THE TYPE NUMBER ON REAL HYPERSURFACES IN A QUATERNION SPACE FORM

By

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0. Introduction

Let $M_n(c)$ be a $4n$-dimensional quaternion space form with the metric $g$ of constant quaternion sectional curvature $8c$. The standard models of quaternion space forms are the quaternion projective space $P_n(Q), (c > 0)$, the quaternion space $Q, (c = 0)$ and the quaternion hyperbolic space $H_n(Q), (c < 0)$. Let $M$ be a connected real hypersurface in $M_n(c)$ with the induced metric.

In particular in [9], J. S. Pak characterized real hypersurfaces in $P_n(Q)$ in terms of the second fundamental form.

When we give a Riemannian manifold and its submanifold, the rank of determined second fundamental form is called the type number.

B. Y. Chen and T. Nagano ([2]) investigated totally geodesic submanifolds in Riemannian symmetric spaces, and as one of their results the following holds

**Theorem A** ([2]). Spheres and hyperbolic spaces are only simply connected irreducible symmetric spaces admitting a totally geodesic hypersurface.

Then it will be an interesting problem to study the type number $t$ of real hypersurfaces in simply connected irreducible symmetric spaces excepted for spheres and hyperbolic spaces.

As a partial answer, it is known that there exists a point such that $t(p) \geq 2$ in any real hypersurface in complex space form with nonzero constant holomorphic sectional curvature and complex dimension $\geq 3$ (cf. [8], [10]). Naturally we can consider the following question.

**Does $M_n(c)$ satisfy the similar fact?**

We answer this question affirmatively, i.e., we shall prove the following
**Main Theorem.** Let $M$ be a connected real hypersurfaces in $M_n(c)$ $(c \neq 0, n \geq 2)$. Then there exists a point $p$ in $M$ such that $t(p) \geq 2$.

1. **Preliminaries**

A quaternion Kähler manifold is a Riemannian manifold $(\overline{M}, g)$ on which there exists a 3-dimensional vector bundle $V$ of tensors of type $(1,1)$ satisfying the following properties:

1. In any open set $W$ in $M$, there is a local base $\{J_i(i = 1, 2, 3)\}$ of $V$ such that
   \[
   J_i^2 = -I, \quad J_iJ_{i+1} = J_{i+2} = -J_{i+1}J_i \pmod{3},
   \]
   where $I$ denotes the identity endomorphism.

   Such a local base $\{J_i(i = 1, 2, 3)\}$ is called a *canonical local base* of the bundle $V$ in $W$.

2. There is a Riemannian metric $g$ on $\overline{M}$ such that
   \[
   g(J_iX, Y) + g(X, J_iY) = 0,
   \]
   for any $X, Y \in \mathfrak{X}(W)$, where $\mathfrak{X}(W)$ is the set of all vector fields on $W$.

3. The Levi-Civita connection $D$ on $\overline{M}$ satisfies following conditions: If $\{J_i(i = 1, 2, 3)\}$ is a canonical local base of $V$ in $W$, then there exists three local 1-forms $p_i$ $(i = 1, 2, 3)$ on $\overline{M}$ such that
   \[
   D_X J_i = p_{i+2}(X)J_{i+1} - p_{i+1}(X)J_{i+2} \pmod{3},
   \]
   for all $X \in \mathfrak{X}(\overline{M})$.

   Let $Q(X)$ be the 4-plane spanned by vectors $X, J_1X, J_2X$ and $J_3X$, for any $X \in T_x\overline{M}, x \in \overline{M}$. If the sectional curvature of any section for $Q(X)$ depends only on $X$, we call it *Q-sectional curvature*.

   A quaternion space form of $Q$-sectional curvature $8c$ is connected quaternion Kähler manifold with constant $Q$-sectional curvature $8c$, which denotes by $M_n(c)$.

   Let $M$ be a real hypersurface in $M_n(c)$ $(n \geq 2, c \neq 0)$. In a neighborhood of each point, we choose a unit normal vector field $N$ in $M_n(c)$. The Levi-Civita connection $D$ in $M_n(c)$ and $V$ in $M$ are related by the following formulas for any $X, Y \in \mathfrak{X}(M)$:

   \[
   D_X Y = \nabla_X Y + \langle AX, Y \rangle N, \]
   \[
   D_X N = -AX,
   \]
where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric on $M$ induced from the metric $g$ on $M_n(c)$ and $A$ is the shape operator of $M$.

It is known that $M$ has an almost contact metric structure induced from the quaternion structure $J_i$ on $M_n(c)$, i.e., we define a tensor $\phi_i$ of type $(1, 1)$, a vector field $\xi_i$ and a 1-form $\eta_i$ on $M$ by the following,

$$(1.7) \quad \langle \phi_i X, Y \rangle = g(J_i X, Y), \quad \langle \xi_i, X \rangle = \eta_i(X) = g(J_i X, N).$$

Then from (1.1) we have

$$(1.8) \quad \langle \phi_i X, Y \rangle + \langle X, \phi_i Y \rangle = 0, \quad \langle \phi_i X, \phi_i Y \rangle = \langle X, Y \rangle - \eta_i(X)\eta_i(Y),$$

$$\phi_i \xi_{i+1} = \xi_{i+2} = -\phi_{i+1} \xi_i \quad (i \text{ mod } 3).$$

From (1.3), we obtain

$$(1.10) \quad \phi_i^2 = -I + \eta_i \otimes \xi_i, \quad \eta_i(\xi_i) = 1, \quad \phi_i \xi_i = 0,$$

$$(1.11) \quad \eta_i(\xi_{i+1}) = \eta_i(\xi_{i+2}) = 0 \quad (i \text{ mod } 3),$$

$$(1.12) \quad \phi_i = \phi_{i+1} \phi_{i+2} - \eta_{i+2} \otimes \xi_{i+1} = -\phi_{i+2} \phi_{i+1} + \eta_{i+1} \otimes \xi_{i+2} \quad (i \text{ mod } 3).$$

Furthermore from (1.2) and (1.7), we get

$$(1.13) \quad (\nabla_X \phi_i) Y = p_{i+1}(X) \phi_{i+2} Y - p_{i+2}(X) \phi_{i+1} Y$$

$$+ \eta_i(Y)AX - \langle AX, Y \rangle \xi_i \quad (i \text{ mod } 3).$$

In terms of (1.4) we have the following Codazzi equation

$$(\nabla_X A) Y - (\nabla_Y A) X = c \sum_{i=1}^{3} (\eta_i(X) \phi_i Y - \eta_i(Y) \phi_i X - 2\langle \phi_i X, Y \rangle \xi_i).$$

2. Formulas

We assume that the rank of $A$ is not larger than $m$ on an open set $W$, then there exists an open set $W_0$ such that $t$ takes the constant $m$. Then the Codazzi equation gives

$$(2.1) \quad -A(\nabla_X Y - \nabla_Y X) = (\nabla_X A) Y - (\nabla_Y A) X$$

$$= c \sum_{i=1}^{3} (\eta_i(X) \phi_i Y - \eta_i(Y) \phi_i X - 2\langle \phi_i X, Y \rangle \xi_i),$$

for any vector fields $X, Y \in \ker A|_{W_0}$. 
Taking the inner product of (2.1) with $Z \in \ker A|_{W_0}$, from (1.7) and $c \neq 0$, we have

\begin{equation}
0 = \sum_{i=1}^{3} \eta_i(X)\langle \phi_i Y, Z \rangle + \eta_i(Y)\langle \phi_i Z, X \rangle - 2\eta_i(Z)\langle \phi_i X, Y \rangle.
\end{equation}

Putting $Z = X$ in (2.2), we obtain

\begin{equation}
\sum_{i=1}^{3} \eta_i(X)\langle \phi_i Y, X \rangle = 0.
\end{equation}

3. Proof of the Main theorem

Since Theorem A, we get $m \geq 1$. Suppose that $m = 1$. Let $\lambda$ be the nonzero principal curvature with principal subspace $T_{\lambda}$. Choose a local orthonormal frame field $U, e_1, \ldots, e_{4n-2}$ on $M$ such that $e_1, \ldots, e_{4n-2}$ is in $\ker A|_{W_0}$ and $U \in T_{\lambda}$. We use the following convention on the range of indices otherwise stated: $r, s, \ldots = 1, \ldots, 4n - 2$.

Putting $Z = e_r$ in (2.2), we get

\begin{equation}
\sum_{i=1}^{3} \eta_i(X)\langle \phi_i Y, e_r \rangle - \eta_i(Y)\langle \phi_i X, e_r \rangle - 2\langle \phi_i X, Y \rangle \eta_i(e_r) = 0.
\end{equation}

**Lemma.** There exists a number $i$ such that $\eta_i(U) \neq 0$.

**Proof.** We assume that

\begin{equation}
\eta_i(U) = 0,
\end{equation}

for any number $i$. Then multiplying (3.1) by $\langle \phi_i U, e_r \rangle$ and summing up for $r$, since (1.8) $(1.12)$ and (3.2) we have

\begin{align*}
- \eta_{i+1}(X)\langle \phi_{i+2} Y, U \rangle + \eta_{i+1}(Y)\langle \phi_{i+2} X, U \rangle \\
+ \eta_{i+2}(X)\langle \phi_{i+1} Y, U \rangle - \eta_{i+2}(Y)\langle \phi_{i+1} X, U \rangle = 0 \quad (i \mod 3).
\end{align*}

Putting $X = e_r$ in above equation and summing up for $r$, from (1.9) $(1.11)$ and (3.2) we obtain

\[\langle \phi_i U, Y \rangle = 0,
\]

together with equation $\langle \phi_i U, U \rangle = 0$, we get

\begin{equation}
\phi_i U = 0.
\end{equation}
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Putting \( X = U \) and \( Y = \xi \) in (1.13) and taking the inner product with \( U \), then using (1.10), (3.2) and (3.3) we get \( \lambda = 0 \), which is a contradiction. \( \square \)

On the other hand, (2.3) implies

\[
\sum_{i=1}^{3} \eta_i(X) \langle \phi, e_r, X \rangle = 0. \tag{3.4}
\]

Multiplying (3.1) by \( \langle \phi, U, e_r \rangle \) and summing up for \( r \), from (1.9), (1.10), (1.12) and equation \( \sum_r \langle \phi, U, e_r \rangle e_r = \phi_i U \), we get

\[
\eta_i(U) \sum_{j=1}^{3} \eta_j^2(X) + \eta_{i+1}(X) \langle U, \phi_{i+2}X \rangle - \eta_{i+2}(X) \langle U, \phi_{i+1}X \rangle = 0 \quad (i \text{ mod } 3).
\]

Putting \( X = e_r \) in above equation and summing up for \( r \), by (1.9) we have

\[
\eta_i(U) \left( \sum_{j=1}^{3} \eta_j^2 \left( \sum e_r \right) - 2 \right) = 0.
\]

According to Lemma, above equation implies

\[
\sum_{j=1}^{3} \eta_j^2 \left( \sum e_r \right) = 2. \tag{3.5}
\]

Multiplying (3.4) by \( \eta_i(e_r) \) and summing up for \( r \), then using (1.9), (1.10) and Lemma we have

\[
\sum_{j=1}^{3} \eta_j(X) \langle U, \phi_j X \rangle = 0. \tag{3.6}
\]

Again multiplying (3.4) by \( \langle \phi, X, e_r \rangle \) and summing up for \( r \) and since (1.8), (1.12) and (3.6) we obtain

\[
\eta_i(X) \left( \|X\|^2 - \sum_{j=1}^{3} \eta_j^2(X) \right) = 0. \tag{3.7}
\]

Suppose that \( \eta_i(X) = 0 \) for any number \( i \). Then we observe \( \eta_i(\xi) = \eta_i(U) = 1 \). This implies \( \xi = U \) for any number \( i \), which is a contradiction. Thus by (3.7) we get

\[
\sum_{j=1}^{3} \eta_j^2(X) = \|X\|^2.
\]
Putting $X = e_r$ in above equation and summing up for $r$, we have

$$
\sum_{j=1}^{3} \eta_j^2 \left( \sum e_r \right) = 4n - 2,
$$

which contradicts (3.5).

It completes the proof of Main Theorem.

**Remark (added in Proof).** J. E. D’Atri [3], J. Berndt [1] and A. Martinez [6] gave some examples of real hypersurfaces in $M_n(c), c \neq 0$. In case $M_n(c)$ is $H_n(Q)$, the type number of these examples is maximum. In case $M_n(c)$ is $P_2(Q)$, there is an example of $t \equiv 4$ in the above. However, we don’t know an example of real hypersurface in $M_n(c), c \neq 0$ such that $t \equiv 2$.

**References**


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