CAUCHY PROBLEMS RELATED TO DIFFERENTIAL OPERATORS WITH COEFFICIENTS OF GENERALIZED HERMITE OPERATORS

By
Xiaowei Xu

1. Introduction

As an example, let us consider the Schrödinger equation:

\[ \{ D_t + D_x^2 + V(x) \} u(t, x) = 0, \]

where \( D_t = (1/i)\partial/\partial t \) and \( D_x = (1/i)\partial/\partial x \). In the harmonic oscillator case, where the potential energy function \( V(x) \) is equal to \( x^2 \), X. Feng ([1]) considered it as follows.

As is well known, the Hermite function

\[ \Phi_\alpha(x) = (\pi/\alpha!)^{1/2} (-1)^k \pi^{-1/4} e^{-x^2/2} \left( \frac{\partial}{\partial x} \right)^k e^{-x^2} \]

is an eigenfunction of the Hermite operator \( H = D_x^2 + x^2 \), corresponding to an eigenvalue \( 2\alpha + 1 \), that is,

\[ H\Phi_\alpha(x) = (2\alpha + 1)\Phi_\alpha(x) \]

for any \( \alpha \in \mathbb{N} = \{0, 1, \ldots\} \). Moreover, \( \{\Phi_\alpha \mid \alpha \in \mathbb{N}\} \) is a complete orthonormal system of \( L^2(R) \), and \( \Phi_\alpha(x) \) belongs to \( S(R) \), where \( S(R) \) is the L. Schwartz space of rapidly decreasing functions in \( R \) ([2]).

Suppose \( u(t, x) \in S'(R) \) for fixed \( t \), where \( S'(R) \) is the conjugate space of \( S(R) \), and set

\[ u_\alpha(t) = \langle u(t, x), \Phi_\alpha(x) \rangle. \]

Then the Cauchy problem

\[ \begin{cases} (D_t + H)u(t, x) = 0 & \quad (0 \leq t \leq T, \ x \in R), \\ u(0, x) = \delta(x) & \quad (x \in R) \end{cases} \]

Received December 8, 1997.
is reduced to the Cauchy problems of ordinary differential equations

\[
\begin{aligned}
(a)_a \left\{ 
& \{D_t + (2a + 2)\} u_a(t) = 0 \quad (0 \leq t \leq T), \\
& u_a(0) = \Phi_a(0).
\end{aligned}
\]

Therefore, the solution \( u(t, x) \) of the problem \((A)\) can be formally represented by making use of solutions \( \{u_a(t)\} \) of the problems \((a)_a\). More precisely,

\[
u(t, x) = \sum_{a \in I_c} u_a(t) \Phi_a(x) = \sum_{a \in I_c} \Phi_a(0) e^{-i(2a+1)t} \Phi_a(x).
\]

X. Feng proved that \( u(t, x) \in S'(R_x) \), owing to the Hermite expression theory in \( S'(R) \) (B. Simon [3]).

How about anharmonic oscillator cases? Suppose that \( \{\phi_a(x)\} \) is a complete orthonormal system in \( L^2(R) \), where \( \phi_a(x) \) is an eigenfunction of the generalized Hermite operator \( L = D^2_x + x^2 + x^4 + 1 \), corresponding to an eigenvalue \( \lambda_a \). Then the solution of the Cauchy problem of the Schrödinger equation

\[
\begin{aligned}
& \{ \{D_t + L\} u(t, x) = 0 \quad (0 \leq t \leq T, \ x \in R), \\
& u(0, x) = \delta(x) \quad (x \in R)
\end{aligned}
\]

can be given by

\[
u(t, x) = \sum_{a \in I_c} \phi_a(0) e^{-i\lambda_a t} \phi_a(x).
\]

Our aim in this paper is to prove \( u(t, x) \in S'(R_x) \). In the following, this problem will be considered in a more general situation.

2. Preparations

Let us define a generalized Hermite operator \( L \) by

\[
L = (L_1, \ldots, L_n),
\]

\[
L_j = D^2_{x_j} + V_j(x_j) \quad (j = 1, 2, \ldots, n),
\]

where \( V_j(s) \) is a \( C^\infty(R) \)-function satisfying the following conditions: there exist \( \delta_j > 0, \ c_0 > 0, \) and \( C_k > 0 \ (k \in I_+) \) such that

\[
\begin{aligned}
& V_j(s) \geq c_0(1 + |s|)^{2\delta_j} \quad (\forall s \in R), \\
& |D^k_s V_j(s)| \leq C_k (1 + |s|)^{2\delta_j} \quad (\forall s \in R).
\end{aligned}
\]

**Lemma 1.** There exist \( \{\phi_{jk}(s)\}_{k \in I_c} \), satisfying the following conditions, where \( \phi_{jk}(s) \) is an eigenfunction of \( L_j \), corresponding to an eigenvalue \( \lambda_{jk} \).
Cauchy problems related to differential

1) $0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots$, and there exists $p_0 > 0$ such that

$$\sum_{k=0}^{\infty} \frac{1}{\lambda_k^{p_0}} < +\infty.$$ 

2) $\phi_j(s)$ is real valued, and $\{\phi_j(s)\}_{k \in I_+}$ is a complete orthonormal system of $L^2(R)$. 

3) $\phi_j(s) \in S(R)$, and there exist $C(l) > 0$ and $p(l) > 0$ for any $l \in I_+$ such that

$$\|\phi_j\|_l := \sum_{a+b \leq l} \sup_{x \in \mathbb{R}} |x^a D_x^b \phi_j(s)| \leq C(l) \lambda_j^{p(l)} \quad (\forall k \in I_+).$$

Lemma 1 is proved in [4] under assumptions slightly different to ours, but it is proved similarly.

Now, for any $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in I^n_+$, we put

$$\phi_{\alpha}(x) = \Pi_{j=1}^n \phi_{j, \alpha_j}(x_j), \quad \lambda_{\alpha} = (\lambda_1, \alpha_1, \ldots, \lambda_n, \alpha_n),$$

$$L^\beta = \Pi_{j=1}^n L_j^{\beta_j}, \quad \lambda_{\alpha}^\beta = \Pi_{j=1}^n \lambda_{j, \alpha_j}^{\beta_j}$$

and denote

$$\Lambda = \{\lambda_{\alpha} | \alpha \in I^n_+\}$$

$$= \{ (\lambda_{10}, \lambda_{20}, \ldots, \lambda_{n0}), (\lambda_{11}, \lambda_{20}, \ldots, \lambda_{n0}), (\lambda_{10}, \lambda_{21}, \ldots, \lambda_{n0}), \ldots \}.$$ 

Using Lemma 1, it is easy to prove

**Lemma 2.** $\phi_{\alpha}(x)$ is an eigenfunction of $L^\beta$ corresponding to an eigenvalue $\lambda_{\alpha}^\beta$, and they satisfy

1) there exists $p_0 > 0$ such that

$$\sum_{\alpha \in I^n_+} |\lambda_{\alpha}|^{-p_0} < +\infty,$$

2) $\{\phi_{\alpha}(x)\}_{\alpha \in I^n_+}$ is a complete orthonormal system of $L^2(R^n)$,

3) $\phi_{\alpha}(x) \in S(R^n)$, and there exist $C(l) > 0$ and $p(l) > 0$ for any $l \in I_+$ such that

$$\|\phi_{\alpha}\|_l \leq C(l) |\lambda_{\alpha}|^{p(l)} \quad (\forall \alpha \in I^n_+).$$

Here we call $\{\phi_{\alpha}(x)\}_{\alpha \in I^n_+}$ a family of generalized Hermite functions.
Let $s$ be a space, whose elements satisfy $a = (a_x)_{x \in I^*_+} = (a_0,\ldots,0,a_1,0,\ldots,0,a_0,1,0,\ldots,0,\ldots) \ (a \in C)$ satisfies

$$|a|_h := \sup_{x \in I^*_+} |a_x| \lambda^h/2 < +\infty \ (\forall h \in I^*_+).$$

$s$ is a Fréchet space with a countable set of seminorms $\{ |a|_h \}_{h \in I^*_+}$. Let $s'$ be the conjugate space of $s$. Namely, $s'$ is a set of all linear continuous mappings from $s$ to $C$. More precisely, let $b \in s'$, that is, $b : s \ni a \rightarrow \langle b,a \rangle \in C$. Then there exist $h > 0$ and $C > 0$ such that it holds

$$|\langle b,a \rangle| \leq C |a|_h \ (\forall a \in s).$$

**Lemma 3.** 1) Let $f(x) \in S(R^n)$ and set

$$a(f) = \{ a_x(f) \}_{x \in I^*_+}, \quad a_x(f) = \langle f, \phi_x \rangle.$$

Then

$$S(R^n) \ni f(x) \rightarrow a(f) \in s$$

is linear continuous. More precisely, there exists $C_h > 0$ for any $h$ such that

$$|a(f)|_h \leq C_h \|f\|_{2n+(\delta+1)h} \quad (\delta = \max_j \delta_j).$$

2) Conversely, let $a \in s$ and set

$$f(x) = \sum_x a_x \phi_x(x).$$

Then

$$s \ni a \rightarrow f \in S(R^n)$$

is linear and continuous. More precisely, there exists $C_l > 0$ for any $l$ such that

$$\|f\|_l \leq C_l |a|_{2p(l)+2p_0}.$$ Moreover, $a(f) = a$ holds.

**Proof.** 1) For any $h \in I^*_+$, it holds

$$|a(f)|_{2h} = \sup_{x \in I^*_+} |a_x(f)| \lambda^h \leq C_h \sup_{x \in I^*_+} (\lambda^h + \cdots + \lambda^{\delta_h} \delta_x) \langle f, \phi_x \rangle$$
and
\[ \lambda_{j,x}^h \langle f, \phi_x \rangle = \langle f, L_j^h \phi_x \rangle = \langle L_j^h f, \phi_x \rangle \leq \| L_j^h f \|_{L^2}. \]

Hence
\[ |a(f)|_{2h} \leq C_h \sum_{j=1}^n \| L_j^h f \|_{L^2}. \]

On the other hand, since
\[ |D_{x_j}^k V_j(x_j)| \leq C_k (1 + |x_j|)^{2b}, \]
we have
\[ \| L_j^h f \|_{L^2} = \| (D_{x_j}^2 + V(x_j))^k f \|_{L^2} \leq C_h \sum_{\beta \leq 2b \| l \|} \| x_j^\beta D_{x_j}^\beta f \|_{L^2} \leq C'_h \| f \|_{2n+2(\delta+1)h}. \]
Therefore, we have
\[ |a(f)|_{2h} \leq C_h \| f \|_{2n+2(\delta+1)h}. \]

2) Conversely, let \( a = \{ a_x \}_{x \in I_+^*} \), then we have from 1) and 3) of Lemma 2,
\[ \sum_{x \in I_+^*} \| a_x \phi_x \|_l = \sum_{x \in I_+^*} |a_x| \| \phi_x \|_l \]
\[ \leq \sum_{x \in I_+^*} |a_x| C(l) |\lambda_x|^{p(l)} \]
\[ \leq C(l) \sup_x |a_x| |\lambda_x|^{p(l)+p_0} \sum_{x \in I_+^*} |\lambda_x|^{-p_0} \]
\[ = C'(l) |a|_{2p(l)+2p_0}, \]
for any \( l \). Therefore, \( \sum_{x \in I_+^*} a_x \phi_x(x) \) is a convergent sequence in \( S(R) \). Hence, set
\[ f(x) = \sum_{x \in I_+^*} a_x \phi_x(x). \]
Then it holds that
\[ \| f \|_l \leq C'(l) |a|_{2p(l)+2p_0}, \]
and
\[ a_f(f) = \langle f, \phi_f \rangle = \sum_{a \in I^+_n} a_a \langle \phi_a, \phi_f \rangle = a_f. \]

Thus \( a(f) = a \) holds. \( \square \)

**Lemma 4.** 1) Let \( T \in S'(R^n) \), and put
\[ b = \{ b_a \}_{a \in I^+_n}, \quad b_a = \langle T, \phi_a \rangle. \]

Then

i) there exists \( h > 0 \) such that \( b = \sup_{a} |\lambda_a|^{-h/2} < \infty \),

ii) \( b : s \ni \forall a \rightarrow \sum a_a b_a \in C \) belongs to \( s' \),

iii) for any \( f \in S(R^n) \), it holds
\[ \langle T, f \rangle = \sum a_a \langle f, \phi_a \rangle, \quad a_a(f) = \langle f, \phi_a \rangle. \]

2) Conversely, let \( b \in s' \). Then \( T : S(R^n) \ni f \rightarrow \langle b, a(f) \rangle \) belongs to \( S'(R^n) \).

**Proof.** 1) i) Since \( T : S(R^n) \ni \phi \rightarrow \langle T, \phi \rangle \in C \) is continuous, there exist \( C > 0 \) and \( l > 0 \) such that
\[ |b_a| = |\langle T, \phi_a \rangle| \leq C \| \phi_a \|_l \leq C C(l)|\lambda_a|^{p(l)}, \]
using 3) of Lemma 2. Hence we have \( |b|_{-2p(l)} < +\infty \).

ii) Let \( h \) be the number in i). Then we have
\[ |b_a| |\lambda_a|^{-h/2} \leq C \quad (\forall a \in I^+_n). \]

Therefore, we have
\[ \sum_{a} |a_a| |b_a| \leq C \sum_{a} |a_a| |\lambda_a|^{h/2} \]
\[ \leq C \sum_{a} |\lambda_a|^{-h_0} \sup_{a} |a_a| |\lambda_a|^{h/2+p_0} \]
\[ = C' |a|_{h+2p_0}, \]
for any \( a = \{ a_a \}_{a \in I^+_n} \in s \), where we used 1) of Lemma 2. Hence
\[ b : s \ni a \rightarrow \langle b, a \rangle = \sum_{a} a_a b_a \in C \]
is a linear continuous mapping, that is, \( b \) belong to \( s' \).
iii) Let \( f(x) \in S(R^n) \). Then we have
\[
f(x) = \sum_x a_x(f) \phi_x(x) \quad \text{in } S(R^n),
\]
where \( a_x(f) = \langle f, \phi_x \rangle \) (\( x \in I^n_+ \)) from 2) of Lemma 3. Hence we have
\[
\langle T, f \rangle = \left( T, \sum_x a_x(f) \phi_x \right) = \sum_x a_x(f) \langle T, \phi_x \rangle = \sum_x a_x(f) b_x.
\]

2) Conversely, for any \( f(x) \in S(R^n) \), we have \( a(f) = \{a_x(f)\} \in s \), and there exists \( C_h > 0 \) for any \( h \in I_+ \) such that
\[
|a(f)|_h \leq C_h \|f\|_{2n+(\beta+1)h},
\]
from 1) of Lemma 3. Let \( b \in s' \). Then there exist \( C > 0 \) and \( h > 0 \) such that
\[
|\langle b, a(f) \rangle| \leq C|a(f)|_h.
\]
Therefore, we have
\[
|\langle b, a(f) \rangle| \leq C\|f\|_{2n+(\beta+1)h}.
\]
Hence
\[
T : S(R^n) \ni f \rightarrow \langle b, a(f) \rangle \in C
\]
is a linear continuous mapping, namely \( T \in S'(R^n) \).

We say that \( u(t,x) \in B^h([0,T], S'(R^n)) \), iff
\[
u : [0,T] \ni t \rightarrow u(t,x) \in S'(R^n)
\]
is continuously differentiable up to order \( h \) in the sense of simple topology of \( S'(R^n) \).

**Lemma 5.** 1) Suppose \( u(t,x) \in B^h([0,T], S'(R^n)) \), and set
\[
u_{a}(t) = \langle u(t,x), \phi_x(x) \rangle.
\]
Then there exist \( C > 0 \) and \( p > 0 \) such that
\[
|D^j u_{a}(t)| \leq C|\lambda_a|^p \quad (x \in I^n_+, \ 0 \leq j \leq h).
\]

2) Conversely, suppose
\[
|D^j u_{a}(t)| \leq C|\lambda_a|^p \quad (x \in I^n_+, \ 0 \leq j \leq h),
\]
and set

\[ u(t, x) = \sum_{\alpha} u_{\alpha}(t)\phi_{\alpha}(x), \]

that is,

\[ u : S(R^n) \ni f \rightarrow \langle u(t, x), f \rangle = \sum_{\alpha} u_{\alpha}(t)\langle \phi_{\alpha}(x), f(x) \rangle = \sum_{\alpha} u_{\alpha}(t) a_{\alpha}(f) \in C \]

for \( t \in [0, T] \). Then \( u(t, x) \in B^h([0, T], S'(R^n)) \).

**Proof.** 1) Suppose \( u(t, x) \in B^h([0, T], S'(R^n)) \), then \( H = \{u(t, x) | t \in [0, T]\} \) is a bounded set in \( S'(R^n) \) in the sense of simple topology. By using the fundamental lemma of Fréchet space ([5]), there exist \( C > 0 \) and \( l_0 > 0 \) such that

\[ \|\langle u(t, x), \phi(x) \rangle\| \leq C\|\phi\|_{l_0} \quad (\forall t \in [0, T], \forall \phi \in S(R^n)). \]

Therefore, it holds

\[ |u_{\alpha}(t)| = |\langle u(t, x), \phi_{\alpha}(x) \rangle| \leq C\|\phi_{\alpha}\|_{l_0} \quad (\alpha \in I^n_+). \]

Besides, since

\[ \|\phi_{\alpha}\|_{l_0} \leq C(l_0)|\lambda_{\alpha}|^{p(l_0)} \]

from 3) of Lemma 2, we have

\[ |u_{\alpha}(t)| \leq CC(l_0)|\lambda_{\alpha}|^{p(l_0)}. \]

In the same way, we have

\[ |D_i^j u_{\alpha}(t)| \leq C|\lambda_{\alpha}|^{p} \quad (\alpha \in I^n_+, j = 0, 1, 2, \ldots, h). \]

2) Conversely, suppose

\[ |D_i^j u_{\alpha}(t)| \leq C|\lambda_{\alpha}|^{p} \quad (\alpha \in I^n_+, j = 0, 1, 2, \ldots, h), \]

and set

\[ u : S(R^n) \ni f \rightarrow \langle u(t, x), f(x) \rangle = \sum_{\alpha \in I^n_+} u_{\alpha}(t) a_{\alpha}(f) \in C, \quad a_{\alpha}(f) = \langle f, \phi_{\alpha} \rangle. \]

Then \( u(t, x) \) belongs to \( S'(R^n) \), from 2) of Lemma 4. Since
Cauchy problems related to differential operators with coefficients of generalized Hermite operators

\[ \sum_{a \in I^*} D^j_{t} u_a(t) a_a(f) \quad (j = 0, 1, \ldots, h) \]

are uniformly convergent sequences in \([0, T]\),

\[
D^j_{t} \langle u(t, x), f(x) \rangle = D^j_{t} \sum_a u_a(t) a_a(f) \\
= \sum_a D^j_{t} u_a(t) a_a(f).
\]

Therefore, we have

\[
|D^j_{t} \langle u(t, x), f(x) \rangle| \leq \sum_a |a_a(f)||D^j_{t} u_a(t)| \\
\leq C \sum_a |a_a(f)||\lambda_a|^p.
\]

By using 1) of Lemma 2, we have

\[
\sum_a |a_a(f)||\lambda_a|^p = \sup_{a \in I^*} |a_a(f)||\lambda_a|^{p+\nu} \sum_a |\lambda_a|^{-\nu} \\
\leq |a(f)|_{2p+2\nu} \sum_{a \in I^*} |\lambda_a|^{-\nu} \\
= C|a(f)|_{2p+2\nu}.
\]

On the other hand, by using 1) of Lemma 3, we have

\[
|a(f)|_{2p+2\nu} \leq C\|f\|_{2n+2(\delta+1)(p+\nu)},
\]

Hence

\[
|D^j_{t} \langle u(t, x), f(x) \rangle| \leq C\|f\|_{2n+2(\delta+1)(p+\nu)} < +\infty \quad (t \in [0, T], j = 0, 1, \ldots, h),
\]

that is, \(u(t, x) \in B^h([0, T], S'(R^n_x))\).

3. Cauchy problems

Let us consider Cauchy problems related to differential operators with coefficients of generalized Hermite operators.
\[ P(D_t, L) = P_m(L)D_t^n + \cdots + P_0(L), \]
\[ P_j(L) = \sum_{|\beta| \leq m_j} a_{j, \beta} L_\beta = \sum_{\beta_1 + \cdots + \beta_n \leq m_j} a_{j, \beta_1, \ldots, \beta_n} L_1^{\beta_1} \cdots L_n^{\beta_n}, \]

where \( a_{j, \alpha} \) are constants and \( \alpha_j \) are non-negative integers. \( P(D_t, L) \) is called an evolution differential operator with coefficients of generalized Hermite operators, iff

(I) there exist \( C_1 > 0 \) and \( p_1 > 0 \) such that

\[ |P_m(\lambda)| \geq C_1 |\lambda|^{-p_1} \quad (\forall \lambda \in \Lambda), \]

(II) there exists \( k > 0 \) such that

\[ I_m \tau_j(\lambda) \geq -k. \quad (\forall \lambda \in \Lambda, 1 \leq j \leq m), \]

where

\[ P(\tau, \lambda) = P_m(\lambda)(\tau - \tau_1(\lambda)) \cdots (\tau - \tau_m(\lambda)). \]

**Theorem 1.** Suppose \( P(D_t, L) \) is an evolution differential operator with coefficients of generalized Hermite operators. Let

\[ f(t, x) \in B^h([0, T], S'(R^n)), \quad g_j(x) \in S'(R^n) \quad (0 \leq j \leq m - 1). \]

Then there exists unique solution \( u(t, x) \), belonging to \( B^{h+m}([0, T], S'(R^n)) \), of the Cauchy problem:

\[ (A) \left\{ \begin{array}{ll} P(D_t, L)u(t, x) = f(t, x) & (0 \leq t \leq T, x \in R^n), \\ D_j u(t, x)|_{t=0} = g_j(x) & (x \in R^n, 0 \leq j \leq m - 1). \end{array} \right. \]

**Proof.** Let

\[ u_\alpha(t) = \langle u(t, x), \phi_\alpha(x) \rangle, \]
\[ f_\alpha(t) = \langle f(t, x), \phi_\alpha(x) \rangle, \quad g_{j, \alpha} = \langle g_j(x), \phi_\alpha(x) \rangle. \]

Then the problem (A) is reduced to the Cauchy problems of ordinary differential equations:

\[ (a) \left\{ \begin{array}{ll} P_m(\lambda_\alpha)D_t^n + \cdots + P_0(\lambda_\alpha)u_\alpha(t) = f_\alpha(t) & (0 \leq t \leq T), \\ D_j u_\alpha(t)|_{t=0} = g_{j, \alpha} & (j = 0, 1, 2, \ldots, m - 1). \end{array} \right. \]
Cauchy problems related to differential

The solutions of \((a)_x\) can be represented as

\[
  u_x(t) = \sum_{j=1}^{m} b_{j-1,x} D_t^{m-j} W(t, \lambda_x) + i P_m'(\lambda_x)^{-1} \int_0^t f_x(s) W(t - s, \lambda_x) \, ds,
\]

where

\[
  W(t, \lambda_x) = \frac{P_m'(\lambda_x)}{2\pi i} \oint_{\gamma} \frac{e^{\lambda z}}{P(z, \lambda_x)} \, dz,
\]

\[
  b_{0,x} = g_{0,x}, \quad b_{j,x} = g_{j,x} - \sum_{i=1}^{j} b_{i-1,x} D_t^{m+j-i} W(0, \lambda_x) \quad (1 \leq j \leq m - 1),
\]

and \(\gamma\) is a closed curve inside of which all zeros of \(P(z, \lambda_x)\) with respect to \(z\) are containd. By evaluating the above representation, there exist \(C > 0\) and \(p > 0\) such that

\[
  |D_t^k u_x(t)| \leq C |\lambda_x|^p \left\{ \left| \sum_{j=0}^{m-1} g_{j,x} \right| + \max_{0 \leq s \leq t} \sup_{0 \leq k \leq m+h} |D_s^l f_x(s)| \right\}
\]

\((0 \leq t \leq T, \ 0 \leq k \leq m + h)\).

Since \(g_j(x) \in S'(R^n)\) \((0 \leq j \leq m - 1)\), there exists \(q_1 > 0\) such that

\[
  \sup_{x \in I^n_+} |g_{j,x}| |\lambda_x|^{-q_1} < +\infty
\]

from 1) of Lemma 4, and since \(f(t,x) \in B^h([0,T], S'(R^n))\), there exists \(q_2 > 0\) such that

\[
  \sup_{x \in I^n_+} \sup_{0 \leq t \leq T} |D_t^l f_x(t)| |\lambda_x|^{-q_2} < +\infty,
\]

from 1) of Lemma 5. Therefore, we have

\[
  |D_t^k u_x(t)| \leq C |\lambda_x|^{p+q} \quad (t \in [0,T]), \ x \in I^n_+, \ 0 \leq k \leq m + h,
\]

where \(q = \max(q_1, q_2)\). Finally, set

\[
  u(t, x) = \sum_x u_x(t) \phi_x(x).
\]

Then \(u(t,x)\) belongs to \(B^{h+m}([0,T], S'(R^n))\), from 2) of Lemma 5, and becomes a solution of the problem \((A)\). The uniqueness of the problem \((A)\) follows from the uniqueness of the problems \((a)_x\).
Remark 1. Let
\[ P(D_t, L) = D_t - \sum_{|\beta| \leq N} a_\beta L^\beta. \]
Then \( P \) is an evolution operator, iff there exists \( k > 0 \) such that
\[ I_m \sum_{|\beta| \leq N} a_\beta \lambda_\beta \geq -k \quad (\forall \lambda_\alpha \in \Lambda). \]

Remark 2. Let
\[ P(D_t, L) = D_t^2 - \sum_{|\beta| \leq N} a_\beta L^\beta. \]
Then \( P \) is an evolution operator, iff there exists \( k > 0 \) such that
\[ \left( \sum_{|\beta| \leq N} R_e a_\beta \lambda_\beta, \sum_{|\beta| \leq N} I_m a_\beta \lambda_\beta \right) \in \Omega_k \quad (\forall \lambda_\alpha \in \Lambda), \]
where
\[ \Omega_k = \{(X, Y) \mid Y^2 \leq kX \text{ or } X^2 + Y^2 \leq k\}. \]
For example,
\[ D_t^2 - \{L_1^2 + \cdots + L_n^2 + i(L_1 - L_2)\}, \]
\[ D_t^2 - \{L_1^3 + \cdots + L_n^3 + i(L_1 - L_2)\}, \]
\[ D_t^2 - \{L_1 L_2^2 + i(L_1 - L_2)\} \]
are evolution operators.

The paper has finished under the kind guidance of Prof. Reiko Sakamoto and Prof. Sadao Miyatake. I am deeply grateful to them.

References

Cauchy problems related to differential equations


Graduate School of Human Culture,
Nara Woman's University