ON A CLASS OF EVEN-DIMENSIONAL MANIFOLDS STRUCTURED BY A $\mathcal{T}$-PARALLEL CONNECTION

By

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Abstract. Geometrical and structural properties are proved for a class of even-dimensional manifolds which are equipped with a $\mathcal{T}$-parallel connection.

1. Introduction

Riemannian manifolds $(M, g)$ structured by a $\mathcal{T}$-parallel connection have been defined in [12]. We recall that if $M$ is such a manifold carrying a globally defined vector field $\mathcal{T}(\mathcal{T}^a)$ and $\theta^a_b$ (resp. $e_a$) are the connection forms (resp. the vectors of an orthonormal basis), the connection forms satisfy

$$\theta^a_b = \langle \mathcal{T}, e_b \wedge e_a \rangle,$$

where $\wedge$ is the wedge product. The equations (1) imply $\nabla_\mathcal{T} e_a = 0$ and this agrees with the definition of a $\mathcal{T}$-parallel connection.

In the present paper we assume that $M$ is of even dimension $2m$. In Section 3 we prove that $M$ is a space-form with the following properties:

(i) $M$ carries a locally conformal symplectic form $\Omega$ having $\mathcal{T}^b = (x)$ as covector of Lee;

(ii) $\mathcal{T}$ is closed torse forming

$$\nabla \mathcal{T} = (c + t) dp - x \otimes \mathcal{T},$$

where $dp$ is the soldering form of $M$, $c$ is a constant, $t = |\mathcal{T}|^2/2$, and $dx = 0$;

(iii) $\mathcal{T}$ defines a relative conformal transformation of $\Omega$ [14] (see also [7]), i.e.

$$d(\mathcal{L}_\mathcal{T} \Omega) = 4(c + f)x \wedge \Omega,$$

where $f$ is the principal scalar field on $M$.

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(iv) the components $\mathcal{F}^a$ $(a = 1, \ldots, 2m)$ of $\mathcal{F}$ are eigenfunctions of the Laplacian $\Delta$ and have all as eigenvalue $f$.

In Section 4 we consider the tangent bundle $TM$ of the manifold $M$ discussed in Section 3. Let $V(v^\alpha)$ be the Liouville vector field [3] on $TM$ and $\psi$ the associated Finslerian 2-form [3]; the following properties are proved

(i) the complete lift $\Omega^c$ [18] of $\Omega$ defines a conformal symplectic structure on $TM$ and $\mathcal{F}$ defines as for $\Omega$ a relative conformal transformation of $\Omega^c$ [14] [7];

(ii) $d(\mathcal{L}_\mathcal{F} \Omega^c) = 2(c + 1)x \cdot \Omega^c$,

and since $\mathcal{L}_\mathcal{F} \Omega^c = \Omega^c$, and $\mathcal{L}_\mathcal{F} \psi = \psi$, both $\Omega^c$ and $\psi$ are homogeneous and of class 1;

(iii) if $X$ is a skew-symmetric Killing vector field [15] having $\mathcal{F}$ as generative, then $\Omega^c$ is invariant by $X$, i.e. $\mathcal{L}_X \Omega^c = 0$, and $X$ defines also an infinitesimal conformal transformation of the canonical symplectic form $\Pi = f\psi$, i.e.

$$\mathcal{L}_X \Pi = -g(X, \mathcal{F})\Pi;$$

(iv) the vertical lift $X^V$ of $X$ defines a relative conformal transformation of the Finslerian form $\psi$, i.e.

$$d(\mathcal{L}_{X^V} \psi) = (dg(X, \mathcal{F}) + g(X, \mathcal{F})X^a) \cdot \psi.$$

2. Preliminaries

Let $(M, g)$ be a Riemannian $C^\infty$-manifold and let $V$ be the covariant differential operator with respect to the metric tensor $g$. We assume that $M$ is oriented and $V$ is the Levi-Civita connection of $g$. Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle, and

$\flat : TM \rightarrow T^*M$ and $\sharp : TM \rightarrow T^*M$

the classical isomorphisms defined by $g$ (i.e. $\flat$ is the index lowering operator, and $\sharp$ is the index raising operator).

Following [11], we denote by

$$A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM),$$

the set of vector valued $q$-forms $(q < \text{dim} \, M)$, and we write for the covariant derivative operator with respect to $V$

$$d^V : A^q(M, TM) \rightarrow A^{q+1}(M, TM).$$ (2)
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It should be noticed that in general \( d^{V^2} = d^V \circ d^V \neq 0 \), unlike \( d^2 = d \circ d = 0 \). If \( p \in M \) then the vector valued 1-form \( dp \in A^1(M, TM) \) is the canonical vector valued 1-form of \( M \), and is also called the soldering form of \( M \) [2]. Since \( \nabla \) is symmetric one has that \( d^V(dp) = 0 \). A vector field \( Z \) which satisfies

\[
d^V(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM), \quad \pi \in \Lambda^1 M, \tag{3}
\]
is defined to be an exterior concurrent vector field [13] (see also [10]). The 1-form \( \pi \) in (3) is called the concurrence form and is defined by

\[
\pi = \lambda Z^\lambda, \quad \lambda \in \Lambda^0 M. \tag{4}
\]

Let \( \emptyset = \{ e_a | a = 1, \ldots, 2m \} \) be a local field of orthonormal frames over \( M \) and let \( \mathcal{C}^\ast = \text{covect}\{ \omega^a \} \) be its associated coframe. Then E. Cartan's structure equations can be written in indexless manner as

\[
\nabla e = \Theta \otimes e, \tag{5}
\]

\[
d\omega = -\Theta \wedge \omega, \tag{6}
\]

\[
d\Theta = -\Theta \wedge \Theta + \Theta. \tag{7}
\]

In the above equations \( \Theta \) (resp. \( \Theta \)) are the local connection forms in the tangent bundle \( TM \) (resp. the curvature 2-forms on \( M \)).

3. Manifolds structured by a \( \mathcal{F} \)-parallel connection

Let \( (M, g) \) be a \( 2m \)-dimensional oriented Riemannian \( C^\infty \)-manifold and

\[
\mathcal{F} = \mathcal{F}^a e_a, \quad \mathcal{F}^\alpha = \mathcal{F}^a \omega^a \tag{8}
\]
be a globally defined vector field and its dual form respectively. Let \( \theta^a_{b} \ (a, b \in \{1, \ldots, 2m\}) \) be the local connection forms in the tangent bundle \( TM \). Then, by reference to [12], \( (M, g) \) is structured by a \( \mathcal{F} \)-parallel connection if the connection forms \( \Theta \) satisfy

\[
\theta^a_{b} = \langle \mathcal{F}, e_b \wedge e_a \rangle, \tag{9}
\]
where \( \wedge \) means the wedge product of vector fields. Making use of Cartan's structure equations (5), we find by (8) and (9) that

\[
\theta^a_{b} = \mathcal{F}^b \omega^a - \mathcal{F}^a \omega^b, \tag{10}
\]
and in consequence of (10), the equations (5) take the form

\[
\nabla e_a = \mathcal{F}^a dp - \omega^a \otimes \mathcal{F}. \tag{11}
\]
Since one has that \( \theta_\nu^a(\mathcal{T}) = 0 \), then following [6] one may say that the connection forms \( \theta_\nu^a \) are relations of integral invariance for \( \mathcal{T} \).

From (11) it also follows that

\[ \nabla_\mathcal{T} e_a = 0, \quad (12) \]

which expresses that all the vectors of the \( \mathcal{O} \)-basis \( \mathcal{O} = \{ e_a \} \) are \( \mathcal{T} \)-parallel and this legitimates our definition regarding the structure of \( M \). Further, making use of E. Cartan’s structure equations (6) one derives that

\[ d\omega^a = \alpha \wedge \omega^a, \quad (13) \]

where we have set \( \alpha = \mathcal{T} \beta \). Hence, by (13) it follows that all the pfaffians \( \omega^a \) of the covector basis \( \mathcal{O}^* \) are exterior recurrent forms [1]. Consequently, the pfaffian \( \alpha \) can be seen to be in fact a closed form, i.e.

\[ d\alpha = 0. \quad (14) \]

Since

\[ \alpha = \mathcal{T} \beta = \sum \mathcal{T}^a \omega^a, \quad (15) \]

one has by (11) \( d\mathcal{T}^a \wedge \omega^a = 0 \), and by reference to [9], one may write

\[ d\mathcal{T}^a = f \omega^a, \quad f \in \Lambda^0 M, \quad (16) \]

and call \( f \) the distinguished scalar on \( M \). By (16) and (14) it can now be seen that \( \alpha \) is also an exact form, and that one may set

\[ \alpha = -\frac{df}{f}. \quad (17) \]

Further, taking the covariant differential of \( \mathcal{T} \), one finds by (11) and (16) that

\[ \nabla \mathcal{T} = (f + 2t) dp - \alpha \otimes \mathcal{T}, \quad (18) \]

where we have set

\[ 2t = \| \mathcal{T} \|^2. \quad (19) \]

Hence, according to [17] (see also [16] [15] [9]), equation (18) expresses that \( \mathcal{T} \) is a torse forming vector field, which in addition, by (11), has the property to be closed; by (19) one may also write

\[ dt = f \alpha. \quad (20) \]
Further, operating on (11) by the exterior covariant operator $d^V$, one gets

$$d^V(\nabla e_a) = \nabla^2 e_a = 2(f + t)\omega^a \wedge dp.$$  \hfill (21)

This reveals that all the constituents of the vector basis $\{e_a\}$ are exterior concurrent vector fields [13] with $2(f + t)$ as exterior concurrent scalar. Under these conditions it suffices to make use of the general formula

$$\nabla^2 Z = Z^a \Theta^b_a \otimes e_b,$$  \hfill (22)

where $Z \in \mathfrak{z}(M)$ and $\Theta^b_a$ are the curvature 2-forms on $M$, to derive

$$\Theta^b_a = 2(f + t)\omega^a \wedge \omega^b.$$  \hfill (23)

It is well known that the equation (23) shows that the manifold $M$ under consideration is a space form of curvature $\kappa = -2(f + t)$

(see also [9]), and we agree to set

$$f + t = c = \text{const..}$$  \hfill (24)

In another perspective, we agree to call the 2-form $\Omega$ of rank $2m$ given by

$$\Omega = \sum \omega^i \wedge \omega^i, \quad i = 1, \ldots, m, \quad \text{i}^* = \text{i} + m,$$  \hfill (25)

the fundamental almost symplectic form of $M$. Taking the exterior derivative of $\Omega$, and in view of (13), one finds that

$$d\Omega = 2\omega \wedge \Omega.$$  \hfill (26)

This affirms the fact that $M$ is endowed with a locally conformal symplectic structure having $\omega$ as covector of Lee. Then, as is known [5], calling the mapping $Z \rightarrow -i_Z\Omega = ^bZ$ the symplectic isomorphism, one has

$$^b\mathcal{F} = \sum (\mathcal{F}^i \omega^i - \mathcal{F}^i \omega_i^*),$$  \hfill (27)

and by (16) one finds that

$$d(^b\mathcal{F}) = 2f\Omega.$$  \hfill (28)

Taking now the Lie derivative of $\Omega$ with respect to the Lee vector field $\mathcal{F}$, yields

$$\mathcal{L}_\mathcal{F}\Omega = 2c\Omega + 2\omega \wedge ^b\mathcal{F},$$  \hfill (29)
and by exterior differentiation one gets
\[ d(\mathcal{L}_\tau \Omega) = 4(f + c)x \wedge \Omega. \] (30)

Hence, following a known definition [14] (see also [7]), the above equation means that \( \mathcal{T} \) defines a relative conformal transformation of \( \Omega \).

Recall now that if \( \tau \in \Lambda^0 M \) is any scalar field, then the Laplacian of \( \tau \) is expressed by
\[ \Delta \tau = \delta df = -\text{div} df = -\text{div} \nabla \tau, \]
where \( \nabla \tau \) is the gradient of \( \tau \). Coming back to the case under discussion, then with the help of (16) one derives that
\[ \nabla \mathcal{T}^a = f \mathcal{T}^a. \] (31)

This shows that \( \mathcal{T}^a \) is an eigenfunction of \( \Delta \) corresponding to the eigenvalue \( f \). Hence one may say that the vector field \( \mathcal{T} \) forms an eigenspace \( E^{2m} \) of eigenvalue \( f \).

**Theorem 3.1.** Let \( M \) be a 2m-dimensional Riemannian manifold structured by a \( \mathcal{T} \)-parallel connection and let \( \mathcal{T}(\mathcal{T}^a) \) be the vector field which defines this connection and \( \mathcal{T}^b \) the dual form of \( \mathcal{T} \). Any such manifold is a space-form and is endowed with a locally conformal symplectic form \( \Omega \) having \( \mathcal{T}^b \) as covector of Lee, i.e.
\[ d\Omega = 2\mathcal{T}^b \wedge \Omega, \]
and \( \mathcal{T} \) defines a relative conformal transformation of \( \Omega \), i.e.
\[ d(\mathcal{L}_\tau \Omega) = 4(c + f)\mathcal{T}^b \wedge \Omega, \]
where \( c \) is a constant and \( f \) is the distinguished scalar on \( M \). The vector field \( \mathcal{T} \) is closed torse forming and its components \( \mathcal{T}^a \) form an eigenspace \( E^{2m} \) of eigenvalue \( f \).

**4. Geometry of the tangent bundle**

Let now \( TM \) be the tangent bundle of the manifold \( M \) discussed in Section 3. Denote as usual by \( V(v^a) \) \( (a \in \{1\ldots 2m\}) \) the Liouville vector field (or the canonical vector field [3]). Under these conditions, one may consider the set \( B^* = \{\omega^a, dv^a\} \) as an adapted cobasis in \( TM \). Following [3] one denotes by \( i_v \) the vertical derivation \( (i_v \text{ is a derivation of degree 0 on } \Lambda TM) \), i.e.
\[ i_v \lambda = 0, \quad i_v dv^a = \omega^a, \quad i_v \omega^a = 0. \] (32)
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Next, the complete lift of $\Omega$ is, as is known from [18], expressed by

$$\Omega^c = \sum (dv^i \wedge \omega^i + \omega^i \wedge dv^i). \tag{33}$$

Then, on behalf of (13), the exterior differential of $\Omega^c$ is given by

$$d\Omega^c = \alpha \wedge \Omega^c. \tag{34}$$

Hence, the complete lift $\Omega^c$ of $\Omega$ defines on $TM$ a conformal symplectic structure, as $\Omega$ does on $M$. Moreover, similarly as for $\Omega$, one can derive that

$$d(\mathcal{L}_V \Omega^c) = 2(c + 1)\alpha \wedge \Omega^c, \tag{35}$$

which proves that $\mathcal{F}$ defines a relative conformal transformation of $\Omega^c$.

Next, as is known [4], the Liouville vector field $V$ is expressed by

$$V = \sum V^a \frac{\partial}{\partial v^a}, \tag{36}$$

and the basic 1-form

$$\mu = \sum V^a \omega^a \tag{37}$$

is called the Liouville 1-form. By (33) one has that

$$i_V \Omega^c = \sum (V^i \omega^i - V^i \omega^i), \tag{38}$$

and by (34) and (38) one gets

$$\mathcal{L}_V \Omega^c = \Omega^c. \tag{39}$$

Equation (39) shows that $\Omega^c$ is a homogeneous 2-form of class 1 [4] on TM.

Further, taking the exterior differential of the Liouville form $\mu$, one derives that

$$d\mu = \alpha \wedge \mu + \psi, \tag{40}$$

where we have set

$$\psi = \sum dv^a \wedge \omega^a. \tag{41}$$

Then, since one first calculates that

$$i_V \psi = \mu, \quad \alpha(V) = 0, \tag{42}$$
one finally gets that

\[ \mathcal{L}_\psi \psi = \psi, \tag{43} \]

which shows that, as \( \Omega^c \), the form \( \psi \) is also a homogeneous 2-form of class 1.

Moreover, by (32) one has that

\[ i_\psi \psi = 0, \tag{44} \]

which together with (43) proves that \( \psi \) is a Finslerian form [3].

In another order of ideas, we recall that the vertical lift \( Z^V \) [18] of any vector field \( Z \) on \( M \) with components \( Z^a \) is expressed by

\[ Z^V = \begin{pmatrix} 0 \\ Z^a \end{pmatrix} = Z^a \frac{\partial}{\partial v^a} \tag{45} \]

Therefore, in the case under consideration, one has

\[ \mathcal{F}^V = \sum \mathcal{F}^a \frac{\partial}{\partial v^a}, \quad a \in \{1, \ldots, 2m\}, \tag{46} \]

and by (41) and (32), one finds that

\[ i_\psi \psi = 0. \tag{47} \]

But, by (40) and (17), one has

\[ i_{\mathcal{F}^V} \psi = \mathcal{F}^V \tag{48} \]

and one derives

\[ \mathcal{L}_{\mathcal{F}^V} \psi = 0, \tag{49} \]

which shows that \( \psi \) is invariant by \( \mathcal{F}^V \).

Next, setting

\[ II = f \psi, \tag{50} \]

it follows from (17) and (32) that

\[ dII = 0. \tag{51} \]

Therefore, the exact symplectic 2-form \( II \) can be viewed as the canonical symplectic form of the manifold \( TM \). Since, as is known from [18], the Killing property for vector fields is invariant by complete liftings, we will now consider a skew-symmetric Killing vector field \( \mathcal{F}^V \) [12] on \( M \) having \( \mathcal{F} \) as generative. Hence, one must write

\[ \nabla X = X \wedge \mathcal{F}, \tag{52} \]
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where $\wedge$ denotes the wedge product of vector fields. Since by (11) one has that

$$\nabla X = \sum dX^a \otimes e_a + g(X, \mathcal{F}) \, dp - X^b \otimes \mathcal{F},$$

(53)

one gets from (52)

$$dX^a + g(X, \mathcal{F})\omega^a = X^a \alpha, \quad (\alpha = \mathcal{F}^b).$$

(54)

Then, since

$$X^b = \sum X^a \omega^a,$$

it follows from (13) that

$$dX^b = 2\alpha \wedge X^b,$$

(55)

which is in agreement with Rosca's lemma [15] concerning skew-symmetric Killing en conformal skew-symmetric Killing vector fields.

Next, since a problem of current interest consists of infinitesimal transformations due to the Lie derivatives, one finds in a first step

$$i_X \Omega^c = \sum (X^i dv^i - X^i dv^i).$$

(56)

Hence, taking the Lie derivative of the complete 2-form $\Omega^c$, one deduces that

$$\mathcal{L}_X \Omega^c = 0,$$

(57)

and this reveals that $\Omega^c$ is invariant by $X$. We also notice that taking the Lie bracket $[\mathcal{F}, X]$ one gets by (53) and (18)

$$[\mathcal{F}, X] = -fX,$$

(58)

and this shows that $\mathcal{F}$ defines an infinitesimal conformal transformation of $X$. Further, by (17), (41), (45) and (51), one calculates that

$$\mathcal{L}_X II = -g(X, \mathcal{F}) II,$$

(59)

and this affirms that $X$ defines an infinitesimal conformal transformation of the canonical symplectic form on $TM$. Finally, let

$$X^V = \sum X^a \frac{\partial}{\partial v^a}$$

be the vertical lift of $X$. By (41) one has that

$$i_X \psi = \sum X^a \omega^a,$$

(60)
and, taking the Lie derivative with respect to \( X^V \), one derives consecutively that
\[
L_{X^V} \psi = g(X, \mathcal{T}) \psi + 3x \wedge X^\gamma, \tag{61}
\]
and
\[
d(L_{X^V} \psi) = (dg(X, \mathcal{T}) + g(X, \mathcal{T})X^\gamma) \wedge \psi. \tag{62}
\]
Hence, (62) shows that the vertical lift \( X^V \) of the Killing vector field \( X \) defines a relative conformal transformation of the Finslerian form \( \psi \).

Theorem 4.1. Let \( TM \) be the tangent bundle manifold, having as basis the \( 2m \)-dimensional space-form manifold \( M(\Omega, \mathcal{T}, \mathcal{F}^\gamma = \alpha) \) discussed in Section 3. The complete lift \( \Omega^c \) of the conformal symplectic form \( \Omega \) defines also on \( TM \) a conformal symplectic structure and the structure vector field \( \mathcal{F} \) defines also a relative conformal transformation of \( \Omega^c \), i.e.
\[
d(\mathcal{L}_\mathcal{F} \Omega^c) = 2(c + 1)x \wedge \Omega^c.
\]
In addition, if \( V \) (resp. \( \psi \)) means the Liouville vector field on \( TM \) (resp. the Finslerian form), one has
\[
\mathcal{L}_V \Omega^c = \Omega^c, \quad \text{and} \quad \mathcal{L}_V \psi = \psi,
\]
which shows that both \( \Omega^c \) and \( \psi \) are homogeneous and of class 1. If \( X \) is a skew-symmetric Killing vector field having \( \mathcal{F} \) as generative, then \( \Omega^c \) is invariant by \( X \), i.e.
\[
\mathcal{L}_X \Omega^c = 0,
\]
and \( X \) defines also an infinitesimal conformal transformation of the canonical symplectic form \( \Pi = \frac{1}{2} \psi \) on \( TM \). Finally, the vertical lift \( X^V \) of \( X \) defines a relative conformal transformation of the Finslerian form \( \psi \).

References

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