal direct sum decompositions:*

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in R_+} \mathfrak{k}_\alpha, \quad \mathfrak{m} = \mathfrak{a} + \sum_{\alpha \in R_+} \mathfrak{m}_\alpha.$$

(2) *For each $\alpha \in \tilde{R}_+ - \tilde{R}_0$ there exist $S_\alpha \in \mathfrak{k}$ and $T_\alpha \in \mathfrak{m}$ such that*

$$\{S_\alpha | \alpha \in \tilde{R}_+, \bar{\alpha} = \lambda\}, \quad \{T_\alpha | \alpha \in \tilde{R}_+, \bar{\alpha} = \lambda\}$$

are respectively orthonormal bases of \mathfrak{k}_λ , \mathfrak{m}_λ and that for $H \in \mathfrak{a}$

$$[H, S_\alpha] = \langle \alpha, H \rangle T_\alpha, \quad [H, T_\alpha] = -\langle \alpha, H \rangle S_\alpha.$$

We denote $m_\lambda = \dim \mathfrak{m}_\lambda = \dim \mathfrak{k}_\lambda$ and call it the multiplicity of λ .

We define a subset D of \mathfrak{a} by

$$D = \bigcup_{\alpha \in R} \{H \in \mathfrak{a} | \langle \alpha, H \rangle = 0\}.$$

Each connected component of $\mathfrak{a} - D$ is called a Weyl chamber. We define the fundamental Weyl chamber by

$$C = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in F(\mathfrak{g}, \mathfrak{k}))\}.$$

Its closure is given by

$$\bar{C} = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle \geq 0 \ (\alpha \in F(\mathfrak{g}, \mathfrak{k}))\}.$$

For each subset $\Delta \subset F = F(\mathfrak{g}, \mathfrak{k})$ we define a subset C^Δ of \bar{C} by

$$C^\Delta = \{H \in \bar{C} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \langle \beta, H \rangle = 0 \ (\beta \in F - \Delta)\}.$$

We easily get the following lemma.

LEMMA 1.2. (1) For $\Delta_1 \subset F$

$$\overline{C^{\Delta_1}} = \bigcup_{\Delta \subset \Delta_1} C^\Delta$$

is a disjoint union. In particular $\bar{C} = \bigcup_{\Delta \subset F} C^\Delta$ is a disjoint union.

(2) For $\Delta_1, \Delta_2 \subset F$, $\Delta_1 \subset \Delta_2$ if and only if $C^{\Delta_1} \subset \overline{C^{\Delta_2}}$.

For $\beta \in F$ we take $H_\beta \in \mathfrak{a}'$ satisfying the following condition.

$$\langle \alpha, H_\beta \rangle = \begin{cases} 1 & (\alpha = \beta) \\ 0 & (\alpha \neq \beta). \end{cases}$$

We have

$$\bar{C} = \mathfrak{c}_m \times \left\{ \sum_{\alpha \in F} t_\alpha H_\alpha \mid t_\alpha \geq 0 \right\}$$

and for $\Delta \subset F$

$$C^\Delta = \mathfrak{c}_m \times \left\{ \sum_{\alpha \in \Delta} t_\alpha H_\alpha \mid t_\alpha > 0 \right\}.$$

For $H \in \mathfrak{m}$ we put

$$Z^H = \{g \in G \mid \text{Ad}(g)H = H\},$$

$$Z_K^H = \{k \in K \mid \text{Ad}(k)H = H\}.$$

$Z_K^H = Z^H \cap K$ holds. Z^H is a closed subgroup of G and Z_K^H is a closed subgroup of K . We can prove the following lemma by the standard argument of compact Lie groups, so we omit its proof.

LEMMA 1.3. Z^H is connected.

For $\Delta \subset F$ we put

$$N^\Delta = \{g \in G \mid \text{Ad}(g)C^\Delta = C^\Delta\}$$

$$Z^\Delta = \{g \in G \mid \text{Ad}(g)|_{C^\Delta} = 1\}$$

$$N_K^\Delta = \{k \in K \mid \text{Ad}(k)C^\Delta = C^\Delta\}$$

$$Z_K^\Delta = \{k \in K \mid \text{Ad}(k)|_{C^\Delta} = 1\}.$$

By the above definitions we have $N_K^\Delta = N^\Delta \cap K$ and $Z_K^\Delta = Z^\Delta \cap K$. Z^Δ is a closed subgroup of G and Z_K^Δ is a closed subgroup of K . If $H \in C^\Delta$, then

$$Z^\Delta \subset Z^H, \quad Z_K^\Delta \subset Z_K^H.$$

We put

$$R^\Delta = R \cap (F - \Delta)_Z$$

$$R_+^\Delta = R^\Delta \cap R_+$$

$$\mathfrak{g}^\Delta = \mathfrak{k}_0 + \mathfrak{a} + \sum_{\alpha \in R_+^\Delta} (\mathfrak{k}_\alpha + \mathfrak{m}_\alpha)$$

and

$$\mathfrak{k}^\Delta = \mathfrak{g}^\Delta \cap \mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in R_+^\Delta} \mathfrak{k}_\alpha$$

$$\mathfrak{m}^\Delta = \mathfrak{g}^\Delta \cap \mathfrak{m} = \mathfrak{a} + \sum_{\alpha \in R_+^\Delta} \mathfrak{m}_\alpha.$$

We have an orthogonal direct sum decomposition:

$$\mathfrak{g}^\Delta = \mathfrak{k}^\Delta + \mathfrak{m}^\Delta.$$

LEMMA 1.4. For $\Delta \subset F$ and $H \in C^\Delta$, we obtain the following equations.

$$(1) R_+^\Delta = \{\alpha \in R_+ \mid \langle \alpha, H \rangle = 0\}$$

$$(2) R^\Delta = \{\alpha \in R \mid \langle \alpha, H \rangle = 0\}$$

$$(3) \mathfrak{g}^\Delta = \{X \in \mathfrak{g} \mid [H, X] = 0\}$$

PROOF. Any $\alpha \in R_+$ can be written as follows:

$$\alpha \in \sum_{\gamma \in F} n_\gamma \gamma \quad (n_\gamma \in \mathbb{Z}, n_\gamma \geq 0).$$

So we obtain

$$\langle \alpha, H \rangle = \sum_{\gamma \in F} n_\gamma \langle \gamma, H \rangle = \sum_{\gamma \in \Delta} n_\gamma \langle \gamma, H \rangle.$$

From this $\langle \alpha, H \rangle = 0$ if and only if $\alpha \in R_+^\Delta$. Therefore we obtain

$$R_+^\Delta = \{ \alpha \in R_+ \mid \langle \alpha, H \rangle = 0 \}.$$

This implies

$$R^\Delta = \{ \alpha \in R \mid \langle \alpha, H \rangle = 0 \}.$$

Any $X \in \mathfrak{g}$ can be written as follows:

$$X = S_0 + \sum_{\alpha \in R_+} a_\alpha S_\alpha + T_0 + \sum_{\alpha \in R_+} b_\alpha T_\alpha,$$

where $S_0 \in \mathfrak{k}_0$ and $T_0 \in \mathfrak{a}$. It follows from Lemma 1.1 and (1) that

$$[H, X] = \sum_{\alpha \notin R_+^\Delta} a_\alpha \langle \alpha, H \rangle T_\alpha - \sum_{\alpha \notin R_+^\Delta} b_\alpha \langle \alpha, H \rangle S_\alpha.$$

From this $[H, X] = 0$ if and only if $X \in \mathfrak{g}^\Delta$. Therefore we obtain

$$\mathfrak{g}^\Delta = \{ X \in \mathfrak{g} \mid [H, X] = 0 \}.$$

LEMMA 1.5. (1) Take $\Delta_1, \Delta_2 \subset F$, $H_1 \in C^{\Delta_1}$, $H_2 \in C^{\Delta_2}$ and $g \in G$. If $\text{Ad}(g)H_1 = H_2$, then $\text{Ad}(g)\mathfrak{g}^{\Delta_1} = \mathfrak{g}^{\Delta_2}$.

(2) If $\Delta \subset F$, then $N^\Delta \subset N(\mathfrak{g}^\Delta)$. For any $H \in C^\Delta$, all of Z^H , Z^Δ , N^Δ and $N(\mathfrak{g}^\Delta)$ are compact subgroups of G and all of their Lie subalgebras coincide with \mathfrak{g}^Δ .

PROOF. (1) By Lemma 1.4 \mathfrak{g}^{Δ_1} is the centralizer of H_1 . $\text{Ad}(g)\mathfrak{g}^{\Delta_1}$ is the centralizer of $\text{Ad}(g)H_1 = H_2$, so is equal to \mathfrak{g}^{Δ_2} .

(2) (1) with $\Delta_1 = \Delta_2 = \Delta$ implies $N^\Delta \subset N(\mathfrak{g}^\Delta)$. Z^H and $N(\mathfrak{g}^\Delta)$ are compact subgroups in G .

Since \mathfrak{g}^Δ is the centralizer of H by Lemma 1.4, the Lie algebra $\mathcal{L}(Z^H)$ of Z^H is equal to \mathfrak{g}^Δ .

We show that the Lie algebras $\mathcal{L}(Z^\Delta)$, $\mathcal{L}(N^\Delta)$ and $\mathcal{L}(N(\mathfrak{g}^\Delta))$ of Z^Δ , N^Δ and $N(\mathfrak{g}^\Delta)$ are all equal to \mathfrak{g}^Δ . By their definitions $Z^\Delta \subset N^\Delta \subset N(\mathfrak{g}^\Delta)$. Z^Δ , $N(\mathfrak{g}^\Delta)$ is a closed subgroup of G , so we have $\mathcal{L}(Z^\Delta) \subset \mathcal{L}(N(\mathfrak{g}^\Delta))$. Any element X of \mathfrak{g}^Δ can be written as follows:

$$X = S_0 + \sum_{\alpha \in R_+^\Delta} a_\alpha S_\alpha + T_0 + \sum_{\alpha \in R_+^\Delta} b_\alpha T_\alpha \quad (S_0 \in \mathfrak{k}_0, T_0 \in \mathfrak{a}).$$

From this we get $X \in \mathcal{L}(Z^\Delta)$, hence $\mathfrak{g}^\Delta \subset \mathcal{L}(Z^\Delta)$. Therefore

$$\mathfrak{g}^\Delta \subset \mathcal{L}(Z^\Delta) \subset \mathcal{L}(N(\mathfrak{g}^\Delta)).$$

Conversely we assume an element

$$X = S_0 + \sum_{\alpha \in R_+} a_\alpha S_\alpha + T_0 + \sum_{\alpha \in R_+} b_\alpha T_\alpha \quad (S_0 \in \mathfrak{k}_0, T_0 \in \mathfrak{a})$$

in \mathfrak{g} is contained in $\mathcal{L}(N(\mathfrak{g}^\Delta))$. Since

$$[H, X] = \sum_{\alpha \in R_+} a_\alpha \langle \alpha, H \rangle T_\alpha - \sum_{\alpha \in R_+} b_\alpha \langle \alpha, H \rangle S_\alpha$$

we obtain $X \in \mathfrak{g}^\Delta$. Therefore we get

$$\mathfrak{g}^\Delta = \mathcal{L}(Z^\Delta) = \mathcal{L}(N(\mathfrak{g}^\Delta)).$$

N^Δ satisfies $Z^\Delta \subset N^\Delta \subset N(\mathfrak{g}^\Delta)$ and $\mathcal{L}(Z^\Delta) = \mathcal{L}(N(\mathfrak{g}^\Delta))$. So N^Δ is also a compact subgroup of G and $\mathcal{L}(N^\Delta) = \mathfrak{g}^\Delta$ holds.

THEOREM 1.6. *For any $\Delta \subset F$ and $H \in C^\Delta$*

$$Z^\Delta = Z^H = N^\Delta, \quad Z_K^\Delta = Z_K^H = N_K^\Delta.$$

PROOF. By the definition we have $Z^\Delta \subset Z^H$ and by the above lemma their Lie algebras coincides. Moreover Z^H is connected by Lemma 1.3, so we obtain $Z^\Delta = Z^H$.

Z^Δ and N^Δ are compact and have the same Lie algebra \mathfrak{g}^Δ . Since Z^Δ is the kernel of the homomorphism

$$\text{Ad} : N^\Delta \rightarrow \text{The permutation group of } C^\Delta,$$

Z^Δ is a normal subgroup of N^Δ and N^Δ/Z^Δ is a finite group. For any $g \in N^\Delta$, the action of $\text{Ad}(g)$ on C^Δ has a finite order, that is, there is an integer N satisfying $\text{Ad}(g)^N|_{C^\Delta} = 1$. Take $H_0 \in C^\Delta$ and put

$$H_1 = \frac{1}{N}(H_0 + \text{Ad}(g)H_0 + \cdots + \text{Ad}(g)^{N-1}H_0).$$

Each $\text{Ad}(g)^i H_0$ is contained in C^Δ and C^Δ is convex, so we get $H_1 \in C^\Delta$. $\text{Ad}(g)H_1 = H_1$ holds and $g \in Z^{H_1} = Z^\Delta$. Hence $N^\Delta \subset Z^\Delta$ and we obtain $Z^\Delta = N^\Delta$.

The second equation follows from the first one.

2. Construction of Retractions

The notation of the preceding section will be preserved. Let B be a compact submanifold of the unit sphere in \mathbb{R}^n . We call $C_B = \{tx|x \in B, t \geq 0\}$ the cone over B . C_B is said to be *area-minimizing* if $C_B^1 = \{tx|x \in B, 0 \leq t \leq 1\}$ has the least area among all surfaces with boundary B .

For an unit vector $H \in \bar{C}$, the orbit $\text{Ad}(K)H$ is a submanifold of the unit sphere in \mathfrak{m} . Then the mapping

$$f : kZ_K^H \mapsto \text{Ad}(k)H,$$

is a diffeomorphism of the homogeneous space K/Z_K^H to $\text{Ad}(K)H$.

PROPOSITION 2.1. *The orbit $\text{Ad}(K)H$ is connected.*

PROOF. Since $\mathfrak{m} = \bigcup_{k \in K_0} \text{Ad}(k) \cdot \mathfrak{a}$, where K_0 be the identity component of K , for any $\text{Ad}(k)H \in \text{Ad}(K)H$ there exists an element $k_1 \in K_0$ such that $\text{Ad}(k_1k)H \in \mathfrak{a}$. From Proposition 2.2 (p. 285) of [2], there exists a member of *Weyl group* whose action on \mathfrak{a} is represented by $\text{Ad}(k_2)$ for some $k_2 \in K_0$ such that $\text{Ad}(k_1k)H = \text{Ad}(k_2)H$. If we put $k_0 = k_1^{-1}k_2 \in K_0$, then $\text{Ad}(k)H = \text{Ad}(k_0)H$ holds. Thus we get $\text{Ad}(K)H = \text{Ad}(K_0)H$ and it is connected.

From now on we assume that (G, K) is irreducible.

Let $F(\mathfrak{g}, \mathfrak{k}) = \{\alpha_1, \dots, \alpha_l\}$ be the fundamental root system and $\tilde{\alpha} = n_1\alpha_1 + \dots + n_l\alpha_l$ be the highest root of $R(\mathfrak{g}, \mathfrak{k})$. Select $\alpha_i \in F(\mathfrak{g}, \mathfrak{k})$ such that $n_i = 1$, we put $\alpha_0 = \alpha_i$ and $A_0 = H_{\alpha_0}/|H_{\alpha_0}|$. It is known that f is an isometry of $K/Z_K^{A_0}$ with the normal homogeneous Riemannian metric multiplied some constant onto $\text{Ad}(K)A_0$ and that $K/Z_K^{A_0}$ is a symmetric space. We call this space a *symmetric R-space*, and f its *canonical imbedding*. Because $\text{Ad}(K)A_0$ is an isolated orbit, $\text{Ad}(K)A_0$ is a minimal submanifold of the unit sphere in \mathfrak{m} by a result of Hsiang [3]. Hence the cone $C_{\text{Ad}(K)A_0}$ is also a minimal submanifold of \mathfrak{m} ([6] p. 97, Prop. 6.1.1). The purpose of this article is to prove $C_{\text{Ad}(K)A_0}$ is an area-minimizing cone.

PROPOSITION 2.2. *Let V and W be two vector spaces with inner products. Suppose $n = \dim W \leq \dim V$. For a linear mapping F of V to W we put*

$$JF = \sup\{|F(u_1) \wedge \dots \wedge F(u_n)|\},$$

where u_1, \dots, u_n runs over all orthonormal vectors of V . If F is not surjective, then

$JF = 0$. If F is surjective, then JF coincides with

$$|F(v_1) \wedge \cdots \wedge F(v_n)|,$$

for an orthonormal base v_1, \dots, v_n of $(\ker F)^\perp$.

Let B be a compact submanifold of the unit sphere in \mathfrak{m} . We call a differentiable retraction $\Phi : \mathfrak{m} \rightarrow C_B$ a *area-nonincreasing* if

$$J(d\Phi_x) \leq 1. \tag{1}$$

for $x \in \mathfrak{m}$.

PROPOSITION 2.3. *The cone C_B over a compact submanifold B of the unit sphere in \mathfrak{m} is area-minimizing if there exists an area-nonincreasing retraction $\Phi : \mathfrak{m} \rightarrow C_B$.*

PROOF. Let S be a surface in \mathfrak{m} with boundary B . Since $C_B^1 \subset \Phi(S)$, we have $\text{vol}(C_B^1) \leq \text{vol}(\Phi(S))$. Let e_1, \dots, e_n be an orthonormal frame of S , then

$$\begin{aligned} \text{vol}(\Phi(S)) &= \int_S |d\Phi(e_1 \wedge \cdots \wedge e_n)| d\mu_S \\ &\leq \int_S J(d\Phi_x) d\mu_S \\ &\leq \int_S d\mu_S \\ &= \text{vol}(S). \end{aligned}$$

Consequently,

$$\text{vol}(C_B^1) \leq \text{vol}(\Phi(S)) \leq \text{vol}(S).$$

This proves the proposition.

We shall now consider a way to construct area-nonincreasing retractions.

LEMMA 2.4. *Suppose ϕ is a mapping of \bar{C} into itself such that $\phi(C^\Delta) \subset \bar{C}^\Delta$ for each $\Delta \subset F(\mathfrak{g}, \mathfrak{k})$. Then ϕ extends to a mapping Φ of \mathfrak{m} as*

$$\Phi(X) = \text{Ad}(k)\phi(H),$$

for each $X = \text{Ad}(k)H \in \mathfrak{m}$ ($k \in K, H \in \bar{C}$).

PROOF. Suppose $k_1, k_2 \in K$ and $H_1, H_2 \in \bar{C}$ satisfy $\text{Ad}(k_1)H_1 = \text{Ad}(k_2)H_2$. Then $\text{Ad}(k_2^{-1}k_1)H_1 = H_2 \in \mathfrak{a}$ and we have $H_1 = H_2$, because each orbit of the Weyl group on \mathfrak{a} intersects \bar{C} in exactly one point ([3], p. 293, Th. 2.22). Let

$$\Delta = \{\alpha \in F \mid \langle \alpha, H_1 \rangle > 0\}.$$

We have $H_1 \in C^\Delta$ and $\phi(H_1) \in \bar{C}^\Delta$ by the assumption of ϕ . Thus Theorem 1.6 implies

$$k_2^{-1}k_1 \in Z_K^{H_1} = Z_K^\Delta,$$

therefore $\text{Ad}(k_2^{-1}k_1)\phi(H_1) = \phi(H_1)$.

From Lemma 2.4, we have the following.

PROPOSITION 2.5. *Let $\phi : \bar{C} \rightarrow \{tA_0 \mid t \geq 0\}$ be a differentiable mapping. Denote $\phi(x) = f(x)A_0$. If f satisfies $f(tA_0) = t$ ($t \geq 0$) and $f|_{\{x_0\}^\perp} \equiv 0$, then ϕ extends to a differentiable retraction $\Phi : \mathfrak{m} \rightarrow C_{\text{Ad}(K)A_0}$.*

In this case Φ is area-nonincreasing if and only if (1) holds for each $x \in C$.

We will compute $J(d\Phi_x)$ of Φ in Proposition 2.5 for $x \in C$.

PROPOSITION 2.6. *We denote $R_+(A_0) = \{\lambda \in R_+ \mid \langle \lambda, A_0 \rangle > 0\}$.*

$$J(d\Phi_x) = |\text{grad}(f)| \prod_{\lambda \in R_+(A_0)} \left(\frac{\langle \lambda, A_0 \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m_\lambda}. \tag{2}$$

PROOF. If $f(x) = 0$, then the both sides of the equation are 0. So we consider the case $f(x) \neq 0$. By the definition of Φ , $d\Phi_x(\mathfrak{a}) \subset \mathbf{R}A_0$. By using the equation

$$\left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp tS_\alpha)x = -\langle \alpha, x \rangle T_\alpha,$$

for $\alpha \in \tilde{R}_+(\mathfrak{g})$, $\tilde{\alpha} \neq 0$, we have

$$d\Phi_x(T_\alpha) = \frac{\langle \alpha, \phi(x) \rangle}{\langle \alpha, x \rangle} T_\alpha.$$

From this we get

$$d\Phi_x \left(\sum_{\lambda \in R_+} \mathfrak{m}_\lambda \right) \subset \sum_{\lambda \in R_+(A_0)} \mathfrak{m}_\lambda,$$

so we can write

$$\begin{aligned} J(d\Phi_x) &= J_1(x) \cdot J_2(x), \\ J_1(x) &= J(d\Phi_x|_{\mathfrak{a}}), \\ J_2(x) &= J(d\Phi_x|_{\sum m_i}). \end{aligned}$$

Take a unit vector v in \mathfrak{a} .

$$\begin{aligned} d\Phi_x(v) &= d\phi_x(v) \\ &= df_x(v)A_0 \\ &= \langle \text{grad}(f), v \rangle A_0. \end{aligned}$$

Therefore

$$\begin{aligned} J_1(x) &= \max\{|d\Phi_x(v)| \mid v \in \mathfrak{a}, |v| = 1\} \\ &= |\text{grad}(f)|. \end{aligned}$$

Secondly for $J_2(x)$, we see the kernel of $d\Phi_x|_{\sum m_i}$. By the above expression of $d\Phi_x(T_\alpha)$ we get

$$\ker(d\Phi_x|_{\sum m_i}) = \sum_{\substack{\lambda \in R_+ \\ \langle \lambda, A_0 \rangle = 0}} m_\lambda.$$

We can take $\{T_\alpha \mid \alpha \in \tilde{R}_+(\mathfrak{g}), \langle \alpha, A_0 \rangle > 0\}$ as an orthonormal base of $\ker(d\Phi_x|_{\sum m_i})^\perp$. It follows that

$$J_2(x) = \left| \bigwedge_{\substack{\alpha \in \tilde{R}_+ \\ \langle \alpha, A_0 \rangle > 0}} d\Phi_x(T_\alpha) \right| = \prod_{\lambda \in R_+(A_0)} \left(\frac{\langle \lambda, A_0 \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m_\lambda},$$

where m_λ is the multiplicity of λ . So we have

$$J(d\Phi_x) = |\text{grad}(f)| \prod_{\lambda \in R_+(A_0)} \left(\frac{\langle \lambda, A_0 \rangle}{\langle \lambda, x \rangle} f(x) \right)^{m_\lambda}.$$

3. Construction of Area-nonincreasing Retractions

THEOREM 3.1. *The cones over*

$$\frac{SO(2l+1)}{SO(2) \times SO(2l-1)}, \quad \frac{SO(l) \times SO(l+n)}{S'(O(l-1) \times O(l+n-1))}$$

corresponding to symmetric pairs

$$(SO(2l+1)^2, SO(2l+1)), \quad (SO(2l+n), SO(l) \times SO(l+n)) \quad (n \geq 2)$$

respectively are area-minimizing, where

$$S'(O(l-1) \times O(l+n-1)) = \left\{ \begin{pmatrix} \varepsilon & & & \\ & A & & \\ & & \varepsilon & \\ & & & B \end{pmatrix} \in SO(l) \times SO(l+n) \left| \begin{array}{l} \varepsilon = \pm 1, A \in O(l-1), \\ B \in O(l+n-1) \end{array} \right. \right\}.$$

PROOF. We consider symmetric pairs of type B_l . Let $\varepsilon_1, \dots, \varepsilon_l$ be an orthonormal basis of the maximal Abelian subspace \mathfrak{a} such that all roots are

$$\pm \varepsilon_i \pm \varepsilon_j \quad (1 \leq i < j \leq l), \quad \pm \varepsilon_i \quad (1 \leq i \leq l).$$

Then for a suitable ordering

$$\begin{aligned} F(\mathfrak{g}, \mathfrak{k}) &= \{\alpha_1, \alpha_2, \dots, \alpha_{l-1}, \alpha_l\}, \\ \alpha_i &= \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i < l), \quad \alpha_l = \varepsilon_l, \\ H_{\alpha_i} &= \varepsilon_1 + \dots + \varepsilon_i \quad (1 \leq i \leq l), \\ \tilde{\alpha} &= \alpha_1 + 2\alpha_2 + \dots + 2\alpha_l = \varepsilon_1 + \varepsilon_2 \end{aligned}$$

and we put

$$A_0 = \frac{H_{\alpha_1}}{|H_{\alpha_1}|} = \varepsilon_1.$$

We have

$$\begin{aligned} R_+(A_0) &= \left\{ \sum_{i=1}^l \alpha_i \right\} \cup \left\{ \sum_{i=1}^{k-1} \alpha_i \mid 2 \leq k \leq l \right\} \\ &\quad \cup \left\{ \tilde{\alpha} - \sum_{i=2}^k \alpha_i \mid 2 \leq k \leq l-1 \right\} \cup \{\tilde{\alpha}\} \end{aligned}$$

Because the multiplicities of roots of same length coincide with each other, we can denote by m_1 the multiplicity of the $\sum_{i=1}^l \alpha_i$ and by m_2 the multiplicity of the rest. For $x = \sum_{i=1}^l x_i H_{\alpha_i} \in \bar{C}$ we define

$$f(x) = \sqrt{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle} = \sqrt{x_1(x_1 + 2x_2 + \cdots + 2x_l)}$$

then it satisfies Proposition 2.5. Using (2), we calculate $J(d\Phi_x)$. Since

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \frac{x_1 + \cdots + x_l}{f}, \\ \frac{\partial f}{\partial x_i} &= \frac{x_1}{f} \quad (2 \leq i \leq l), \end{aligned}$$

we get

$$J_1(x) = |\text{grad}(f)| = \left| \sum_{i=1}^l \frac{\partial f}{\partial x_i} \alpha_i \right| = \sqrt{\frac{\langle \alpha_1, x \rangle^2 + \langle \tilde{\alpha}, x \rangle^2}{2\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}}.$$

We also obtain

$$J_2(x) = \left(\frac{f}{\langle \sum_{i=1}^l \alpha_i, x \rangle} \right)^{m_1} \times \prod_{k=2}^{l-1} \left(\frac{f^2}{\langle \sum_{i=1}^k \alpha_i, x \rangle \langle \tilde{\alpha} - \sum_{i=2}^k \alpha_i, x \rangle} \right)^{m_2}.$$

Therefore

$$\begin{aligned} J(d\Phi_x) &= \sqrt{\frac{\langle \alpha_1, x \rangle^2 + \langle \tilde{\alpha}, x \rangle^2}{2\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}} \left(\frac{\sqrt{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}}{\langle \sum_{i=1}^l \alpha_i, x \rangle} \right)^{m_1} \\ &\quad \times \prod_{k=2}^{l-1} \left(\frac{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}{\langle \sum_{i=1}^k \alpha_i, x \rangle \langle \tilde{\alpha} - \sum_{i=2}^k \alpha_i, x \rangle} \right)^{m_2}. \end{aligned}$$

For any k ,

$$\begin{aligned} &\left\langle \sum_{i=1}^k \alpha_i, x \right\rangle \left\langle \tilde{\alpha} - \sum_{i=2}^k \alpha_i, x \right\rangle \\ &= \left\langle \sum_{i=1}^k \alpha_i, x \right\rangle \langle \tilde{\alpha}, x \rangle - \left\langle \sum_{i=1}^k \alpha_i, x \right\rangle \left\langle \sum_{i=2}^k \alpha_i, x \right\rangle \\ &= \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle + \left\langle \sum_{i=2}^k \alpha_i, x \right\rangle \langle \tilde{\alpha}, x \rangle - \left\langle \sum_{i=1}^k \alpha_i, x \right\rangle \left\langle \sum_{i=2}^k \alpha_i, x \right\rangle \\ &= \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle + \left\langle \sum_{i=2}^k \alpha_i, x \right\rangle \left(\langle \tilde{\alpha}, x \rangle - \left\langle \sum_{i=1}^k \alpha_i, x \right\rangle \right) \\ &\geq \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle, \end{aligned}$$

hence

$$\frac{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}{\langle \sum_{i=1}^k \alpha_i, x \rangle \langle \tilde{\alpha} - \sum_{i=2}^k \alpha_i, x \rangle} \leq 1.$$

On the other hand

$$\begin{aligned} & \left\langle \sum_{i=1}^l \alpha_i, x \right\rangle^2 - \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle \\ &= \left\langle \sum_{i=1}^l \alpha_i, x \right\rangle^2 - \langle \alpha_1, x \rangle \left\langle \sum_{i=1}^l \alpha_i + \sum_{i=2}^l \alpha_i, x \right\rangle \\ &= \left\langle \sum_{i=1}^l \alpha_i, x \right\rangle \left\langle \sum_{i=2}^l \alpha_i, x \right\rangle - \langle \alpha_1, x \rangle \left\langle \sum_{i=2}^l \alpha_i, x \right\rangle \\ &= \left\langle \sum_{i=2}^l \alpha_i, x \right\rangle^2 \geq 0. \end{aligned}$$

Therefore

$$\frac{\sqrt{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}}{\langle \sum_{i=1}^l \alpha_i, x \rangle} \leq 1.$$

We consider the case $m_1 \geq 2$ and let

$$A = J_1(x) \left(\frac{\sqrt{\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle}}{\langle \sum_{i=1}^l \alpha_i, x \rangle} \right)^2.$$

If $A \leq 1$ then $J(d\Phi_x) \leq 1$.

$$\begin{aligned} A^2 &= \frac{\langle \alpha_1, x \rangle^2 + \langle \tilde{\alpha}, x \rangle^2}{2\langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle} \frac{\langle \alpha_1, x \rangle^2 \langle \tilde{\alpha}, x \rangle^2}{\langle \sum_{i=1}^l \alpha_i, x \rangle^4} \\ &= \frac{8(\langle \alpha_1, x \rangle^3 \langle \tilde{\alpha}, x \rangle + \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle^3)}{(\langle \tilde{\alpha}, x \rangle + \langle \alpha_1, x \rangle)^4}. \end{aligned}$$

Here subtracting the numerator from the denominator we have

$$\begin{aligned} & (\langle \tilde{\alpha}, x \rangle + \langle \alpha_1, x \rangle)^4 - 8(\langle \alpha_1, x \rangle^3 \langle \tilde{\alpha}, x \rangle + \langle \alpha_1, x \rangle \langle \tilde{\alpha}, x \rangle^3) \\ &= (\langle \tilde{\alpha}, x \rangle - \langle \alpha_1, x \rangle)^4 \geq 0. \end{aligned}$$

Therefore if $m_1 \geq 2$, we get $J(d\Phi_x) \leq 1$.

There are two kind of symmetric pairs: (i) $m_1 = n$ ($n \geq 2$), $m_2 = 1$ and (ii) $m_1 = m_2 = 2$.

(i) $(G, K) = (SO(2l + n), SO(l) \times SO(l + n))$. It is defined by the involution:

$$\theta(g) = I_{l,l+n}gI_{l,l+n} \quad (g \in SO(2l + n)), \quad I_{l,l+n} = \begin{pmatrix} -I_l & 0 \\ 0 & I_{l+n} \end{pmatrix}.$$

Then

$$\mathfrak{g} = \mathfrak{so}(2l + n) = \{X \in M_{2l+n}(\mathbf{R}) \mid X + {}^tX = 0\},$$

$$\mathfrak{k} = \mathfrak{so}(l) \times \mathfrak{so}(l + n) = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x \in \mathfrak{so}(l), y \in \mathfrak{so}(l + n) \right\},$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & x \\ -{}^tx & 0 \end{pmatrix} \mid x \in M_{l,l+n}(\mathbf{R}) \right\} \cong M_{l,l+n}(\mathbf{R}).$$

Since \mathfrak{m} is isomorphic to $M_{l,l+n}(\mathbf{R})$, we identify them. The action of K on \mathfrak{m} through this identification is

$$\text{Ad} \begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix} \cdot x = kx'k'.$$

We define an $\text{Ad}(K)$ -invariant inner product on \mathfrak{m} by

$$\langle X, Y \rangle = \text{Trace}({}^tXY) \quad (X, Y \in \mathfrak{m}).$$

Let

$$\begin{aligned} \mathfrak{a} &= \left\{ \begin{pmatrix} 0 & t \\ -{}^tt & 0 \end{pmatrix} \mid t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_l \end{pmatrix}, t_i \in \mathbf{R} \right\} \\ &\cong \left\{ t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_l \end{pmatrix} \right\}. \end{aligned}$$

Then \mathfrak{a} is a maximal Abelian subspace in \mathfrak{m} . Next, we consider a root space decomposition of \mathfrak{m} with respect to \mathfrak{a} . Let E_{pq} be a matrix whose (p, q) -entry is 1 and all other entries are 0. Then

$$\varepsilon_p = E_{pp},$$

$$m_{\varepsilon_p} = \sum_{q=l+1}^{l+n} RE_{pq},$$

$$m_{\varepsilon_p - \varepsilon_q} = R(E_{pq} + E_{qp}),$$

$$m_{\varepsilon_p + \varepsilon_q} = R(E_{pq} - E_{qp}).$$

Since $A_0 = \varepsilon_1 = E_{11}$, we have

$$Z_K^{A_0} = S'(O(l-1) \times O(l+n-1)).$$

Hence the corresponding symmetric R -space is

$$\frac{SO(l) \times SO(l+n)}{S'(O(l-1) \times O(l+n-1))}.$$

(ii) $(G, K) = (SO(2l+1)^2, SO(2l+1))$. It is defined by the involution such that

$$\theta(g_1, g_2) = (g_2, g_1) \quad ((g_1, g_2) \in SO(2l+1)).$$

Then

$$\mathfrak{g} = \mathfrak{so}(2l+1) \times \mathfrak{so}(2l+1),$$

$$\mathfrak{k} = \{(x, x) | x \in \mathfrak{so}(2l+1)\},$$

$$\mathfrak{m} = \{(x, -x) | x \in \mathfrak{so}(2l+1)\} \cong \mathfrak{so}(2l+1).$$

Since \mathfrak{m} is isomorphic to $\mathfrak{so}(2l+1)$, we identify them. The action of K on \mathfrak{m} through this identification is

$$\text{Ad}(k) \cdot x = kx^t k \quad (k \in SO(2l+1), x \in \mathfrak{so}(2l+1)).$$

We define an $\text{Ad}(K)$ -invariant inner product on \mathfrak{m} by

$$\langle X, Y \rangle = \frac{1}{2} \text{Trace}({}^t XY) \quad (X, Y \in \mathfrak{m}).$$

Let

$$\alpha = \left\{ t = \begin{pmatrix} t'_1 & & & \\ & \ddots & & \\ & & t'_i & \\ & & & 0 \end{pmatrix} \middle| t'_i = \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix}, t_i \in \mathbf{R} \right\} \subset \mathfrak{m} \cong \mathfrak{so}(2l+1).$$

Then \mathfrak{a} is a maximal Abelian subspace in \mathfrak{m} . Next, we consider a root space decomposition of \mathfrak{m} with respect to \mathfrak{a} . Let $G_{pq} = E_{pq} - E_{qp}$. Then

$$\begin{aligned}\varepsilon_p &= G_{2p-1,2p}, \\ \mathfrak{m}_{\varepsilon_p} &= \mathbf{R}G_{2p-1,2l+1} + \mathbf{R}G_{2p,2l+1}, \\ \mathfrak{m}_{\varepsilon_p - \varepsilon_q} &= \mathbf{R}(G_{2p-1,2q-1} - G_{2p,2q}) + \mathbf{R}(G_{2p-1,2q} + G_{2p,2q-1}), \\ \mathfrak{m}_{\varepsilon_p + \varepsilon_q} &= \mathbf{R}(G_{2p-1,2q-1} + G_{2p,2q}) + \mathbf{R}(G_{2p-1,2q} - G_{2p,2q-1}).\end{aligned}$$

Since $A_0 = \varepsilon_1 = G_{12}$, we have

$$Z_K^{A_0} = SO(2) \times SO(2l-1).$$

Hence the corresponding symmetric R -space is

$$\frac{SO(2l+1)}{SO(2) \times SO(2l-1)}.$$

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Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki
305-8571
Japan
daigo@math.tsukuba.ac.jp
kanno@math.tsukuba.ac.jp
tasaki@math.tsukuba.ac.jp