ON THE GAUSS MAP OF B-SCROLLS

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Abstract. B-scrolls over null curves in the 3-dimensional Lorentz-Minkowski space $L^3$ are characterized as the only ruled surfaces with null rulings whose Gauss map satisfies the condition $\Delta G = \Lambda G$, $\Lambda$ being an endomorphism of $L^3$. This note completes the classification of such surfaces given by S. M. Choi in Tsukuba J. Math. 19 (1995), 285-304.

1. Introduction

Let $M$ be a connected surface in Euclidean 3-space $\mathbb{R}^3$ and let $G: M \rightarrow S^2 \subset \mathbb{R}^3$ be its Gauss map. It is well known (see [9]) that $M$ has constant mean curvature if and only if $\Delta G = \|dG\|^2G$, $\Delta$ being the Laplace operator on $M$ corresponding to the induced metric on $M$ from $\mathbb{R}^3$. As a special case one can consider Euclidean surfaces whose Gauss map is an eigenfunction of the Laplacian, i.e., $\Delta G = \lambda G$, $\lambda \in \mathbb{R}$. In [3], C. Baikoussis and D. E. Blair asked for ruled surfaces in $\mathbb{R}^3$ whose Gauss map satisfies $\Delta G = \Lambda G$, where $\Lambda$ stands for an endomorphism of $\mathbb{R}^3$. They showed that the only ones are planes and circular cylinders. Recently, S. M. Choi in [5], investigates the Lorentz version of the above result and she essentially obtains the same result. Namely, the only ruled surfaces in $L^3$ whose Gauss map satisfies $\Delta G = \Lambda G$ are the planes $R^2$ and $L^2$, as well as the cylinders $S^1 \times R^1$, $R^1 \times S^1$ and $H^1 \times R^1$.

It should be pointed out that all surfaces obtained above have diagonalizable shape operator. However, it is well known that a self-adjoint linear operator on a 2-dimensional Lorentz vector space has a matrix of exactly three types, two of them being non-diagonalizable. This makes a chief difference with regard to the

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Riemannian submanifolds that has been greatly exploited (see, for example, [1], [2] and [7]). To illustrate the current situation, we bring here the famous example of L. K. Graves (see [8]), the so called B-scroll. This is a surface which can be parametrized as a “ruled surface” in \( L^3 \) with null directrix curve and null rulings, i.e., \( X(s, t) = x(s) + tB(s) \), \( x(s) \) being a null curve and \( B(s) \) a null vector field along \( x(s) \) satisfying \( \langle x', B \rangle = -1 \).

The main purpose of this short note is to complete Choi’s classification of ruled surfaces in \( L^3 \) whose Gauss map satisfies the condition \( \Delta G = \Lambda G \). Actually, we will show that B-scrolls over null curves are the only ruled surfaces in \( L^3 \) with null rulings satisfying the above condition.

We would like to thank to the referee for bringing to our attention the preprint [6], where some related topics are considered.

2. Setup

Let \( x : I \subset \mathbb{R} \to L^3 \) be a regular curve in \( L^3 \) and \( B : I \subset \mathbb{R} \to L^3 \) a vector field along \( x \). Consider the ruled surface parametrized by \( X(s, t) = x(s) + tB(s) \).

Let us write down, as usually, \( X_s := \partial X/\partial s = x' + tB' \) and \( X_t := \partial X/\partial t = B \).

Observe that, at \( t = 0 \), \( X_s(s, 0) = x'(s) \) and \( X_t(s, 0) = B(s) \). Then \( X(s, t) \) is a regular surface in \( L^3 \) provided that the plane \( \Pi = \text{span}\{x', B\} \) is non degenerate in \( L^3 \). In fact, the matrix of the metric of \( X(s, t) \) is given by

\[
g(s, t) = \begin{pmatrix}
\langle x', x' \rangle + 2t \langle x', B' \rangle + t^2 \langle B', B' \rangle & \langle x', B \rangle + t \langle B', B \rangle \\
\langle x', B \rangle + t \langle B', B \rangle & \langle B, B \rangle
\end{pmatrix},
\]

so that when the plane \( \Pi \) is spacelike (respectively, timelike) \( X(s, t) \) parametrizes a spacelike surface (respectively, timelike surface) on the domain

\[
\{(s, t) \in I \times \mathbb{R} : \det g(s, t) > 0 \quad (\text{respectively, } \det g(s, t) < 0)\}.
\]

According to the causal character of \( x' \) and \( B \), there are four possibilities:

1. \( x' \) and \( B \) are non-null and linearly independent.
2. \( x' \) is null and \( B \) is non-null with \( \langle x', B \rangle \neq 0 \).
3. \( x' \) is non-null and \( B \) is null with \( \langle x', B \rangle \neq 0 \).
4. \( x' \) and \( B \) are null with \( \langle x', B \rangle \neq 0 \).

Let us first see that, with an appropriate change of the curve \( x \), cases (2) and (3) can be locally reduced to (1) and (4), respectively. Let \( X(s, t) \) be in case (2). Reparametrizing the null curve \( x \) and normalizing the rulings \( B \) if necessary, we may assume that

\[
\langle B, B \rangle = e = \pm 1, \quad \text{and} \quad \langle x', B \rangle = -1,
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so that

\begin{equation}
(2.1) \quad g(s, t) = \det g(s, t) = e(2t<x', B'> + t^2<B', B'>) - 1 < 0.
\end{equation}

We are looking for a curve $\gamma(s) = x(s) + t(s)B(s)$ in the surface with $<\gamma', \gamma'> = e$ and such that $\gamma'$ and $B$ are linearly independent. Writing $\gamma' = x' + t'B + tB'$, the condition $<\gamma', \gamma'> = e$ is equivalent to the following differential equation for $t = t(s)$

\begin{equation}
(2.2) \quad (t')^2 - 2et' + g(s, t) = 0.
\end{equation}

From (2.1) the discriminant of (2.2) is positive and we can locally integrate (2.2) to obtain $t$. Besides, $\gamma'$ and $B$ are linearly independent because $<\gamma', \gamma'> = <B, B'> = e$ and $<\gamma', B> = -1 + t'e \neq \pm e$ due to (2.2). This shows that $X(s, t)$ can be reparametrized as in case (1) taking $\gamma$ as the directrix curve. On the other hand, if $X(s, t)$ is in case (3), reparametrizing the null curve $x$ and normalizing the rulings $B$ if necessary, we may assume that

$<x', x'> = e = \pm 1, \quad \text{and} \quad <x', B> = -1.$

We are now looking for a curve $\gamma(s) = x(s) + t(s)B(s)$ in the surface with $<\gamma', \gamma'> = 0$ and $<\gamma', B> \neq 0$. Writing $\gamma' = x' + t'B + tB'$, the condition $<\gamma', \gamma'> = 0$ now becomes

\begin{equation}
(2.3) \quad 2t' = e + 2t<x', B'> + t^2<B', B'>.
\end{equation}

Equation (2.3) can be locally integrated to obtain $t$. Moreover, $<\gamma', B> = <x', B> \neq 0$. Thus, using the curve $\gamma$ as the directrix, $X(s, t)$ can be reparametrized as in case (4). Since case (1) has been discussed in [5], we will pay attention to the latter one which we aim to characterize in terms of the Laplacian of its Gauss map.

Therefore, let $M$ be a ruled surface in $L^3$ parametrized by $X(s, t) = x(s) + tB(s)$, where the directrix $x(s)$, as well as the rulings $B(s)$, are null. Furthermore, and without loss of generality, we may assume $<x', B> = -1$. First of all, we will do a detailed study of this kind of surfaces.

The matrix of the metric on $M$ writes, with respect to coordinates $(s, t)$, as follows

\[
\begin{pmatrix}
2t<x', B'> + t^2<B', B'> & -1 \\
-1 & 0
\end{pmatrix}.
\]
In terms of local coordinates \((y_1, \ldots, y_n)\), the Laplacian \( \Delta \) of a manifold is defined by (see [4, p. 100])

\[
\Delta = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial y_i} \left( g^{ij} \frac{\partial}{\partial y_j} \right),
\]

where \( g = \text{det}(g_{ij}) \) and \((g_{ij})\) denotes the components of the metric with respect to \((y_1, \ldots, y_n)\). Then the Laplacian on the surface \( M \) is nothing but

\[
\Delta = -2 \frac{\partial^2}{\partial s \partial t} - 2\{\langle x', B' \rangle + t\langle B', B' \rangle\} \frac{\partial}{\partial t} - \{2t\langle x', B' \rangle + t^2\langle B', B' \rangle\} \frac{\partial^2}{\partial t^2}.
\]

Now we will recall the notion of cross product in \( L^3 \). There is a natural orientation in \( L^3 \) defined as follows: an ordered basis \( \{X, Y, Z\} \) in \( L^3 \) is positively oriented if \( \text{det}[XYZ] > 0 \), where \([XYZ]\) is the matrix with \( X, Y, Z \) as row vectors. Now let \( \omega \) be the volume element on \( L^3 \) defined by \( \omega(X, Y, Z) = \text{det}[XYZ] \). Then given \( X, Y \in L^3 \), the cross product \( X \times Y \) is the unique vector in \( L^3 \) such that \( \langle X \times Y, Z \rangle = \omega(X, Y, Z) \), for any \( Z \in L^3 \).

Then the Gauss map can be directly obtained from \( X_s \times X_t \) getting

\[
G(s, t) = x'(s) \times B(s) + tB'(s) \times B(s).
\]

By putting \( C = x' \times B \), then \( \{x', B, C\} \) is a frame field along \( x \) of \( L^3 \). In this frame, we easily see that \( B' \times B = -fB \), \( f \) being the function defined by \( f = \langle x', B' \times B \rangle \). Thus

\[
(2.4) \quad G(s, t) = -tf(s)B(s) + C(s).
\]

Also, and for later use, we find out that

\[
(2.5) \quad B' = -\langle x', B' \rangle B - fC
\]

and

\[
(2.6) \quad C' = -fx' - \langle x', x'' \times B \rangle B.
\]

As for the shape operator \( S \) we have that

\[
(2.7) \quad G_t := \frac{\partial G}{\partial t} = B' \times B = -fB = -fX_t
\]

and

\[
(2.8) \quad G_s := \frac{\partial G}{\partial s} = -\langle x', x'' \times B \rangle + tf')X_t - fX_s.
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So $S$ writes down as

$$
\begin{pmatrix}
  f \\
  tf' + \langle x', x'' \times B \rangle & f
\end{pmatrix}.
$$

A straightforward computation yields

$$(2.9) \quad \Delta G = 2\{f' + tf' \langle B', B' \rangle \}B - 2f^2C.$$

We now present a very typical example.

**Example.** Let $x(s)$ be a null curve in $L^3$ with Cartan frame $\{A, B, C\}$, i.e., $A, B, C$ are vector fields along $x$ in $L^3$ satisfying the following conditions:

$$
\langle A, A \rangle = \langle B, B \rangle = 0, \quad \langle A, B \rangle = -1,
$$

$$
\langle A, C \rangle = \langle B, C \rangle = 0, \quad \langle C, C \rangle = 1,
$$

and

$$
\begin{aligned}
x' &= A, \\
C' &= -aA - \kappa(s)B,
\end{aligned}
$$

$a$ being a constant and $\kappa(s)$ a function vanishing nowhere. Then the map

$$
X : L^2 \to L^3
$$

$$(s, t) \to x(s) + tB(s)
$$

defines a Lorentz surface $M$ in $L^3$ that L. K. Graves [8] called a B-scroll. It is not difficult to see that a unit normal vector field is given by

$$
G(s, t) = -atB(s) + C(s),
$$

and the shape operator writes down, relative to the usual frame $\{\partial X/\partial s, \partial X/\partial t\}$, as

$$
S = \begin{pmatrix}
  a & 0 \\
  k(s) & a
\end{pmatrix}.
$$

Thus the B-scroll has non-diagonalizable shape operator with minimal polynomial $P_S(u) = (u - a)^2$. It has constant mean curvature $\kappa = a$ and constant Gaussian curvature $K = a^2$ and satisfies $\Delta G = \lambda G$, where $\lambda = 2a^2$. 
3. **Main results**

It seems natural to state the following problem: *is a B-scroll the only ruled surface in \( L^3 \) with null rulings satisfying the equation \( \Delta G = \Lambda G \)?*  
Then our major result states as follows.

**Theorem 1.** *B-scrolls over null curves are the only ruled surfaces in \( L^3 \) with null rulings satisfying the equation \( \Delta G = \Lambda G \).*

From here and Choi’s result we have got the complete classification of ruled surfaces in the 3-dimensional Lorentz-Minkowski space whose Gauss map satisfies \( \Delta G = \Lambda G \).

**Corollary 2.** *A ruled surface \( M \) in \( L^3 \) satisfies the equation \( \Delta G = \Lambda G \) if and only if \( M \) is one of the following surfaces:*

1. \( R^2 \), \( L^2 \) and the cylinders \( S^1 \times R^1 \), \( R^1 \times S^1 \) and \( H^1 \times R^1 \);
2. a B-scroll over a null curve.

**Proof of the Theorem.** Suppose that the Gauss map of \( M \) satisfies the equation \( \Delta G = \Lambda G \). From Choi’s result we may suppose that \( M \) has null rulings, so we only have to study the case (4). We are going to show that the function \( f = \langle x', B' \times B \rangle \) is constant or, equivalently, that the open set \( \mathcal{U} = \{ s \in I : f(s)f'(s) \neq 0 \} \) is empty. Otherwise, for \( s \in \mathcal{U} \), differentiating with respect to \( t \) in \( \Delta G = \Lambda G \), we have

\[
2f'\langle B', B' \rangle B = -f\Lambda B,
\]

where we have used equations (2.4), (2.7) and (2.9). By (2.5) we obtain \( \langle B', B' \rangle = f^2 \), so that from (3.10) we see that \( -2f^2 \) is an eigenvalue of \( \Lambda \), unless \( f = 0 \). Then \( f \) is a constant function, which is a contradiction that finishes the proof.

**References**


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