ON THE STRUCTURE OF TAKAHASHI MANIFOLDS

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Abstract. We study the topological structure of the closed orientable 3-manifolds obtained by Dehn surgeries along certain links, first considered by Takahashi in [23]. The interest about such manifolds arises from the fact that they include well-known families of 3-manifolds, previously studied by several authors, as the Fibonacci manifolds [7], [10], [11], the Fractional Fibonacci manifolds [14], and the Sieradski manifolds [5], [6], respectively. Our main result states that the Takahashi manifolds are 2-fold coverings of the 3-sphere branched along the closures of specified 3-string braids. We also describe many of the above-mentioned manifolds as n-fold cyclic branched coverings of the 3-sphere.

1. Introduction and main results

The goal of the paper is to study the topological structure of the closed connected orientable 3-manifolds obtained by Dehn surgeries along certain chains of unknotted oriented circles in the oriented 3-sphere. Our results complete in a sense the ones of a previous paper of Takahashi [23]. It turns out that the above manifolds contemporarily include well-known families of manifolds, treated in the literature (see references), as the (Fractional) Fibonacci manifolds and the Sieradski manifolds. So we can re-obtain several results of the quoted papers as simple corollaries of our main theorem. To state it we first consider the link $L_{2n}$ resp. $L_n'$ with $2n$ resp. $n$ components, $n \geq 2$, each of which is unknotted oriented and linked with exactly two adjacent components as shown in Figure 1a resp. 1b.

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Figure 1a: the link $L_{2n}$.

2n components

n components

Figure 1b: the link $L'_{n}$.
Let us denote by $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$ resp. $M'(a_1/b_1, \ldots, a_n/b_n)$ the closed connected orientable 3-manifold obtained by Dehn surgery along $L_{2}n$ resp. $L_{n}'$ with surgery coefficients $p_i/q_i$ and $r_i/s_i$ resp. $a_i/b_i$, $i = 1, 2, \ldots, n$, according to Figure 1. In [23] Takahashi gave finite presentations of the fundamental group of the manifolds $M(p_i/q_i; r_i/s_i)$, so for convenience we refer to such manifolds as the Takahashi manifolds.

These presentations actually coincide with the standard ones of the Fibonacci groups

$$F(2, 2n) = \langle x_1, x_2, \ldots, x_{2n} : x_i x_{i+1} = x_{i+2} \ (\text{indices mod } 2n) \rangle$$

resp. the Fractional Fibonacci groups

$$F^{k/l}(2, 2n) = \langle x_1, x_2, \ldots, x_{2n} : x_i^{l} x_{i+1}^{k} = x_{i+2}^{l} \ (\text{indices mod } 2n) \rangle$$

when $p_i/q_i = 1$ and $r_i/s_i = -1$ resp. $p_i/q_i = k/l$ and $r_i/s_i = -k/l$ for every $i = 1, 2, \ldots, n$. It is well-known that the above presentations correspond to spines of closed orientable 3-manifolds, called the Fibonacci manifolds and the Fractional Fibonacci manifolds, respectively. It was also proved that the Fibonacci manifolds resp. the Fractional Fibonacci manifolds are two-fold cyclic coverings of the 3-sphere branched over the Turk's head links $Th_n$ resp. the links $Th_{n}^{k/l}$, that are the closures of the 3-string braids $(\sigma_1 \sigma_2^{-1})^n$ resp. $(\sigma_1^{k/l} \sigma_2^{-k/l})^n$ (see [7], [10], [11] and [14]). Our main theorem extends these results to the case of Takahashi manifolds.

**Theorem 1.** For any coprime integers $p_i$ and $q_i$ resp. $r_i$ and $s_i$, $i = 1, 2, \ldots, n$, and for any integer $n \geq 2$, the Takahashi manifold $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$ is the two-fold cyclic covering of the 3-sphere branched along the closure of the rational 3-string braid

$$\sigma_1^{p_1/q_1} \sigma_2^{r_1/s_1} \cdots \sigma_1^{p_n/q_n} \sigma_2^{r_n/s_n}.$$

We also obtain finite presentations of the fundamental group of the manifolds $M'(a_1/b_1, \ldots, a_n/b_n)$, and further prove that these manifolds are still examples of Takahashi manifolds. Our presentations coincide with the standard ones of the Sieradski groups

$$S(n) = \langle x_1, x_2, \ldots, x_n : x_i x_{i+2} = x_{i+1} \ (\text{indices mod } n) \rangle$$

when $a_i/b_i = -1$, for every $i = 1, 2, \ldots, n$. It was proved that $S(n)$ corresponds to a spine of the $n$-fold cyclic covering of the 3-sphere branched over the trefoil knot (see [5], also for other types of generalizations).
The following extends this result to the case of manifolds $M'(a_i/b_i)$.

**Theorem 2.** For any coprime integers $a_i$ and $b_i$, $i = 1, 2, \ldots, n$, and for any integer $n \geq 2$, the manifold $M'(a_1/b_1, \ldots, a_n/b_n)$ is homeomorphic to the Takahashi manifold $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$, where $r_i/s_i = 1$ and $p_i/q_i = a_i/b_i + 2$, and so it is the two-fold cyclic covering of the 3-sphere branched along the closure of the rational 3-string braid

$$\sigma_1^{a_1/b_1+2} \sigma_2 \cdots \sigma_1^{a_n/b_n+2} \sigma_2.$$

Finally we remark that the link $L_{2n}$ is hyperbolic (see [1], p. 222) so according to the Thurston-Jørgensen theory of hyperbolic surgery (see [24]) we get the following result:

**Theorem 3.** For any integer $n \geq 2$, and for all but a finite number of pairs $(p_i, q_i)$ and $(r_i, s_i)$, the Takahashi manifolds $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$ are hyperbolic.

2. The Takahashi manifolds

The following was proved by Takahashi in [23].

**Theorem 4.** The fundamental group of the 3-manifold $M(p_i/q_i; r_i/s_i)$ obtained by Dehn surgery along the oriented link $L_{2n}$ with surgery coefficients $p_i/q_i$ and $r_i/s_i$, $i = 1, 2, \ldots, n$, admits the finite presentation

$$\Pi_1(M(p_i/q_i; r_i/s_i)) = \langle x_1, x_2, \ldots, x_{2n} : x_2^{s_i} x_i^{p_i+1} x_2^{r_i} \sigma_2^{x_{2i} x_2^{r_i} x_2^{p_i+1}} \rangle \quad (\text{indices mod } n).$$

Generalizing an example given in [23] (case $n = 3$) yields the following

**Theorem 5.** The fundamental group of the 3-manifold $M'(a_1/b_1, \ldots, a_n/b_n)$ obtained by Dehn surgery along the oriented link $L'_n$ with surgery coefficients $a_i/b_i$, $i = 1, 2, \ldots, n$, admits the finite presentation

$$\Pi_1(M'(a_1/b_1, \ldots, a_n/b_n)) = \langle x_1, x_2, \ldots, x_n : x_i^{a_i+b_i} x_i^{b_i} x_i^{-b_i} x_i^{b_i+1} = 1 \rangle \quad (\text{indices mod } n).$$
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**Proof.** Let

\[ \Pi_1(S^3 \setminus L'_n) = \langle u_1, u_2, \ldots, u_n, w_1, w_2, \ldots, w_n : w_iu_{i-1} = u_{i-1}u_i \ (R_i) \]

\[ u_iw_{i+1} = w_{i+1}w_i \ (Q_i) \]

(indices mod $n$)

be the Wirtinger presentation of the link group of $L'_n$ where the generators $u_i, w_i$ are taken as shown in Figure 1b. If $m_i$ and $l_i$ denote the meridian and the longitude, respectively, of the $i$-th component of $L'_n$, then we have

\[ m_i = u_i, \quad l_i = w_{i+1}u_{i-1}, \quad [m_i, l_i] = 1. \]

The presentation of $\Pi_1(M'(a_1/b_1, \ldots, a_n/b_n))$ comes from the one of $\Pi_1(S^3 \setminus L'_n)$ by adding the relations $m_i^{a_i}l_i^{b_i} = 1$, for any $i = 1, 2, \ldots, n$.

Since $a_i$ and $b_i$ are coprime integers, there exist two integers $c_i$ and $d_i$ such that

\[ b_ic_i - a_id_i = 1 \]

for every $i = 1, 2, \ldots, n$.

Setting

\[ x_i = m_i^{c_i}l_i^{d_i}, \]

it follows that

\[ m_i = x_i^{b_i} \]

\[ l_i = x_i^{-a_i} \]

\[ u_i = x_i^{b_i}, \]

and hence

\[ w_i = l_{i-1}u_{i-2}^{-1} = x_{i-1}^{-a_{i-1}}x_{i-2}^{-b_{i-2}} \ (S_i). \]

Now relations $R_i$ and $S_i$ directly imply

\[ x_i^{a_i+b_i}x_{i+1}^{b_i}x_i^{-b_i}x_{i-1}^{-b_i} = 1, \]

where the indices are taken mod $n$ as usual. Finally, using these relations and $S_i$, one can verify that relations $Q_i$ become identities for every $i = 1, 2, \ldots, n$. Thus the proof is completed.

\[ \square \]

Now we are going to prove that the finite group presentations of Theorems 4 and 5 correspond to spines of the represented manifolds. For that, we first recall
some definitions about RR-systems (see [20]). Let $D$ be a regular hexagon in the plane $E^2$. For each pair of opposite faces construct a finite set (possibly empty), station say, of parallel line segments, called tracks, through $D$ with endpoints on these opposite faces. Let $\{D_i : i = 1, 2, \ldots, s\}$ be a set of disjoint regular hexagons in $E^2$. A route is an arc whose interior lies in $E^2 \setminus \bigcup_{i=1}^{s} D_i$ connecting endpoints of tracks. A RR-system is the union in $E^2$ of a finite set of hexagons with stations and a finite set of disjoint routes in $S^2 \setminus \bigcup_{i=1}^{s} D_i$ such that each endpoint of every track intersects exactly one route in one of its endpoints. A RR-system gives rise to a family of group presentations whose generators $x_i \ (i = 1, 2, \ldots, s)$ are in one-to-one correspondence with the hexagons $D_i$. In each hexagon we start from some vertex of the boundary and proceed clockwise (according to an orientation of $S^2$) along an edge which corresponds to a station $m_i$ of $D_i$. Orient the tracks of this station so that the positive direction is toward this edge. Label the stations corresponding to the second and third edges encountered by $m_i + n_i$ and $n_i$ respectively, and orient the tracks of these stations toward the respective edges. By walking along each closed arc (made by tracks and routes) we write a word on generators $x_i \ (i = 1, 2, \ldots, s)$ in the following way: as we enter in each hexagon $D_i$ we give the name of the station as exponent of $x_i$ with sign $+1$ resp. $-1$ if our direction of travel concords resp. opposes the orientation of the tracks (see [20] for more details). Osborne and Stevens proved in [20] that a finite group presentation with the same number of generators and relations corresponds to a spine of a closed connected orientable 3-manifold if and only if it arises from an RR-system. Since the group presentation of Theorem 4 resp. 5 is induced by the RR-system depicted in Figure 2 (as communicated us by Hog-Angeloni [12]) resp. 3, we get the following

**Theorem 6.** The finite group presentation

$$\langle x_1, x_2, \ldots, x_{2n} : x_2^r x_1 x_2^r = x_1, x_2, \ldots, x_{2n} : x_i x_{i+1} = x_{i+2} \ (\text{indices mod } 2n) \rangle$$

resp.

$$\langle x_1, x_2, \ldots, x_n : x_i^{a_i+b_i} x_{i+1}^{-b_i} x_i^{b_i} x_{i-1}^{-1} = 1 \rangle$$

corresponds to a spine of the 3-manifold $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$ resp. $M'(a_1/b_1, \ldots, a_n/b_n)$.

We observe that if $p_i = q_i = s_i = 1$ and $r_i = -1$, for any $i = 1, 2, \ldots, n$, then $\Pi_1(M(1, \ldots, 1; -1, \ldots, -1)) = \langle x_1, x_2, \ldots, x_{2n} : x_i x_{i+1} = x_{i+2} \ (\text{indices mod } 2n) \rangle$ is the Fibonacci group $F(2, 2n)$, first introduced by Conway in [8].
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If \( p_i = k, r_i = -k \) and \( q_i = s_i = l \), for any \( i = 1, 2, \ldots, n \), then
\[
\Pi_1(M(k/l, \ldots, k/l; -k/l, \ldots, -k/l)) = \langle x_1, x_2, \ldots, x_{2n} : x_i^{k}x_{i+1}^k = x_{i+2}^l \rangle
\]
(indices mod 2n)
is the Fractional Fibonacci group \( F^{k/l}(2, 2n) \), studied by Kim and Vesnin in [14].

If \( a_i = -1 \) and \( b_i = 1 \), for any \( i = 1, 2, \ldots, n \), then
\[
\Pi_1(M'(a_1, b_1, \ldots, a_n, b_n)) = \langle x_1, x_2, \ldots, x_n : x_i^2x_{i+2} = x_{i+1}^2 \rangle
\]
is the Sieradski group (see [22] and [5]).

Now we apply the Kirby-Rolfsen calculus on links with coefficients (see [15], [16] and [21]) to prove the following result.

**Theorem 7.** The manifold \( M'(a_1/b_1, \ldots, a_n/b_n) \) is homeomorphic to the Takahashi manifold \( M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n) \) if \( r_i/s_i = 1 \) and \( p_i/q_i = a_i/b_i + 2 \) for any \( i = 1, 2, \ldots, n \).

**Proof.** Let us consider the link \( L_{2n} \) of Figure 1a with surgery coefficients \( r_i/s_i = 1 \), for any \( i = 1, 2, \ldots, n \) and twist about each component of \( L_{2n} \) with coefficient \( r_i/s_i = 1 \) in the left-hand sense (\( \tau = -1 \)). We obtain the link \( L' \) with \( n \) components of Figure 1b and surgery coefficients \( a_i/b_i = p_i/q_i - 2 \), for any \( i = 1, 2, \ldots, n \). The sequence of surgery moves is illustrated in Figure 4.

**3. Branched coverings**

In this section we are going to prove Theorem 1. For this we use a well-known result of Montesinos (Theorem 1 of [19]) which states that a closed orientable 3-manifold, obtained by Dehn surgery along a strongly-invertible link \( L \) of \( n \) components, is a 2-fold cyclic covering of \( S^3 \) branched over a link of at most \( n + 1 \) components. Following [4] and [9], let \( \sigma_{t/h}^i \) denote the rational \( t/h \)-tangle, whose incoming arcs are \( i \)-th and \( (i + 1) \)-th strings (Here \( t \) and \( h \) are coprime integers). If \( t/h \) is written as a continued fraction
\[
t/h = \frac{1}{c_1 + \frac{1}{\ddots + \frac{1}{c_2}}}
\]
and \( t, h, c_1, \ldots, c_z \) are positive integers with \( c_z \geq 2 \), then the rational \( t/h \)-tangle is defined as in Figure 5.
Figure 4.
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Figure 5: the rational $t/h$-tangle with $t, h > 0$.

**Proof of Theorem 1.** The link $L_{2n}$ is strongly-invertible. In fact there exists an involution $p : S^3 \to S^3$ whose axis $r$ intersects each component of the link $L_{2n}$ in two points (see Figure 6a).

The Montesinos theorem assures that $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$ is a two-fold covering of the 3-sphere branched along the link constructed as follows.

Figure 6a: the strongly-invertible link $L_{2n}$. 
Let $N_i$ be a tubular neighbourhood of the $i$-th component of the link $L_{2n}$, for each $i = 1, 2, \ldots, 2n$. If $\pi : S^3 \to S^3/p$ is the canonical projection, then $\pi(N_i)$ is the trivial tangle which consists of a 3-ball $B_i$ where $\pi(r \cap N_i)$ are two arcs. Let us denote by $B'_j, j=1$ resp. $B'_j, j=2,\ldots, n$ with the underlying 3-ball $B_i$. The manifold $M(p_1/q_1, \ldots, p_n/q_n; r_1/s_1, \ldots, r_n/s_n)$ is the 2-fold branched covering of

$$\left( \bigcup_{i=1}^{2n} B'_i \right) \cup_{\pi} \left( S^3 \setminus \bigcup_{i=1}^{2n} N_i \right)$$

where the branch set is a link formed by arcs of tangles $B'_i$ and $\pi(r \cap (S^3 \setminus U_{i=1}^{2n} N_i))$. Using Reidemeister moves, one can easily see that the branch set is

$$\sigma_1^{p_1/q_1} \sigma_2^{r_1/s_1} \ldots \sigma_1^{p_n/q_n} \sigma_2^{r_n/s_n}$$

as shown in Figure 6b.

**Corollary 8.** If $p_i/q_i = p/q$ and $r_i/s_i = r/s$, for every $i = 1, 2, \ldots, n$, then the Takahashi manifold $M(p/q,r/s) = M(p/q, \ldots, p/q; r/s, \ldots, r/s)$ is the two-fold covering of the 3-sphere branched over the link $(\sigma_1^{p/q} \sigma_2^{r/s})^n$, and then $n$-fold cyclic covering of the 3-sphere branched over the link $(\sigma_1^{p/q} \sigma_2^{r/s})^2$.

In particular we obtain as corollaries some results proved in [14], [11], [10], and [7].
COROLLARY 9. If \( p_i/q_i = k/l \) and \( r_i/s_i = -k/l \), for every \( i = 1, 2, \ldots, n \), then the Takahashi manifold \( M(k/l, -k/l) = M(k/l, \ldots, k/l; -k/l, \ldots, -k/l) \) is the Fractional Fibonacci manifold defined in [14], and so it is the two-fold covering of the 3-sphere branched over the link \( (\sigma_1^{k/l} \sigma_2^{-k/l})^n \) and the \( n \)-fold cyclic covering of the 3-sphere \( S^3 \) branched over the link \( (\sigma_1^{k/l} \sigma_2^{-k/l})^2 \).

Some particular case of Corollary 9 was also treated in [17] and [18].

COROLLARY 10. If \( p_i/q_i = 1 \) and \( r_i/s_i = -1 \), for every \( i = 1, 2, \ldots, n \), then the Takahashi manifold \( M(1, -1) = M(1, \ldots, 1; -1, \ldots, -1) \) is the Fibonacci manifold considered in [10], [7], [11], and so it is the two-fold covering of the 3-sphere branched over the link \( (\sigma_1 \sigma_2^{-1})^n \) and the \( n \)-fold cyclic covering of the 3-sphere branched over the figure-eight knot \( (\sigma_1 \sigma_2^{-1})^2 \).

Now Theorems 1 and 7 directly imply Theorem 2, and the following corollaries (compare also with [5]).

COROLLARY 11. If \( a_i/b_i = k/l \), for any \( i = 1, 2, \ldots, n \), then the Takahashi manifold \( M'(k/l, \ldots, k/l) \) is the 2-fold covering of \( S^3 \) branched over the closed 3-string braid \( (\sigma_1^{k/l+2} \sigma_2)^n \), and the \( n \)-fold cyclic covering of \( S^3 \) branched over the link \( (\sigma_1^{k/l+2} \sigma_2)^2 \).

COROLLARY 12. If \( a_i/b_i = -1 \), for any \( i = 1, 2, \ldots, n \), then we have the Sieradski manifold \( M'(-1, \ldots, -1) \) which is the 2-fold covering of \( S^3 \) branched over the torus link \( (\sigma_1 \sigma_2)^n = K(n, 3) \) and the \( n \)-fold cyclic covering of \( S^3 \) branched over the trefoil knot \( (\sigma_1 \sigma_2)^2 \).

We note that the 3-string braid \( \sigma_1^{p_1} \sigma_2^{p_1'} \cdots \sigma_1^{p_n} \sigma_2^{p_n'} \) is a 6-plat (see [2]) so it may be represented as a 3-bridge link. By Theorem 5 of [3] we obtain the following

COROLLARY 13. The manifold \( M(p_1, \ldots, p_n; r_1, \ldots, r_n) \) has Heegaard genus \( \leq 2 \). In particular, the Fibonacci manifolds and the Sieradski manifolds have Heegaard genus \( \leq 2 \).

4. Orbifolds

Let \( L(1/q, 1/s, n)(2) \) resp. \( L(1/q, 1/s, 2)(n) \) be the orbifold whose underlying space is \( S^3 \) and whose singular set is the link \( L(1/q, 1/s, n) := \sigma_1^{1/q} \sigma_2^{1/s} \cdots \sigma_1^{1/q} \sigma_2^{1/s} \).
Let $y(l/q, l/s)(2, ri)$ be the orbifold whose underlying space is the 3-sphere and whose singular set is the two-component link $\mathcal{L}(l/q, l/s)$ shown in Figure 7a, with branch indices 2 and $n$ on its components (which are equivalent).

The following extends Theorem 3.2 of [14].

**Theorem 14.** *The following diagram of cyclic branched coverings holds:*
M(1/q, 1/s) \rightarrow M(1/q, 1/s)
\downarrow \quad \downarrow ^n
L(1/q, 1/s, n)(2) \rightarrow L(1/q, 1/s, 2)(n)
\downarrow ^n \quad \downarrow ^2
\mathcal{L}(1/q, 1/s)(2, n) \rightarrow \mathcal{L}(1/q, 1/s)(2, n).

**Proof.** The statement follows from the following easily verifiable facts:

1) The manifold $M(1/q, 1/s)$ admits a $(\mathbb{Z}_2 \oplus \mathbb{Z}_n)$-action which is induced by the natural $(\mathbb{Z}_2 \oplus \mathbb{Z}_n)$-symmetry of the link $L_{2n}$;

2) The quotient orbifolds $M(1/q, 1/s)/(\mathbb{Z}_2 \oplus \mathbb{Z}_n)$, $M(1/q, 1/s)/\mathbb{Z}_2$, and $M(1/q, 1/s)/\mathbb{Z}_n$ are equivalent to $\mathcal{L}(1/q, 1/s)(2, n)$, $L(1/q, 1/s, n)(2)$ and $L(1/q, 1/s, 2)(n)$, respectively.

Hence we have the following sequences of maps

$$M(1/q, 1/s) \xrightarrow{\quad ^2 \quad} L(1/q, 1/s, n)(2) \xrightarrow{\quad ^n \quad} \mathcal{L}(1/q, 1/s)(2, n)$$

and

$$M(1/q, 1/s) \xrightarrow{\quad ^n \quad} L(1/q, 1/s, 2)(n) \xrightarrow{\quad ^2 \quad} \mathcal{L}(1/q, 1/s)(2, n)$$

which induce the subgroup embeddings

$$\Pi_1(M(1/q, 1/s)) \subset \Pi_1(L(1/q, 1/s, n)(2)) \subset \Pi_1(\mathcal{L}(1/q, 1/s)(2, n))$$

and

$$\Pi_1(M(1/q, 1/s)) \subset \Pi_1(L(1/q, 1/s, 2)(n)) \subset \Pi_1(\mathcal{L}(1/q, 1/s)(2, n)),$$

where

$$[\Pi_1(\mathcal{L}(1/q, 1/s)(2, n)) : \Pi_1(L(1/q, 1/s, n)(2))]$$

$$= [\Pi_1(L(1/q, 1/s, n)(2)) : \Pi_1(M(1/q, 1/s))] = n$$

and

$$[\Pi_1(L(1/q, 1/s, n)(2)) : \Pi_1(M(1/q, 1/s))]$$

$$= [\Pi_1(\mathcal{L}(1/q, 1/s)(2, n)) : \Pi_1(L(1/q, 1/s, 2)(n))] = 2.$$
For $q = l$ and $s = -l$ we re-obtain Theorem 3.2 of [14] since $\mathcal{L}(1/l, -1/l)$ coincides with the link $\mathcal{L}^{1/l}$ defined in [14], and shown in Figure 7b for convenience.

References

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