THE INJECTIVITY RADIUS AND THE FUNDAMENTAL GROUP OF COMPACT HOMOGENEOUS RIEMANNIAN MANIFOLDS OF POSITIVE CURVATURE

By

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1. Introduction

In this paper we characterize a homogeneous spherical space form whose fundamental group is a binary dihedral group by means of the injectivity radius of the exponential map. As an application of this result we show that the fundamental group of a non-simply connected homogeneous 1/4-pinched Riemannian manifold is isomorphic to a finite subgroup of the special unitary group $SU(2)$ of degree 2. Moreover we show that the fundamental group of a non-simply connected homogeneous 1/4-pinched Riemannian manifold whose dimension is of $4j+1$ ($j \geq 1$) is a cyclic group. As is well known, every nontrivial finite subgroup of $SU(2)$ is a cyclic, binary dihedral or binary polyhedral group ([21]). Therefore the fundamental group of a non-simply connected homogeneous 1/4-pinched Riemannian manifold is isomorphic to one of the finite groups stated above.

Let $M$ be an $m$-dimensional ($m \geq 3$) complete, connected Riemannian manifold and $N$ an $n$-dimensional ($0 \leq n \leq m - 1$) connected, compact submanifold (without boundary) embedded in $M$. Let $Exp_N : v(N) \to M$ denote the normal exponential map of $N$. Here $v(N)$ is the total space of the normal bundle of $N$ in $M$. In the case where $N$ is a point of $M$, $v(N)$ stands for the tangent space to $M$ at that point. We denote by $i(N)$ the injectivity radius of $Exp_N$. It is defined as the supremum of the set of all $r > 0$ for which $Exp_N : v_r(N) \to M$ is an embedding, where $v_r(N)$ is the set of all normal vectors to $N$ of length less than $r$. The injectivity radius $Inj(M)$ of $M$ is then defined as $Inj(M) = \inf \{ i(x) \mid x \in M \}$. If $M$ is compact, then $Inj(M)$ is positive.

A binary dihedral group $D_k^*$ is a finite group generated by two elements $a$
and $b$ with fundamental relations $a^{2k} = e$, $a^k = b^2$ and $aba = b$, where $2k$ is the order of $a$ ($k \geq 2$) and $e$ denotes the unit element of $D_2^n$. In case of $k = 2$, $D_2^n$ is isomorphic to the quaternion group $Q_8$. As stated above, $SU(2)$ contains binary dihedral groups.

There are homogeneous spherical space forms whose fundamental groups are binary dihedral groups. We can characterize such spherical space forms in terms of the injectivity radius of the exponential map.

Let $M$ denote an $m$-dimensional ($m \geq 3$) connected, compact, non-simply connected Riemannian manifold with sectional curvature $K_M \geq 1$. An upper bound of $\text{Inj}(M)$ is closely related to the fundamental group $\pi_1(M)$ of $M$. The diameter sphere theorem due to Grove and Shiohama ([8]) shows that $\text{Inj}(M) \leq \pi/2$. Toponogov's diameter theorem implies that if $\text{Inj}(M) = \pi/2$, then $M$ is isometric to the $m$-dimensional real projective space $RP^m$ with constant curvature $1$. Shiohama showed in [18] the following result. If $\pi_1(M) \cong Z_3$, then $\text{Inj}(M) \leq \pi/3$ and equality holds if and only if $M$ is isometric to the lens space of constant curvature $1$. We showed in [11] that if the order of $\pi_1(M)$ is not a prime, then $\text{Inj}(M) \leq \pi/4$ and equality holds if and only if $M$ is a homogeneous Riemannian manifold of constant curvature $1$ and $\pi_1(M)$ is isomorphic to either $Z_4$ or $Q_8$. Here if $\pi_1(M) \cong Q_8$, then we have $m = 4j - 1$ ($j \geq 1$).

Let $M$ be as above, and let $N$ be an $n$-dimensional ($n \geq 1$) connected, compact, totally geodesic submanifold embedded in $M$ such that $2n \leq m - 1$. We note here that if $2n \geq m$, then the first relative homotopy class $\pi_1(M, N)$ is trivial, i.e., the homomorphism $\iota_1: \pi_1(N) \to \pi_1(M)$ induced from the inclusion $\iota: N \to M$ is surjective ([6]). In the case where $\pi_1(M, N)$ is nontrivial, i.e., $\iota_2(\pi_1(N)) \neq \pi_1(M)$, an upper bound of $i(N)$ is closely related to $\pi_1(M)$ and $\pi_1(M, N)$. We now assume that $M$ is a homogeneous Riemannian manifold. Then we showed in [11] that if $N$ is a simple closed geodesic of $M$ which is homotopically nontrivial and if $\pi_1(M)$ is not a cyclic group, then $i(N) \leq \pi/4$. Here if equality holds, then $M$ is of constant curvature $1$, $m = 4j - 1$ ($j \geq 1$) and $\pi_1(M)$ is a binary dihedral group. In this result we can eliminate the assumption that $N$ is homotopically nontrivial. Moreover this result is also true for the case $n \geq 2$. In this paper we show the following.

**Theorem A.** Let $M$ be an $m$-dimensional ($m \geq 3$) connected, compact, non-simply connected homogeneous Riemannian manifold with sectional curvature $K_M \geq 1$. Let $N$ be an $n$-dimensional ($n \geq 1$) compact, connected, totally geodesic submanifold embedded in $M$ such that $2n \leq m - 1$. Assume that $\pi_1(M)$ is not a cyclic group and that $\pi_1(M, N)$ is nontrivial. Then $i(N) \leq \pi/4$. Here if equality
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holds, then $M$ is of constant curvature $1$, $m = 4j - 1$ ($j \geq 1$) and $\pi_1(M)$ is isomorphic to a binary dihedral group.

As applications of Theorem A we have the following two results.

**Theorem B.** Let $M$ be an $m$-dimensional ($m \geq 2$) connected, compact, non-simply connected homogeneous Riemannian manifold whose sectional curvature $K_M$ satisfies $1 \leq K_M \leq 4$. Then $\pi_1(M)$ is isomorphic to a finite subgroup of $SU(2)$.

**Theorem C.** Let $M$ be an $m$-dimensional connected, compact, non-simply connected homogeneous Riemannian manifold whose sectional curvature $K_M$ satisfies $1 \leq K_M \leq 4$. If $m = 4j + 1$ ($j \geq 1$), then $\pi_1(M)$ is a cyclic group.

In the case where $M$ is a non-simply connected homogeneous spherical space form, Theorems B and C are classical results ([21]; p. 229). However, even if $M$ is such a spherical space form, our proof for these theorems is different from one given in [21].

The proof of the theorems stated above will be given in Sections 3 and 4. We give in Section 5 examples of connected, compact, non-simply connected homogeneous Riemannian manifolds with $1 \leq K \leq 4$.

2. **Preliminaries**

Throughout this paper we always assume that all geodesics on Riemannian manifolds are parameterized by arc-length, unless otherwise stated.

In this section we prepare lemmas which will be used in the proof of the theorems stated in Section 1.

Throughout this section let $M$ be an $m$-dimensional ($m \geq 3$) connected, compact, non-simply connected Riemannian manifold whose sectional curvature $K_M$ satisfies $K_M \geq 1$ and let $p : V \to M$ denote the universal Riemannian covering. $V$ is a complete Riemannian manifold with sectional curvature $K_V \geq 1$. We will denote by $d$ the distance function on $V$ which is induced from the Riemannian metric of $V$. Let $\Gamma$ denote the deck transformation group of $V$ corresponding to the fundamental group $\pi_1(M)$ of $M$. $\Gamma$ acts freely on $V$.

By the theorem of Bonnet-Myers, the diameter $d(V)$ of $V$ is not greater than $\pi$. Hence $V$ is compact and $\Gamma$ is a finite group. Toponogov's diameter theorem shows that $d(V) = \pi$ holds if and only if $V$ is isometric to the Euclidean unit $m$-sphere $S^m$. By the diameter sphere theorem of Grove and Shiohama ([8]), the diameter $d(M)$ of $M$ is not greater than $\pi/2$. Rigidity theorem due to Gromoll
and Grove ([7]) implies that if \( m \) is odd and \( d(M) = \pi/2 \), then \( M \) is of constant curvature 1.

A nonempty subset \( C \) of \( V \) is called totally \( r \)-convex in \( V (r > 0) \) if for every geodesic \( \gamma : [0, a] \to V \) with \( \gamma(0), \gamma(a) \in C \) and \( 0 < a < r \) we have \( \gamma([0, a]) \subset C \).

Let \( C \) be a connected, compact, totally \( r \)-convex set in \( V \) whose boundary \( \partial C \) is nonempty. The interior of \( C \) is a totally geodesic submanifold embedded in \( V \). We set \( C^a = \{ x \in C \mid d(x, \partial C) \geq a \} \ (a \geq 0) \) and \( p = \max \{ d(x, \partial C) \mid x \in C \} \). Then the set \( \bigcap_{0 \leq a \leq p} C^a \) consists of one point \( s_C \), which is called the soul of \( C \) ([3]). If \( C \) is invariant under an isometry \( \varphi \) of \( V \), then \( s_C \) is a fixed point of \( \varphi \) because \( \varphi \) leaves \( \partial C \) invariant. Hence we have

**Lemma 2.1.** Let \( C \) be a connected, compact, totally \( r \)-convex proper subset in \( V \). If \( C \) is invariant under a fixed point free isometry of \( V \), then \( \dim C \geq 1 \) and \( \partial C = \emptyset \).

Let \( C \) be a compact totally \( \pi \)-convex proper subset in \( V \). If \( C \) is not arcwise connected, then there exist two points \( x, y \in C \) such that \( d(x, y) \geq \pi \). Then by the theorem due to Bonnet-Myers we get \( d(x, y) = \pi \). Hence \( V \) is isometric to \( S^m \) and \( C \) consists of exactly two points. Thus we have

**Lemma 2.2.** Let \( C \) be a compact totally \( \pi \)-convex proper subset in \( V \). If \( C \) contains at least three points, then \( C \) is arcwise connected and \( \dim C \geq 1 \).

Let \( C \) be a connected, compact, totally \( \pi \)-convex proper subset in \( V \) with \( \dim C \geq 1 \). Then any two points of \( C \) can be connected by a minimizing geodesic in \( V \) which is contained in \( C \).

Let \( A \) be a nonempty compact proper subset in \( V \). We set

\[
B = \{ x \in V \mid d(x, A) \geq \pi/2 \}, \quad C = \{ x \in V \mid d(x, B) \geq \pi/2 \}.
\]

Then we shall show the following lemma.

**Lemma 2.3.** Let \( A, B \) and \( C \) be as above. Let \( \Gamma_1 \) be a subgroup of \( \Gamma \) such that \( \Gamma_1 \neq \{ I_V \} \). Assume that \( m(\geq 3) \) is odd. Suppose that \( A \) is invariant under \( \Gamma_1 \) and that \( C \) and \( B \) contain connected, compact submanifolds \( N_1 \) and \( N_2 \) with \( 1 \leq \dim N_1 \), \( \dim N_2 \leq m - 2 \), respectively. Then we have

1. \( B \) and \( C \) are totally \( \pi \)-convex in \( V \) and \( \partial B = \partial C = \emptyset \).
2. If \( x \in B \) and \( y \in C \), then \( d(x, y) = \pi/2 \).
3. \( V \) is isometric to \( S^m \).
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**Proof.** By using the comparison theorem of Toponogov, we can show that both $B$ and $C$ are totally $\pi$-convex in $V$ ([9], [10]). Since $N_1 \subset C$ and $N_2 \subset B$, Lemma 2.2 shows that both $B$ and $C$ are arcwise connected. Since $A$ is invariant under $\Gamma_1$, $B$ and $C$ are also invariant under $\Gamma_1$. Then Lemma 2.1 implies that $\delta B = \delta C = \emptyset$. By using again the comparison theorem of Toponogov, we obtain that $d(x, y) = \pi/2$ for any $x \in B$ and $y \in C$. Let $p_1 : V \to V/\Gamma_1$ be the Riemannian covering of the quotient Riemannian manifold $V/\Gamma_1$. Since $B$ and $C$ are invariant under $\Gamma_1$, the distance between $p_1(B)$ and $p_1(C)$ in $V/\Gamma_1$ is equal to $\pi/2$. Hence we have $d(V/\Gamma_1) = \pi/2$. Since $m$ is odd, by the rigidity theorem ([7]) $V/\Gamma_1$ is of constant curvature 1, and hence $V$ is isometric to $S^m$.

**Lemma 2.4.** Let $N_0$ be an $n$-dimensional connected, compact, totally geodesic submanifold (without boundary) embedded in $V$ with $1 \leq n \leq m - 2$. Let $\Gamma_1$ be a subgroup of $\Gamma$ such that $\Gamma_1 \neq \{I_V\}$. Assume that $m \geq 3$ is odd and that $N_0$ is invariant under $\Gamma_1$. If there exists a point $x_0 \in V$ such that $d(x_0, N_0) \geq \pi/2$, then $V$ is isometric to $S^m$.

**Proof.** We set

$$A_1 = N_0, \quad B_1 = \{x \in V \mid d(x, A_1) \geq \pi/2\}, \quad C_1 = \{x \in V \mid d(x, B_1) \geq \pi/2\}.$$

Then $x_0 \in B_1$ and $A_1 \subset C_1$. Both $B_1$ and $C_1$ are invariant under $\Gamma_1$ because $\Gamma_1$ leaves $A_1$ invariant. By using the comparison theorem of Toponogov, we conclude that both $B_1$ and $C_1$ are totally $\pi$-convex in $V$ ([9], [10]). We shall show that $B_1$ is arcwise connected. To do that, we assume that $B_1$ is not arcwise connected. Since $B_1$ is totally $\pi$-convex, $V$ is isometric to $S^m$ and $B_1$ consists of exactly two points. Hence $C_1$ is isometric to a great $(m - 1)$-sphere $S_1$ in $S^m$. Then $A_1$ is isometric to a great $n$-sphere in $S^m$ which is contained in $S_1$ because $A_1$ is totally geodesic in $V$ and is contained in $C_1$. Since $n \leq m - 2$, there exists a point $x \in C_1$ such that $d(x, A_1) = \pi/2$, which shows $x \in B_1 \cap C_1$. This is a contradiction. Thus $B_1$ is arcwise connected. Since $B_1$ is invariant under $\Gamma_1$, by Lemma 2.1 $B_1$ has no boundary and $\dim B_1 \geq 1$. Similarly $C_1$ has no boundary. By Frankel's theorem ([5]), we have $\dim B_1 + \dim C_1 \leq m - 1$. Since $\dim B_1, \dim C_1 \geq 1$, we obtain that $\dim B_1, \dim C_1 \leq m - 2$. Applying Lemma 2.3 to the present situation, we conclude that $V$ is isometric to $S^m$.

For each $\varphi \in \Gamma$ we set $T(\varphi) = \min\{d(x, \varphi(x)) \mid x \in V\}$. Let $\varphi \in \Gamma \setminus \{I_V\}$. Suppose that the displacement function $d(\cdot, \varphi(\cdot)) : V \to R$ takes the minimum at $x_0 \in V$. Let $\sigma$ be a minimizing geodesic segment from $x_0$ to $\varphi(x_0)$ and $\tilde{\sigma} : R \to V$ the
geodesic extension of \( \sigma \) in the both directions. Then \( \varphi \) translates \( \tilde{\sigma} \), i.e., \( \varphi(\tilde{\sigma}(t)) = \tilde{\sigma}(t + T(\varphi)) \) for all \( t \in \mathbb{R} \). Furthermore \( \tilde{\sigma} : [0, kT(\varphi)] \to V \) is a closed geodesic where \( k \) is the order of \( \varphi \).

Let \( Z(\Gamma) \) be the centralizer of \( \Gamma \) in the full isometry group of \( V \). \( M \) is a homogeneous Riemannian manifold if and only if \( Z(\Gamma) \) acts transitively on \( V \) ([21]; p. 73). We now assume that \( M \) is a homogeneous Riemannian manifold. Then \( V \) is also a homogeneous Riemannian manifold. Each \( \varphi \in \Gamma \) is a Clifford transformation of \( V \), i.e., the displacement function \( d(\cdot, \varphi(\cdot)) : V \to \mathbb{R} \) is a constant function ([21]). Hence for any \( \varphi \in \Gamma \) we have \( T(\varphi) = d(y, \varphi(y)), \ y \in V \). Therefore for any \( x \in M \) and for any \( [\gamma] \in \pi_1(M, x) \setminus \{e\} \) we can choose a closed geodesic as a representation of \( [\gamma] \).

**Lemma 2.5.** Let \( M \) and \( V \) be as above. Assume that \( M \) is a homogeneous Riemannian manifold. Let \( \varphi \in \Gamma \setminus \{I_V\} \). Let \( \sigma : [0, a] \to V \) be a simple closed geodesic satisfying the conditions: (1) \( \sigma(0) = \sigma(a) \), \( T(\varphi) < a \); (2) \( \sigma \) is invariant under \( \varphi \); (3) \( \sigma : [0, T(\varphi)] \to V \) is a minimizing geodesic segment between \( \sigma(0) \) and \( \varphi(\sigma(0)) \). Let \( \Gamma_\sigma \) be the subgroup of \( \Gamma \) whose any element leaves \( \sigma \) invariant. If \( T(\varphi) \leq T(\psi) \) for any \( \psi \in \Gamma_\sigma \setminus \{I_V\} \), then \( \Gamma_\sigma \) is the cyclic group generated by \( \varphi \).

**Proof.** Let \( \Gamma_1 \) be the cyclic group generated by \( \varphi \) and \( k \) its order where \( k \geq 2 \). We put \( x = \sigma(0) \). Then \( \varphi \) translates \( \sigma \) and we have \( \varphi^j(x) = \sigma(jT(\varphi)) \) for each \( j \) \( (0 \leq j \leq k - 1) \) where \( \varphi^j = \varphi \circ \cdots \circ \varphi \) (\( j \) times). Hence we have \( a = kT(\varphi) \). For each \( j \) \( (0 \leq j \leq k - 1) \) \( \sigma : [jT(\varphi), (j + 1)T(\varphi)] \to V \) is a minimizing geodesic segment between \( \varphi^j(x) \) and \( \varphi^{j+1}(x) \). Suppose that there is a \( \psi \in \Gamma_\sigma \setminus \Gamma_1 \). Since \( \sigma \) is invariant under \( \psi \) and \( \Gamma \) acts freely on \( V \), we have \( \psi(x) \in \sigma((jT(\varphi), (j + 1)T(\varphi))) \) for some \( j \) \( (0 \leq j \leq k - 1) \). Since \( \varphi_1 = \varphi^{-j} \circ \psi \) leaves \( \sigma \) invariant and \( \varphi_1(x) \in \sigma((0, T(\varphi))) \), we get \( T(\varphi_1) < T(\varphi) \). This is a contradiction. Hence we have \( \Gamma_\sigma = \Gamma_1 \).

The following lemmas are well known results (for the proof, see [1], [2]).

**Lemma 2.6.** Let \( W \) be an \( m \)-dimensional \( (m \geq 3) \) connected, complete Riemannian manifold with sectional curvature \( K_W \leq \lambda^2 \) \( (\lambda > 0) \) and \( N \) a connected, compact, totally geodesic submanifold embedded in \( W \) such that \( 1 \leq \dim N \leq m - 2 \). Let \( \gamma : [0, \infty) \to W \) be a geodesic. Then we have

1. If \( \gamma(a) \) is the first conjugate point to \( \gamma(0) \) along \( \gamma \), then \( a\lambda \geq \pi \).
2. If \( \gamma(0) \in N \) and the tangent vector \( \gamma'(0) \) is orthogonal to \( N \) and if \( \gamma(a) \) is the first focal point of \( N \) along \( \gamma \), then \( 2a\lambda \geq \pi \).
Lemma 2.7. Let $W$ be as in Lemma 2.6. Let $x$ and $y$ be distinct points of $W$. Suppose that there exist distinct minimizing geodesics $\sigma_1, \sigma_2 : [0, a] \to W$ from $x$ to $y$. If $a = i(x)$ and $\alpha < \pi$, then $\sigma_1'(a) = -\sigma_2'(a)$. 

Lemma 2.8. Let $W$ and $N$ be as in Lemma 2.6. Let $x \in W \setminus N$. Suppose that there exist distinct minimizing geodesics $\sigma_1, \sigma_2 : [0, a] \to W$ from $x$ to $N$. If $a = i(N)$ and $2a\lambda < \pi$, then $\sigma_1'(0) = -\sigma_2'(0)$. 

The following theorem will be used in the proof of Theorems B and C.

Theorem 2.1 ([2], [4], [12]). Let $W$ be an $m$-dimensional ($m \geq 2$) connected, complete, simply connected Riemannian manifold with $1 \leq K_W \leq 4$. Then we have

1. $\text{Inj}(W) \geq \pi/2$.
2. If $\text{d}(W) = \pi/2$ and $m \geq 3$ is odd, then $W$ is isometric to the Euclidean $m$-sphere $S^m(4)$ with constant curvature $4$.

3. Proof of Theorem A

Throughout this section let $M$ be an $m$-dimensional ($m \geq 3$) connected, compact, non-simply connected homogeneous Riemannian manifold whose sectional curvature $K_M$ satisfies $K_M \geq 1$ and $N$ an $n$-dimensional ($n \geq 1$) connected, compact, totally geodesic submanifold (without boundary) embedded in $M$.

Let $\iota : \pi_1(N) \to \pi_1(M)$ be the homomorphism which is induced from the inclusion $i : N \to M$. Let $\pi_1(M, N)$ denote the first relative homotopy class. For the sake of convenience we write $\pi_1(M, N) = 0$ if $\iota$ is surjective and $\pi_1(M, N) \neq 0$ otherwise. As stated in Section 1, if $2n \geq m$, then we have $\pi_1(M, N) = 0$.

Let $p : V \to M$ denote the universal Riemannian covering. $V$ is also compact homogeneous Riemannian manifold with $K_V \geq 1$. We denote by $\Gamma$ the deck transformation group of $V$ corresponding to $\pi_1(M)$. Let $\Gamma_0$ be the subgroup of $\Gamma$ which corresponds to $\iota_2(\pi_1(N))$. If $\pi_1(M, N) \neq 0$, then $p^{-1}(N)$ has at least two connected components and we have $2i(N) \leq d(N_1, N_2)$ for any distinct connected components $N_1$ and $N_2$ of $p^{-1}(N)$. Let $N_0$ be a connected component of $p^{-1}(N)$. Let $\varphi \in \Gamma$. Then $\varphi$ is contained in $\Gamma_0$ if and only if $N_0$ is invariant under $\varphi$.

For $\varphi_1, \ldots, \varphi_k \in \Gamma$ we will denote by $\Gamma(\varphi_1, \ldots, \varphi_k)$ the subgroup of $\Gamma$ generated by $\varphi_1, \ldots, \varphi_k$.

In order to prove Theorem A, we prepare several lemmas.
Lemma 3.1. Assume that $\Gamma$ is not cyclic and $\Gamma_0 = \{I_V\}$. Then $i(N) \leq \pi/4$. Here if equality holds, then $M$ is of constant curvature 1 and $\Gamma \cong Q_8$.

Proof. Assuming that $i(N) \geq \pi/4$, we shall show that $i(N) = \pi/4$ and $K_M \equiv 1$. Let $N_0$ be a connected component of $p^{-1}(N)$ and fix it. By the assumption on $\Gamma_0$, we have $\varphi(N_0) \cap N_0 = \varnothing$ for all $\varphi \in \Gamma \setminus \{I_V\}$. Since $\Gamma$ is not cyclic, $m$ is odd by Synge's theorem and $\Gamma$ contains a proper subgroup. The assumption that $i(N) \geq \pi/4$ implies that $d(N_0, \psi(N_0)) \geq \pi/2$ for all $\psi \in \Gamma \setminus \{I_V\}$. Let $\Gamma_1$ be an arbitrary proper subgroup of $\Gamma$. We set
\[ A = \bigcup_{\varphi \in \Gamma_1} \varphi(N_0), \quad B = \{x \in V \mid d(x, A) \geq \pi/2\}, \quad C = \{x \in V \mid d(x, B) \geq \pi/2\}. \]
Then $A$ is invariant under $\Gamma_1$ and we have $\psi(N_0) \subset B$ for all $\psi \in \Gamma \setminus \Gamma_1$. We can apply Lemma 2.3 to the present situation. Hence the assertions (1), (2) and (3) in Lemma 2.3 hold for the present situation. Thus $M$ is of constant curvature 1. Let $\psi \in \Gamma \setminus \Gamma_1$. By Lemma 2.3 (2), we have $d(x, y) = \pi/2$ for any $x \in N_0$ and $y \in \psi(N_0)$. This shows that $T(\psi) = \pi/2$ and $d(N_0, \psi(N_0)) = \pi/2$. Hence we have $i(N) = \pi/4$.

In the following we assume that $i(N) = \pi/4$. We shall show that $\Gamma \cong Q_8$. It follows from the argument above that there exists a $\varphi \in \Gamma$ with $T(\varphi) = \pi/2$ and $T(\psi) = \pi/2$ holds for all $\psi \in \Gamma \setminus \Gamma(\varphi)$. From now on we identify $V$ with $S^m$ and view $\Gamma$ as a finite subgroup of the orthogonal group $O(m + 1)$. By homogeneity of $V$, $\Gamma$ is a Clifford transformation group of $S^m$. We take a $\varphi_1 \in \Gamma$ with $T(\varphi_1) = \pi/2$ and fix it. For each $x \in S^m$, $\varphi_1$ translates the great circle in $S^m$ passing through $x$ and $\varphi_1(x)$. Hence $\varphi_1$ has the properties that $\varphi_1^2 = -I$ and $\Gamma(\varphi_1) \cong Z_4$, where $I$ denotes the unit $(m + 1)$-matrix. Each $\psi \in \Gamma \setminus \Gamma(\varphi_1)$ has the properties that $\Gamma(\psi) = \pi/2$, $\psi^2 = -I$ and $\Gamma(\psi) \cong Z_4$. Let $\varphi_2 \in \Gamma \setminus \Gamma(\varphi_1)$ and fix it. We have the relations $(\varphi_1 \varphi_2)^2 = \varphi_1^2 = \varphi_2^2 = -I$ since $\varphi_1 \varphi_2 \notin \Gamma(\varphi_1)$. By using these relations, we obtain that $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$ and $\varphi_2 \varphi_1 \varphi_2 = \varphi_1$. This shows that $\Gamma(\varphi_1, \varphi_2) \cong Q_8$. We put $\Gamma_2 = \Gamma(\varphi_1, \varphi_2)$. We now assume that $\Gamma \neq \Gamma_2$. Take a $\varphi \in \Gamma \setminus \Gamma_2$. Since $\varphi_1 \varphi, \varphi_2 \varphi$ and $\varphi_1 \varphi_2 \varphi$ are not contained in $\Gamma_2$, we obtain that $(\varphi_1 \varphi_2 \varphi)^2 = (\varphi_1 \varphi)^2 = (\varphi_2 \varphi)^2 = \varphi_1^2 = \varphi^2 = -I$. The relation $(\varphi_1 \varphi)^2 = \varphi_1$ implies $\varphi_1 \varphi_1 = \varphi$. By using the relations that $(\varphi_1 \varphi_2 \varphi)^2 = \varphi_1^2$, $\varphi_1 \varphi \varphi_1 = \varphi$ and $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$, we get $(\varphi_2 \varphi)^2 = I$. This is a contradiction. Hence we have $\Gamma = \Gamma_2$, which shows that $\Gamma \cong Q_8$.

Lemma 3.2. Suppose that $\Gamma$ is not cyclic and that $\Gamma_0$ is a proper subgroup of $\Gamma$. Then $i(N) \leq \pi/4$. Here if equality holds, then $M$ is of constant curvature 1 and furthermore, identifying $V$ with $S^m$ and viewing $\Gamma$ as a finite subgroup of $O(m + 1)$, we have
(1) If $\psi \in \Gamma \backslash \Gamma_0$, then $\psi^2 = -I \in \Gamma_0$ and $\Gamma(\psi) \cong \mathbb{Z}_4$.
(2) If $\psi \in \Gamma \backslash \Gamma_0$ and $\varphi \in \Gamma_0$, then $\varphi \psi \varphi = \psi$.

**Proof.** As in the proof of Lemma 3.1 we fix a connected component $N_0$ of $p^{-1}(N)$. Then $N_0$ is invariant under $\Gamma_0$. Suppose that $i(N) \geq \pi/4$. We shall show that $i(N) = \pi/4$ and $K_M \equiv 1$. We set

$$A = N_0, \quad B = \{x \in V \mid d(x, A) \geq \pi/2\}, \quad C = \{x \in V \mid d(x, B) \geq \pi/2\}.$$

Let $\psi \in \Gamma \backslash \Gamma_0$. We have $d(N_0, \psi(N_0)) \geq \pi/2$ because $i(N) \geq \pi/4$. Thus we have $\psi(N_0) \in B$ for all $\psi \in \Gamma \backslash \Gamma_0$. The order of $\Gamma$ is greater than 2 since $\Gamma_0$ is a proper subgroup of $\Gamma$. Hence $m \geq 3$ is odd by Synge's theorem. By applying Lemma 2.3 to the present situation, we conclude that $i(N) = \pi/4$ and $K_M \equiv 1$. From now on we assume that $i(N) = \pi/4$. By Lemma 2.3, both $B$ and $C$ are totally geodesic submanifolds of $V$ without boundary and we obtain $d(x, y) = \pi/2$ for any $x \in B$ and $y \in C$. We identify $V$ with $S^m$ and view $\Gamma$ as a finite subgroup of $O(m+1)$.

Then $N_0$ is a great $n$-sphere in $S^m$ and $B$ is a great $(m-n-1)$-sphere in $S^m$. Hence we have $N_0 = C$ by the definition of $C$. Let $\psi \in \Gamma \backslash \Gamma_0$. Since $\psi(N_0) \in B$ and $\psi$ is a Clifford transformation, we have $T(\psi) = \pi/2$. Let $x \in N_0$. Then $\psi$ translates the great circle in $S^m$ passing through $x$ and $\psi(x)$. Hence $\psi^2(x) = -x \in N_0$. Since $\Gamma$ acts freely on $S^m$ and $\psi^2$ leaves $N_0$ invariant, we obtain that $\psi^2 = -I \in \Gamma_0$ and $\Gamma(\psi) \cong \mathbb{Z}_4$. This shows (1). Next we shall show (2). Let $\psi \in \Gamma \backslash \Gamma_0$ and $\varphi \in \Gamma_0$. We may assume that $\varphi \neq \pm I$. Since $\varphi \psi \varphi = \psi^2$, we get $\varphi \psi \varphi = \psi$.

**Lemma 3.3.** Suppose that $\Gamma$ and $\Gamma_0$ satisfy the same hypotheses as in Lemma 3.2 and that $i(N) = \pi/4$. Then $\Gamma_0$ is a cyclic group of order $2k$ ($k \geq 1$).

**Proof.** By Lemma 3.2 we can identify $V$ with $S^m$. Then $\Gamma$ can be viewed as a finite subgroup of $O(m+1)$. Let $N_0$ be a connected component of $p^{-1}(N)$ and fix it. Then $N_0$ is a great $n$-sphere in $S^m$ and is invariant under $\Gamma_0$. We may assume that $N_0 = S^n = S^m \cap R^{n+1}$. We take a $\psi_1 \in \Gamma_0 \backslash \{I\}$ such that $T(\psi_1) \leq T(\varphi)$ for all $\varphi \in \Gamma_0 \backslash \{I\}$. We shall show that $\Gamma_0 = \Gamma(\psi_1)$. We first assume that $\psi_1 = -I$. Since $T(\psi_1) = \pi$, we have $T(\varphi) = \pi$ for all $\varphi \in \Gamma_0 \backslash \{I\}$. This implies that $\Gamma_0 = \{I, \psi_1\} = \Gamma(\psi_1)$. We next assume that $\psi_1 \neq -I$. In case of $n = 1$, by Lemma 2.5 $\Gamma_0$ is the cyclic group generated by $\psi_1$. From now on, let $n \geq 2$. Let $C_1$ be the great circle in $S^n$ which contains $x_0$ and $\psi_1(x_0)$. Then $C_1$ is invariant under $\psi_1$. Since $T(\psi_1) < \pi$, the order of $\Gamma_0$ is greater than 2. Hence $n (\geq 3)$ is odd by Synge's theorem. Let $n = 2q + 1$, $q \geq 1$. Since $-I \in \Gamma_0$ and $(-I)(C_1) = C_1$, by
Lemma 2.5 there exists the smallest positive integer \( k \geq 2 \) such that \((\psi_1)^k = -I\). Hence the order of \( \Gamma(\psi_1) \) is equal to \(2k\). Let \( x_0 \in N_0 \) and \( \psi_2 \in \Gamma \setminus \Gamma_0 \), and fix them. Lemma 3.2 shows that \((\psi_2)^2 = -I \in \Gamma_0 \) and \( \varphi \psi_2 \varphi = \psi_2 \) for all \( \varphi \in \Gamma_0 \). Let \( \varphi \in \Gamma_0 \setminus \{ \pm I, \psi_1 \} \). Since \( \psi_1 \psi_2 \varphi \) and \( \psi_2 \psi_2 \) are not contained in \( \Gamma_0 \), by Lemma 3.2 (1) we have the relation \((\psi_1 \psi_2 \varphi)^2 = (\psi_1 \psi_2 \psi_2)^2\). By combining this with the relation \( \varphi \psi_2 \varphi = \psi_2 \), we obtain \( \varphi \psi_1 = \psi_1 \psi_2 \). Then there exists a complex vector \( \xi \in S^n \subset C^{q+1} \) which is a common eigenvector of \( \varphi \) and \( \psi_1 \). Let \( C_2 \) be the great circle in \( S^n \) determined by \( \xi \) and \( \bar{\xi} \) where \( \bar{\xi} \) is the conjugate vector of \( \xi \) in \( C^{q+1} \). Then \( C_2 \) is invariant under \( \varphi \) and \( \psi_1 \). By Lemma 2.5 we have \( \varphi \in \Gamma(\psi_1) \). Hence we have \( \Gamma_0 = \Gamma(\psi_1) \). Thus \( \Gamma_0 \) is the cyclic group of order \(2k\) generated by \( \psi_1 \) \((k \geq 1)\).

**Lemma 3.4.** Suppose that \( \Gamma \) and \( \Gamma_0 \) satisfy the same hypotheses as in Lemma 3.2 and that \( i(N) = \pi/4 \). Then \( \Gamma \cong D^*_s \) \((s \geq 2)\).

**Proof.** By Lemma 3.2 we may identify \( V \) with \( S^m \) and view \( \Gamma \) as a finite subgroup of \( O(m+1) \). We take a \( \psi_1 \in \Gamma_0 \setminus \{ I \} \) such that \( T(\psi_1) \leq T(\varphi) \) for all \( \varphi \in \Gamma_0 \setminus \{ I \} \). As we have shown in Lemma 3.3, \( \Gamma_0 \) is a cyclic group generated by \( \psi_1 \) with order \(2k \) \((k \geq 1)\) and \((\psi_1)^k = -I \). Let \( \psi_2 \in \Gamma \setminus \Gamma_0 \). It follows from Lemma 3.2 that \((\psi_2)^2 = -I \), \( \psi_1 \psi_2 \psi_1 = \psi_2 \) and \( \Gamma(\psi_2) \cong \mathbb{Z}_4 \). We first consider the case \( k \geq 2 \). Then we have \( \Gamma(\psi_1, \psi_2) \cong D^*_s \). We shall show that \( \Gamma = \Gamma(\psi_1, \psi_2) \).

To do that, we assume that there exists a \( \varphi \in \Gamma \setminus \Gamma(\psi_1, \psi_2) \). Since \( \psi_2 \notin \Gamma(\psi_1) \) and \( \psi_2 \psi_2 \notin \Gamma(\psi_1, \psi_2) \), by Lemma 3.2 we obtain that \( \psi_1 \psi_2 \psi_1 = \psi_2 \), \( \psi_1 \varphi \psi_1 = \varphi \) and \( \psi_1 \psi_2 \psi_2 \psi_1 = \psi_2 \varphi \). By using these relations, we get \((\psi_1)^2 = I \). This is a contradiction because \((\psi_1)^{2k} = I \) and \( k \geq 2 \). Thus we have \( \Gamma = \Gamma(\psi_1, \psi_2) \).

Next let us consider the case \( k = 1 \). Then \( \psi_1 = -I \) and \( \Gamma_0 = \Gamma(\psi_2) \). We take a \( \psi_3 \in \Gamma \setminus \Gamma(\psi_2) \). By Lemma 3.2 (1) we obtain that \((\psi_2 \psi_3)^2 = (\psi_2)^2 = (\psi_3)^2 = -I \). These relations yield that \( \psi_2 \psi_3 \psi_2 = \psi_3 \) and \( \psi_3 \psi_2 \psi_3 = \psi_2 \). Hence we have \( \Gamma(\psi_2, \psi_3) \cong \mathbb{Q}_8 \). By the same way as in the proof of Lemma 3.1, we can show that \( \Gamma = \Gamma(\psi_2, \psi_3) \). Thus we have \( \Gamma \cong D^*_s \) \((s \geq 2)\).

**Lemma 3.5.** Suppose that \( \Gamma \) is not a cyclic group and that \( \Gamma_0 \neq \Gamma \). If \( i(N) = \pi/4 \), then \( m = 4j - 1 \) \((j \geq 1)\).

**Proof.** By Lemmas 3.1 and 3.2, \( V \) is isometric to \( S^m \) and \( \Gamma \) is isomorphic to \( D^*_s \) \((s \geq 2)\). In the case where \( \Gamma_0 \) is trivial, \( \Gamma \) is isomorphic to \( \mathbb{Q}_8 \). If \( \Gamma_0 \) is nontrivial, then \( \Gamma_0 \) is a cyclic group of order \(2k \) \((k \geq 1)\) by Lemma 3.3. We identify \( V \) with \( S^m \) and view \( \Gamma \) as a finite subgroup of \( O(m+1) \). As we have shown in the proofs of Lemmas 3.1 and 3.4, we can choose a generator \( \{ \varphi_1, \varphi_2 \} \)
of $\Gamma$ as follows. In the case where $\Gamma_0 = \{I\}$ or $\Gamma_0 = \{I,-I\}$, $\varphi_1$ and $\varphi_2$ have the properties that $T(\varphi_1) = T(\varphi_2) = \pi/2$ and $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$, $\varphi_2 \varphi_1 \varphi_2 = \varphi_1$. If the order of $\Gamma_0$ is greater that 2, then $\varphi_1$ is a generator of $\Gamma_0$ and $T(\varphi_1) = \pi/k$ $(k \geq 2)$, $T(\varphi_2) = \pi/2$. In this case $\varphi_1$ and $\varphi_2$ satisfy the relations $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$, $(\varphi_1)^k = (\varphi_2)^2 = -I$. Let $x \in S^m$. For $\varphi_i$ $(i = 1,2)$ let $C_i$ be the great circle in $S^m$ passing through $x$ and $\varphi_i(x)$. Then $C_i$ is invariant under $\varphi_i$, $i = 1,2$. Let $C_3 = \varphi_2(C_1)$. Since $T(\varphi_2) = \pi/2$ and $\varphi_2 \notin \Gamma(\varphi_1)$, we have $C_1 \cap C_3 = \emptyset$. The relations $\varphi_1 \varphi_2 \varphi_1 = \varphi_2$ and $(\varphi_2)^2 = -I$ imply that $\varphi_1(C_3) = C_3$ and $\varphi_2(C_3) = (-I)(C_1) = C_1$. Let $W_i$ be the 2-dimensional subspace in $R^{m+1}$ such that $C_i = W_i \cap S^m$ $(i = 1,2,3)$. We set $W_4 = W_1 \oplus W_3$. Then $W_2$ is contained in $W_4$ and both $\varphi_1$ and $\varphi_2$ leave $W_4$ invariant. Since $\Gamma$ is generated by $\varphi_1$ and $\varphi_2$, $W_4$ is $\Gamma$-invariant. Hence for any $x \in S^m$ there exists a $\Gamma$-invariant 4-dimensional subspace of $R^{m+1}$ containing $x$. Thus $R^{m+1}$ can be expressed as a direct sum of $\Gamma$-invariant 4-dimensional subspaces, which implies that $m = 4j - 1$ $(j \geq 1)$.

**Proof of Theorem A.** Lemmas 3.1 and 3.2 show that $i(N) \leq \pi/4$. Suppose $i(N) = \pi/4$. Then $M$ is of constant curvature 1. Moreover Lemmas 3.1, 3.4 and 3.5 imply that $\pi_1(M) \cong D^s_1$ $(s \geq 2)$ and $m = 4j - 1$ ($j \geq 1$).

4. Proof of Theorems B and C

First of all we state a theorem which will be used in the proof of Theorem C.

**Theorem 4.1 ([11]).** Let $M$ be an $m$-dimensional $(m \geq 3)$ connected, compact, non-simply connected Riemannian manifold with sectional curvature $K_M \geq 1$. Suppose that the order of $\pi_1(M)$ is not a prime. Then $\text{Inj}(M) \leq \pi/4$. If equality holds, then $M$ is of constant curvature 1 and $\pi_1(M)$ is isomorphic to either $Z_4$ or $Q_8$. Here if $\pi_1(M) \cong Q_8$, then $m = 4j - 1$ ($j \geq 1$).

Throughout this section let $M$ denote an $m$-dimensional $(m \geq 3)$ connected, compact, non-simply connected homogeneous Riemannian manifold whose sectional curvature $K_M$ satisfies $1 \leq K_M \leq 4$. If $m$ is even, then $\pi_1(M)$ is isomorphic to $Z_2$ by Synge's theorem. In the following we assume that $m$ $(\geq 3)$ is odd, unless otherwise stated. Then $M$ is orientable. Let $p : V \rightarrow M$ be the universal Riemannian covering and $\Gamma$ the deck transformation group corresponding to $\pi_1(M)$. Let $G$ denote the identity connected component of the full isometry group of $M$. $G$ is a compact Lie group with respect to the compact open topology. $G$ also acts on $M$ transitively. We take an $x_0 \in M$ and fix it in the
following. Let $H$ be the isotropy subgroup of $G$ at $x_0$. The action $\Psi : G \times M \to M \ ((\varphi, x) \mapsto \varphi(x))$ on $M$ on the left induces a diffeomorphism $\hat{\Psi} : G/H \to M (\varphi H \mapsto \varphi(x_0))$.

**Lemma 4.1.** Let $G$ and $H$ be as above. If $\dim H = 0$, then $S^3$ is a covering space of $M$ and $\Gamma$ is isomorphic to a finite subgroup of $SU(2)$.

**Proof.** We identify $M$ with $G/H$. By assumption, $H$ is a finite subgroup of $G$. Hence the natural projection $p_1 : G \to G/H$ is a covering map. Let $\hat{G}$ be the universal covering Lie group of $G$ with covering homomorphism $p_2$. Then $\hat{p} := p_1 \circ p_2 : \hat{G} \to G/H$ is a universal covering map and $\Gamma$ is isomorphic to $p_2^{-1}(H)$. Hence $\hat{G}$ is compact. Let $\hat{g}$ be the Riemannian metric on $\hat{G}$ induced from that of $G/H$ by $\hat{p}$. Then $\hat{g}$ is a left invariant metric on $\hat{G}$ and each sectional curvature $K$ of $(\hat{G}, \hat{g})$ satisfies $1 \leq K \leq 4$. By a theorem due to Wallach ([20]; Theorem 2.1), $\hat{G}$ is isomorphic to $SU(2)$ as a Lie group. This completes the proof.

In what follows we assume that $\dim H \geq 1$. Any nontrivial one-parameter subgroup of $H$ induces a nontrivial Killing vector field on $M$ which vanishes at $x_0$. Let $X$ be a nontrivial Killing vector field on $M$ vanishing at $x_0$. Let $L$ be the set of all points of $M$ at which $X$ vanishes. Each connected component of $L$ is a compact totally geodesic submanifold (without boundary) embedded in $M$ whose codimension is even ([13]; p. 59). Hence the dimension of each connected component of $L$ is odd since $m$ is odd.

Under the condition that $1 \leq K_M \leq 4$, $L$ has the following properties.

**Lemma 4.2.** Let $M$ and $L$ be as above. Then

1. $L$ is connected.
2. $L$ is totally $\pi/2$-convex in $M$.
3. $i(L) \geq \pi/4$.

**Proof.** Suppose that $L$ is disconnected. Let $L_1, \ldots, L_s$ ($s \geq 2$) be the distinct connected components of $L$. By exchanging indices if necessary, we may assume that $d(L_1, L_2) \leq d(L_i, L_j)$, $1 \leq i < j \leq s$. Let $\sigma : [0, a] \to M$ be a minimizing geodesic segment between $L_1$ and $L_2$ such that $\sigma(0) \in L_1$ and $\sigma(a) \in L_2$ where $a = d(L_1, L_2)$. Then $X$ is a Jacobi field along $\sigma$ which vanishes at $\sigma(0)$ and $\sigma(a)$. We note here that $X$ does not vanish at $\sigma(t)$, $0 < t < a$. Since $K_M \leq 4$, by Lemma 2.6 (1) we get $a \geq \pi/2$. Hence we have $d(M) = a = \pi/2$ because $d(M) \leq \pi/2$. By the
rigidity theorem ([7], [11]), $M$ is of constant curvature 1 because $K_M \geq 1$ and $m$ is odd. Since $\sigma(a)$ is the first conjugate point to $\sigma(0)$ along $\sigma$, it must be $a = \pi$. This is a contradiction, which implies (1).

Let $\gamma : [0, b] \to M$ be a geodesic segment such that $\gamma(0), \gamma(b) \in L$ and $\gamma([0, b]) \not\subset L$. Since $X$ is a nontrivial Jacobi along $\gamma$, we have $b \geq \pi/2$ by Lemma 2.6 (1). This proves (2).

To show (3), we suppose that $r := i(L) < \pi/4$. Let $x$ be a cut point of $L$ with $d(x, L) = r$. It follows from Lemma 2.6 (2) that for each geodesic $\gamma : [0, \infty) \to M$ emanating orthogonally from $L$ $\gamma(r)$ is not a focal point of $L$ along $\gamma$. Hence there exist distinct minimizing geodesics $\sigma_1, \sigma_2 : [0, r] \to M$ from $x$ to $L$. By Lemma 2.8 we have $\sigma_2'(0) = -\sigma_1'(0)$. Thus there exists a geodesic $\sigma : [0, 2r] \to M$ such that $\sigma(0), \sigma(2r) \in L$ and $\sigma([0, 2r]) \cap L = \emptyset$. Since $L$ is totally $\pi/2$-convex in $M$, we have $2r \geq \pi/2$, which is a contradiction. This shows (3).

Let $L$ be as above. By homogeneity of $M$, $L$ is a homogeneous Riemannian manifold ([14]; p. 60).

**Lemma 4.3.** $M$ contains an embedded, connected, compact, totally geodesic submanifold $N$ (without boundary) with the following properties:

1. $\dim N$ is either 1 or 3.
2. It is totally $\pi/2$-convex in $M$.
3. $i(N) \geq \pi/4$.
4. If $\dim N = 3$, then any nontrivial Killing vector field on $N$ nowhere vanishes.

**Proof.** Let $L$ be as above. As stated above, $\dim L$ is odd and $\operatorname{codim} L$ is even. In the case where $\dim L = 1$, we let $N = L$. Then the claim follows from Lemma 4.2. In the following we assume that $\dim L \geq 3$. We first consider the case where any nontrivial Killing field on $L$ nowhere vanishes. Then the isotropy subgroup of the isometry group of $L$ at $x_0$ is a discrete group. Lemma 4.1 shows $\dim L = 3$. Setting $N = L$, we obtain a submanifold with the required properties. Next let us consider the case where there exists a nontrivial Killing vector field $X_1$ on $L$ vanishing at some point. Let $L_1$ be the set of all points of $L$ at which $X_1$ vanishes. Then $\dim L_1$ is odd and $\dim L_1 \geq 1$. $L_1$ has the properties (1), (2) and (3) in Lemma 4.2 as a submanifold of $L$. Since $L$ is totally $\pi/2$-convex in $M$, so is $L_1$. Moreover $L_1$ is a connected, compact, totally geodesic submanifold (without boundary) embedded in $M$. By the same way as in the proof of Lemma 4.2 (3), we obtain $i(L_1) \geq \pi/4$ as a submanifold of $M$. If $\dim L_1 \geq 3$, then in $L_1$ we can
carry out the same argument as above. By repeating the argument above, we obtain a submanifold $N$ of $M$ which has the required properties.

From now on let $N$ denote a connected, compact, totally geodesic submanifold (without boundary) embedded in $M$ with the properties stated in Lemma 4.3. Since $M$ is homogeneous, we may assume that $x_0 \in N$. $N$ is also a homogeneous Riemannian manifold. Let $G_1$ be the identity connected component of the isometry group of $N$ and $H_1$ the isotropy subgroup of $G_1$ at $x_0$. $G_1$ is a compact Lie group and acts transitively on $N$.

With the notations stated above, we have

**Lemma 4.4.** Assume that $\dim N = 3$. Then

1. $H_1$ is a finite group.
2. $N$ is covered by $S^3$.
3. $\pi_1(N)$ is isomorphic to a finite subgroup of $SU(2)$.

**Proof.** If $\dim H_1 \geq 1$, then each nontrivial one-parameter subgroup of $H$ induces a nontrivial Killing vector field which vanishes at $x_0$. This contradicts Lemma 4.3 (4). Hence $\dim H_1 = 0$ and $H_1$ is a finite group. Then (2) and (3) follow from (1) and Lemma 4.1.

The following is evident.

**Lemma 4.5.** If $\dim N = 1$ and $\pi_1(M, N) = 0$, then $\Gamma$ is a cyclic group.

From Theorem A and Lemma 4.3 we have

**Lemma 4.6.** If $\pi_1(M, N) \neq 0$, then $\Gamma$ is isomorphic to either a cyclic group or a binary dihedral group. Moreover if $\pi_1(M, N) \neq 0$ and $\Gamma$ is a binary dihedral group, then $m = 4j - 1$ ($j \geq 1$).

**Proof.** Suppose that $\Gamma$ is not cyclic. Theorem A and Lemma 4.3 (3) imply that $i(N) = \pi/4$. Then Theorem A shows that $\Gamma \cong D_s^*$ ($s \geq 2$) and $m = 4j - 1$ ($j \geq 1$).

**Lemma 4.7.** Suppose that $\dim N = 3$ and $\pi_1(M, N) = 0$. Then $\pi_1(M) \cong \pi_1(N)$.

**Proof.** Let $t_2 : \pi_1(N, x_0) \to \pi_1(M, x_0)$ be the homomorphism induced from the inclusion $i : N \to M$. We take an $x_1 \in p^{-1}(x_0)$ and fix it. By assumption, it suffices to show that $t_2$ is injective. To do that, we suppose that $\ker t_2 \neq \{e\}$. Let $[\gamma] \in \ker t_2 \setminus \{e\}$. By assumption, $\tilde{N} := p^{-1}(N)$ is connected and $\Gamma$-invariant. Let $\tilde{\gamma} :
[0, a] → V be the lift of γ emanating from x₁. Since γ is homotopic to the point curve x₀ in M, ̇γ is a loop in ̂N. Hence ̂N is not simply connected since ̇γ is homotopically nontrivial in ̂N. Thus the intrinsic diameter of ̂N is not greater than π/2 ([8]). Therefore we have d(z, w) ≤ π/2 for any z, w ∈ ̂N. Let x, y be points of V such that d(x, y) = d(V). Then we have d(x, y) ≥ π/2 because 1 ≤ K_M ≤ 4 (Theorem 2.1 (1)). We shall show that d(x, y) = π/2. By homogeneity of V we may assume that x ∈ ̂N. If y ∈ ̂N, then d(x, y) = π/2. Let y ̸∈ ̂N. If d(y, ̂N) ≥ π/2, then by applying Lemma 2.4 to the present situation we conclude that V is isometric to S^n. Since ̂N is totally geodesic, it is isometric to S^3, which contradicts that ̂N is non-simply connected. Thus we have d(y, ̂N) < π/2. Let σ₁ : [0, a] → V be a minimizing geodesic between ̂N and y such that σ₁(0) ∈ ̂N and σ₁(a) = y where 0 < a < π/2. Let σ₂ : [0, b] → ̂N be a minimizing geodesic in ̂N from σ₁(0) to x where 0 < b ≤ π/2. Then σ₁(0) is orthogonal to σ₂(0). By applying Toponogov’s comparison theorem to the hinge (σ₁, σ₂, π/2), we obtain d(x, y) ≤ π/2. Hence it must be d(x, y) = π/2. Thus we have d(V) = π/2. By Berger’s minimal diameter theorem (Theorem 2.1 (2)), V is isometric to m-sphere S^m(4) with constant curvature 4. Then ̂N is isometric to 3-sphere S^3(4) with constant curvature 4 because ̂N is totally geodesic in V. This is a contradiction. Thus we have ker i₁ = {e}, which shows that π₁(M) ≃ π₁(N).

**Lemma 4.8.** Suppose that dim N = 3 and that there exists a φ ∈ G such that φ(N) ≠ N and φ(N) ∩ N ≠ ∅. Moreover assume that Γ is not cyclic. Then m = 4j - 1 (j ≥ 2).

**Proof.** Let N₁ be a connected component of φ(N) ∩ N. From the property of N (Lemma 4.3 (2)), N₁ is totally π/2-convex in M. Moreover N₁ is a compact, totally geodesic submanifold (without boundary) embedded in M. Since dim N = 3 and φ(N) ≠ N, we have dim N₁ ≤ 2. By the same way as in the proof of Lemma 4.2 (3), the inequality i(N₁) ≥ π/4 holds as a submanifold of M. We first assume that dim N₁ = 0. By homogeneity of M we obtain that Inj(M) ≥ π/4. Since Γ is not cyclic, Theorem 4.1 shows that Inj(M) = π/4 and m = 4j - 1 (j ≥ 2). If dim N₁ = 1, then π₁(M, N₁) ≠ 0 because Γ is not cyclic. If dim N₁ = 2, then the order of π₁(N₁) is at most two, which implies π₁(M, N₁) ≠ 0. Hence we have π₁(M, N₁) ≠ 0 if 1 ≤ dim N₁ ≤ 2. Then Theorem A shows that m = 4j - 1 (j ≥ 2).

As a consequence of Lemma 4.8, we have

**Lemma 4.9.** Assume that m = 5 and dim N = 3. Then Γ is a cyclic group.
PROOF. Suppose that $\Gamma$ is not cyclic. Let $x \in M \setminus \mathcal{N}$. By homogeneity of $M$, there exists a $\phi \in G$ such that $\phi(x_0) = x$. Clearly, we have $\phi(\mathcal{N}) \neq \mathcal{N}$. It follows from our assumption and Frankel's theorem ([5]) that $\phi(\mathcal{N}) \cap N \neq \emptyset$. Then Lemma 4.8 shows that $m = 4j - 1$ ($j \geq 2$), which is a contradiction. Thus $\Gamma$ is a cyclic group.

We shall prove Theorems B and C. We use the same notations as above.

PROOF OF THEOREM B. Let $\mathcal{N}$ be as above. By Lemmas 4.5 and 4.6 it suffices to consider the case where $\dim \mathcal{N} = 3$ and $\pi_1(M, \mathcal{N}) = 0$. It follows from Lemmas 4.4 (3) and 4.7 that $\Gamma$ is isomorphic to a finite subgroup of $SU(2)$.

PROOF OF THEOREM C. We suppose that $\Gamma$ is not cyclic. Let $\mathcal{N}$ be as above. Since $m = 4j + 1$ ($j \geq 1$), Lemmas 4.5 and 4.6 imply that $\dim \mathcal{N} = 3$ and $\pi_1(M, \mathcal{N}) = 0$. By Lemma 4.9 we may assume that $m = 4j + 1 \geq 9$. It follows from Lemma 4.8 that $\phi(\mathcal{N}) \cap N = \emptyset$ or $\phi(\mathcal{N}) = \mathcal{N}$ for all $\phi \in G$. Thus we have $\phi(\mathcal{N}) = \mathcal{N}$ for all $\phi \in H$. Let $T \subset T_{x_0} M$ be the tangent space to $N$ at $x_0$. Let $\phi, \psi \in G$ be such that $\phi(x_0) = \psi(x_0)$. Since $\phi(\mathcal{N}) = \psi(\mathcal{N})$, we have $(d\phi)_{x_0}(T) = (d\psi)_{x_0}(T)$. Hence the action $\Psi : G \times M \to M$ ($(\phi, x) \mapsto \phi(x)$) induces a smooth field of 3-planes on $M$. This field of 3-planes can be lifted to $V$. Since $V$ is homeomorphic to $S^{4j+1}$ by the sphere theorem ([2], [8]), there exists a continuous field of 3-planes on $S^{4j+1}$. But this is a contradiction because $S^{4j+1}$ does not admit a continuous field of 3-planes ([19]; p. 144). Therefore $\Gamma$ is a cyclic group.

5. Examples

We give examples of connected, compact, non-simply connected homogeneous Riemannian manifolds whose sectional curvature $K$ satisfies $\delta A \leq K \leq A$, where $A$ and $\delta$ are positive constants and $1/4 \leq \delta < 1$. These manifolds are obtained as quotient spaces of Berger spheres.

5.1. By using the formula given in [15] we see that $SU(2)$ admits a left invariant Riemannian metric whose sectional curvature $K$ satisfies $\delta A \leq K \leq A$. Let $\Gamma$ be a nontrivial finite subgroup of $SU(2)$. Then the quotient space $M := SU(2)/\Gamma$ is a homogeneous Riemannian manifold with sectional curvature $\delta A \leq K_M \leq A$.

5.2. Let $HP^m$ be the quaternion projective space with the standard Riemannian metric whose sectional curvature $K$ satisfies $1 \leq K \leq 4$ where $m \geq 2$. 

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The symplectic group $Sp(m+1)$ acts transitively on $HP^m$ as an isometry group. Fix an $x \in HP^m$. The isotropy subgroup of $Sp(m+1)$ at $x$ is $Sp(m) \times Sp(1)$. Let $V_r$ denote the geodesic hypersphere in $HP^m$ with radius $r$ and center $x$, $0 < r < \pi/2$. $V_r$ is diffeomorphic to $S^{4m-1}$. The principal curvatures of $V_r$ with respect to the inner unit normal are $2 \cot 2r$ and $\cot r$ whose multiplicity are 3 and $4m - 4$ respectively. Let $K_\sigma$ be an arbitrary sectional curvature of $V_r$ with the metric induced from $HP^m$. By using the equation of Gauss, we obtain $1 + 4 \cot^2 2r \leq K_\sigma \leq 4 + \cot^2 r$. Thus there exists an $r$ such that $0 < r < \pi/2$ and $4(1 + 4 \cot^2 r) \geq 4 + \cot^2 r$. Let $r$ be such a positive. Since $HP^m$ is a two point homogeneous Riemannian manifold, $Sp(m) \times Sp(1)$ acts transitively on $V_r$ as an isometry group. Let $\Gamma_0$ be a nontrivial finite subgroup of $Sp(1)$. Then $\Gamma := \{I\} \times \Gamma_0$ acts freely on $V_r$. Since $Sp(m) \times \{I\}$ acts on $V_r$ transitively and $Sp(m) \times \{I\} \subset Z(\Gamma)$, the quotient space $M = V_r/\Gamma$ is a homogeneous Riemannian manifold ([21]; p. 73). Then all sectional curvature $K_M$ of $M$ satisfy $\delta A \leq K_M \leq A$, where $A = 4 + \cot^2 r$ and $\delta = (1 + 4 \cot^2 2r)/(4 + \cot^2 r)$.

5.3. For the complex projective space $CP^m$ with $1 \leq K \leq 4$ the same method as in 5.2 gives us non-simply connected homogeneous Riemannian manifolds $M$ with $\delta A \leq K_M \leq A$ whose fundamental groups are cyclic groups.

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References


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