GLOBAL SOLVABILITY FOR THE GENERALIZED DEGENERATE KIRCHHOFF EQUATION WITH REAL-ANALYTIC DATA IN $\mathbb{R}^n$

By
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1. Introduction

Kirchhoff equation was proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string and it is expressed as follows

$$\partial_t^2 u(t, x) - \left( \varepsilon^2 + \frac{1}{2l} \int_0^l |\partial_x u(t, x)|^2 dx \right) \partial_x^2 u(t, x) = 0,$$

where $t > 0$, $l > 0$, $\varepsilon > 0$ and $x \in [0, l]$. In 1940 S. Bernstein [B] proved the global solvability for analytic initial data and local solvability for $C^m$-class initial data to the following initial boundary value problem:

$$\begin{cases}
\partial_t^2 u(t, x) - \left( a + b \int_0^{2\pi} |\partial_x u(t, x)|^2 dx \right) \partial_x^2 u(t, x) = 0 & (t > 0, x \in [0, 2\pi]), \\
u(t, x) = 0 & (t \geq 0, x = 0, 2\pi), \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x),
\end{cases}$$

where $a > 0$ and $b > 0$. In 1971, T. Nishida [Nd] proved Bernstein’s result in case of $a = 0$. Equation (1.2) can be regarded as the following more generalized equation:

$$\begin{cases}
\partial_t^2 u(t, x) - M \left( \int_{\Omega} |\nabla_x u(t, x)|^2 dx \right) \Delta_x u(t, x) = 0 & (t > 0, x \in \Omega), \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in \Omega \subset \mathbb{R}^n,
\end{cases}$$

with boundary condition

$$u(t, x) = \varphi \quad \text{on} \ [0, \infty) \times \partial \Omega.$$
In case of (1.2), $\Omega = [0, 2\pi]$, $\phi = 0$ and $M(\eta) = a + b\eta$. In 1975, S. I. Pohožaev [P] proved the existence and uniqueness of time global real-analytic solution for the problem (1.3)−(1.4) under the assumption of $n \geq 1$ and $M(\eta) \in C^1([0, \infty))$ where $\Omega$ is bounded and $\phi = 0$.* On the other hand, in case that $\Omega = \mathbb{R}^n$, Y. Yamada [Yd] proved the existence and uniqueness of global solution of (1.3) in 1980. In 1984, K. Nishihara [Nh] showed the global existence of the quasi-analytic solution in case that $M(\eta)$ is locally Lipschitz continuous and non-degenerate. In that year, A. Arosio and S. Spagnolo [AS] proved the existence of time global $2\pi$-periodic solution for real-analytic data in case that $\Omega = [0, 2\pi]^n$ under some assumptions for $M(\eta) \in C^0$. In 1992, P. D'Ancona and S. Spagnolo [DS] relaxed the assumptions in [AS] to any $M(\eta) \in C^0$. Moreover, the equation (1.3)−(1.4) can be generalized as

$$\left\{ \begin{array}{l} \partial_t^2 u(t, x) + M((Au(t, \cdot), u(t, \cdot))) Au(t, x) = f(t, x) \quad (t > 0, x \in \Omega), \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), \quad x \in \Omega \subset \mathbb{R}^n, \end{array} \right.$$ (1.5)

with boundary condition

$$u(t, x) = \phi \quad \text{on } [0, \infty) \times \partial \Omega.$$ (1.6)

Here $A$ is a degenerate elliptic operator of second order defined as $Au(t, x) = \sum_{i,j=1}^n D_{x_i} (a_{ij}(x) D_{x_j} u(t, x))$, $D_{x_i} = \left(1/\sqrt{-1}\right)(\partial/\partial x_i)$. Suppose that $[a_{ij}(x)]_{i,j=1,...,n}$ is a real-analytic symmetric matrix which satisfies that

$$a(x, \xi) = \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq 0$$ (1.7)

and there are $c_0 > 0$ and $\rho_0 > 0$ such that

$$|D_{x_i}^2 a_{ij}(x)| \leq c_0 \rho_0^{-|x|}|x|! \quad i, j = 1, \ldots, n,$$ (1.8)

for $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n) \in \mathbb{N}^n$, $(Au, u)$ is an inner product of $Au(x)$ and $u(x)$ in $L^2_{x}(\Omega)$ and $M(\eta)$ satisfies

$$M(\eta) \in C^0([0, \infty)) \quad \text{and} \quad M(\eta) \geq 0.$$ (1.9)

If $a_{ij}(x) = \delta_{ij}$ and $f(t, x) \equiv 0$, then equation (1.5) coincides with equation (1.3), where $\delta_{ij}$ is Kronecker's delta. In 1994 K. Kajitani and K. Yamaguti [KY] proved the existence and uniqueness of time global real-analytic solution for (1.5) in case

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*In fact he proved the existence and uniqueness of time global solution to more general problem on some suitable Hilbert space.
that $\Omega = \mathbb{R}^n$, $u_0(x), u_1(x) \in L^2(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$, $M(\eta) \in C^1([0, \infty))$, $M(\eta) \geq 0$, and $a_\eta(x) \geq 0$ are $C^0(\mathbb{R}^n)$ functions, respectively, where $C^0(\mathbb{R}^n)$ is the set of real analytic functions in $\mathbb{R}^n$. In 1995 K. Yamaguti [Yg] extended the result of [KY] for quasi-analytic data under the assumption of $M(\eta) > 0$.

Our main theorem in this paper is an extension of the result of [KY] in case of $M(\eta) \in C^0$. At first we introduce some definitions in order to state our main theorem.

**Definition 1.1.** For $s \in \mathbb{R}$ and $\rho > 0$, we define the function space $H^s_\rho$ by

$$H^s_\rho = \{u(x) \in L^2_\xi(\mathbb{R}^n); \langle \xi \rangle^s e^{\rho(\xi)\hat{u}(\xi)} \in L^2_{\xi}(\mathbb{R}^n)\},$$

(1.10)

where $\xi = (\xi_1, \ldots, \xi_n)$, $\langle \xi \rangle = (1 + \xi_1^2 + \cdots + \xi_n^2)^{1/2}$, and $\hat{u}(\xi)$ stands for Fourier transform of $u$. If we introduce the inner product $(\cdot, \cdot)_{H^s_\rho}$ of $H^s_\rho$ such that

$$(u, v)_{H^s_\rho} = (e^{\rho(\cdot)}\hat{u}(\cdot), e^{\rho(\cdot)}\hat{v}(\cdot)),$$

(1.11)

then $H^s_\rho$ is a Hilbert space, where $(\cdot, \cdot)_s$ is an inner product of $H^s$ which is the normal Sobolev space (See [Ku]). For $\rho < 0$ we define $H^s_\rho$ as the dual space of $H^{-s}_\rho$.

**Definition 1.2.** For $\rho \in \mathbb{R}$, define the operator $e^{\rho(D)}$ from $H^s_\rho$ into $H^s$ as follows:

$$e^{\rho(D)}u(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi + \rho(\xi)\hat{u}(\xi)}d\xi,$$

(1.12)

for $u \in H^s_\rho$, where $x = (x_1, \ldots, x_n)$, $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ and $d\xi = (2\pi)^{-n}d\xi$. Note that $(e^{\rho(D)})^{-1} = e^{-\rho(D)}$ is a mapping from $H^s$ into $H^s_\rho$.

Hilbert space $H^s_\rho$ and the operator $e^{\rho(D)}$ were introduced in [Ka] and [KY]. In this paper we define the new space $H^s_{\rho, \delta, \kappa}$ as a weighted subspace of $H^s_\rho$.

**Definition 1.3.** For $s, \rho, \delta \in \mathbb{R}$ and $\kappa > 0$, we define $H^s_{\rho, \delta, \kappa}$ as

$$H^s_{\rho, \delta, \kappa} = \{u(x) \in \mathcal{S}'(D); \langle D \rangle^s \{\langle x \rangle^\delta e^{\rho(D)}u(x)\} \in L^2_\kappa(\mathbb{R}^n)\},$$

(1.13)

where $\langle x \rangle_\kappa = (\kappa^2 + x_1^2 + \cdots + x_n^2)^{1/2}$ and $\mathcal{S}'$ is the dual space of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions in $\mathbb{R}^n$. And we define the inner product $(\cdot, \cdot)_{H^s_{\rho, \delta, \kappa}}$ of $H^s_{\rho, \delta, \kappa}$ as follows:

$$(u, v)_{H^s_{\rho, \delta, \kappa}} = (\langle \cdot \rangle^\delta e^{\rho(D)}u(\cdot), \langle \cdot \rangle^\delta e^{\rho(D)}v(\cdot))_\kappa.$$

(1.14)
The principal method of the proof of this theorem is based on [Ka] and [KY]. In this paper we introduce the new space $H^{s}_{p,\delta,\kappa}$ which is a weighted subspace of $H^{s}_{p}$ for $\delta > 0$, and we consider the global solvability for the equation in it. For positive real numbers $p$ and $\kappa$ and for non-negative real numbers $s$ and $\delta$, the function spaces $H^{s}_{p}$ and $H^{s}_{p,\delta,\kappa}$ are included the intersection of $L^2(\mathbb{R}^{n})$ and $C^{\alpha}(\mathbb{R}^{n})$. Our main theorem in this paper is the global existence of the real-analytic solution which has initial condition in $H^{s}_{p,\delta,\kappa}$.

**Main Theorem.** Assume that (1.7), (1.8) and (1.9) are valid. Let $0 < \rho_{1} < \rho_{0}/\sqrt{n}$, $\delta > 0$, $\kappa > 0$ and put $\rho(t) = \rho_{1}e^{-\gamma t}$ for $\gamma > 0$. Then there exists $\gamma > 0$ such that for any $u_{0} \in H^{2}_{\rho_{1},\delta,\kappa}$, $u_{1} \in H^{1}_{\rho_{1},\delta,\kappa}$ and for any $f(t, x)$ satisfying $\langle x \rangle^{q \delta}e^{\rho(t)\langle x \rangle^{q}}f(t, x) \in C^{1}(\{0, \infty\}; H^{1})$, the Cauchy problem (1.5) with $\Omega = \mathbb{R}^{n}$ has a solution $u(t, x)$ that satisfies $\langle x \rangle^{q \delta}e^{\rho(t)\langle x \rangle^{q}}u(t, x) \in \bigcap_{j=0}^{2}C^{2-j}(\{0, \infty\}; H^{1})$.

2. Preliminaries

In this section we introduce some propositions and lemmas to prove the following lemmas and our main theorem.

**Proposition 2.1.** Assume that $a(x, \xi) \in S^{2}$ is non-negative. Then there are positive constants $C_{1}$ and $C_{2}$ such that

$$\Re(\text{Op}(a)u, u) \geq -C_{1}\| u \|_{s}$$

(2.1) and

$$\sum_{\|a\|=1}^{\|a\|} \left\{\| \text{Op}(a_{1})u \|_{s_{1}}^{2} + \| \text{Op}(a^{(s)})u \|_{s}^{2} \right\} \leq C_{2}\| u \|_{s}^{2} + \Re(\text{Op}(a)u, u)$$  

(2.2)

for $u \in H^{s+2}$, where $S^{m}$ is the symbol-class of pseudo-differential operator of order $m$ (See [Ku]), $\text{Op}(a)$ is the pseudo-differential operator defined as

$$\text{Op}(a)u = \int_{\mathbb{R}^{n}} e^{i\langle x \rangle \cdot a(x) \xi} \hat{u}(\xi) d\xi$$

for $u(x) \in \mathcal{S}$, where $\| \cdot \|_{s}$ is a norm of $H^{s}$.

For a proof of this proposition, refer to [FP].

**Proposition 2.2.** (i) Let $a(x, \xi) \in C^{\infty}(\mathbb{R}^{n} \times \mathbb{R}^{n})$ be a ‘double order’ symbol in the ‘double order symbol space’ $SG_{1}^{(m_{1}, m_{2})}$.

$$SG_{1}^{(m_{1}, m_{2})} = \{a(x, \xi) \in C^{\infty}(\mathbb{R}^{n} \times \mathbb{R}^{n}); a_{(\beta)}^{(s)}(x, \xi) = O(\langle \xi \rangle^{-|s|} \langle x \rangle^{m_{2} - |\beta|})\}$$

(2.3)
for \((m_1, m_2) \in \mathbb{R} \times \mathbb{R}\) where \(a^{(2)}_{(\beta)}(x, \xi) = \partial^2_x D_x^\beta a(x, \xi)\), and if we \(a(x, \xi)\) define the operator \(Op(a)\) by

\[
(Op(a))(x, D)f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \tilde{a}(x, \xi) \tilde{f}(\xi) \hat{\partial}_\xi, \quad f \in \mathcal{S},
\]

(2.4)

then \(Op(a)\) is the bounded linear operator from \(H^s_{p, \xi} \cap \mathbb{R}\) into \(H^{s_1-m_1} \cap \mathbb{R}\) for each \(s_1, s_2 \in \mathbb{R}\).

(ii) If \(s > s'\) and \(\delta > \delta'\), then the embedding \(H^s_{p, \xi} \hookrightarrow H^{s'}_{p, \xi} \cap \mathbb{R}\) is compact.

(iii) Let \(c(x, \xi)\) be the symbol of the product \(Op(a)Op(b)\) of \(a \in SG^{(l_1, l_2)}\) and \(b \in SG^{(m_1, m_2)}\), then \(c(x, \xi)\) has the asymptotic expansion:

\[
c(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} a^{(\alpha)}(x, \xi) b^{(\alpha)}(x, \xi).
\]

(2.5)

This proposition is introduced in \([S]\).

**Lemma 2.3.** (i) Let \(u \in H^s_{p, 0} = H^s\), then for \(p > 0\),

\[
||D^\alpha u||_s \leq ||u||_{H^s_{p, \xi}} p^{-|\alpha|} |\alpha|!
\]

(2.6)

and

\[
|D^\alpha u(x)| \leq C_n ||u||_{H^s_{p, \xi}} p^{-|\alpha|+n+|\alpha|-s}(|x| + n + |\alpha|)!
\]

(2.7)

for \(x \in \mathbb{R}^n\) and \(\alpha \in \mathbb{N}^n\).

(ii) Let \(u(x)\) be a function in \(H^\infty\) and \(s \in \mathbb{R}\). If \(u(x)\) satisfies

\[
||D^\alpha u||_{H^s_{p, \xi}} \leq c_0 p^{-|\alpha|} |\alpha|!
\]

(2.8)

for every multi-index \(\alpha \in \mathbb{N}^n\), then \(u(x) \in H^s_{p, \xi}\) for \(p < p_1/\sqrt{n}\).

For a proof of this lemma, refer to \([KY]\).

**Lemma 2.4.** Let \(\delta \geq 0, c > 0\) and \(\varepsilon \in (0, 1]\), then \((x)_c^\delta\) is a real-analytic function satisfying

\[
|D^\alpha (x)_c^\delta| \leq (8c^{-1}) |\alpha| (1 + \varepsilon)^\delta |\alpha|! (x)_c^\delta - |\alpha|,\]

(2.9)

for \(x \in \mathbb{R}^n\). Moreover if \(0 \leq \delta \leq 1\), then

\[
|D^\alpha (x)_c^\delta| \leq 4 |\alpha|! (x)_c^\delta - |\alpha|\]

(2.10)

for \(x \in \mathbb{R}^n\).
For a proof, refer to [Ka].

Let \( a(x) \) be a real-analytic function in \( \mathbb{R}^n \) satisfies that there are \( c_0 > 0 \) and \( \rho_0 > 0 \) such that

\[
|D_x^a a(x)| \leq c_0 \rho_0^{-|a|} |x|!
\]

(2.11)

for any \( x \in \mathbb{R}^n \) and any multi-index \( \alpha \in \mathbb{N}^n \). Define the multiplier \( a \cdot (a \cdot u)(x) = a(x) u(x) \). Let us define \( a(\rho; x, D)u(x) = e^{\rho(D)} a \cdot e^{-\rho(D)} u(x) \) for \( u(x) \in L^2(\mathbb{R}^n) \) and denote by \( a(\rho; x, \xi) \) its symbol.

**Proposition 2.5.** (i) \( a(\rho; x, D) \) is a pseudo-differential operator of order 0 and its symbol has the following expansion:

\[
a(\rho; x, \xi) = a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r(\rho; x, \xi),
\]

where

\[
a_1(x, \xi) = -\sum_{j=1}^n D_x a(x) \partial_{\xi_j} \langle \xi \rangle
\]

and \( a_2 \) and \( r \) respectively satisfy

\[
|a_2^{(a)}(\rho; x, \xi)| \leq C_{2\rho_0} |\xi|^{-1-|a|},
\]

(2.14)

\[
|r^{(a)}(\rho; x, \xi)| \leq C_{2\rho_0} |\xi|^{-1-|a|}
\]

(2.15)

for \( x, \xi \in \mathbb{R}^n, |\rho| < \rho_0/\sqrt{n} \) and \( a, \beta \in \mathbb{N}^n \).

(ii) If \( \rho = \rho(t) \in C^0([0, T]) \) for \( T > 0 \), then \( a(\rho(t); x, \xi) \in C^0([0, T]; S^0) \).

For a proof of (i), refer to [KY] and for (ii) refer to [Ka].

**Corollary 2.6.** Define the operator \( A_\Lambda \) by

\[ A_\Lambda u(x) = e^{\rho(D)} (A e^{-\rho(D)} u(x)) \]

(2.16)

for \( A = \sum_{i,j=1}^n D_i (a_{ij}(x) D_j) \). Then \( A_\Lambda \) and \( \langle x \rangle^\delta A_\Lambda \langle x \rangle^{-\delta} \) are pseudo-differential operators of order 2 and their symbols have the following expansions respectively;

\[
\sigma(A_\Lambda)(x, \xi) = \sum_{i,j=1}^n (a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r(\rho; x, \xi)) \xi^j \bar{\xi}^i,
\]

(2.17)

\[
\sigma(\langle x \rangle^\delta A_\Lambda \langle x \rangle^{-\delta})(x, \xi) = \sum_{i,j=1}^n (a(x) + \rho a_1(x, \xi) + \rho^2 a_2(\rho; x, \xi) + r(\rho; x, \xi)) \xi^j \bar{\xi}^i,
\]

(2.18)
where \( \sigma(P)(x, \xi) \) denotes the symbol of a pseudo-differential operator \( P(x, D) \), \( a_1 = a_{1y} \) and \( a_2 = a_{2y} \) are defined in Proposition 2.5, and both \( r_1 = r_{1y} \) and \( r_2 = r_{2y} \) belong to \( S^{-1} \). Moreover, for \( \rho(t) \in C^0([0, T]) \), \( \sigma(A_A)(t, x, \xi) \) and \( \sigma((\langle x \rangle^\delta_A)(x)^{-\delta})(t, x, \xi) \) belong to \( C^0([0, T]; S^2) \).

**Proof.** It is obvious by Proposition 2.2 and Proposition 2.5.

**Lemma 2.7.** If \( u(x) \in H^s_{\rho, \delta, k} \) for \( \delta > 0 \), then \( u(x) \) is a real-analytic function whose radius of convergence is \( \rho_1 \), where \( \rho_1 \leq \min\{\kappa/8, \rho_0\} \) and \( 0 < \rho_0 < \rho \).

**Proof.** Note that \( (\langle x \rangle^\delta_A)(x) \in H^s_{\rho} \) if \( u(x) \in H^s_{\rho, \delta, k} \):

\[
|D_x^\alpha u(x)| = |D_x^\alpha ((\langle x \rangle^{-\delta})(\langle x \rangle^\delta_A)(x) u(x))| \\
\leq \sum_{\alpha' \leq \alpha} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) |D_x^{\alpha - \alpha'} (\langle x \rangle^{-\delta})(\langle x \rangle^\delta_A)(x) u(x)| \\
\leq C_1 \sum_{\alpha' \leq \alpha} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) |x'|! |x - \alpha'|! \left( \frac{\kappa}{8} \right)^{|x - x'|} \rho_0^{-|x'|} \\
\leq C_2 \rho_1^{-|x|!} |x|!, \tag{2.19}
\]

where \( \rho_1 \leq \min\{\kappa/8, \rho_0\} \), \( 0 < \rho_0 < \rho \) and we used Lemma 2.3, Lemma 2.4 and the estimate;

\[
\sum_{\alpha' \leq \alpha} \left( \begin{array}{c} \alpha \\ \alpha' \end{array} \right) |x'|! |x - \alpha'|! \eta_1^{-|x'|} \eta_2^{-|x - x'|} \leq \frac{\eta_1}{\eta_1 - \eta_2} \eta_2^{-|x|!} |x|!, \tag{2.20}
\]

for \( 0 < \eta_2 < \eta_1 \). \( \square \)

3. Existence of solutions for the linear problem

In this section, we consider the local existence for the following linear Cauchy problem:

\[
\begin{align*}
\partial_t^2 u(t, x) + m(t)Au(t, x) &= f(t, x), \\
u(0, x) &= u_0(x), \quad \partial_t u(0, x) = u_1(x),
\end{align*}
\tag{3.1}
\]

where \( m(t) \) is a non-negative continuous function in \([0, \infty)\).

At first we introduce a proposition to prove the existence of the linear problem (3.1).

Let \( P(t) = [p_{ij}(t, x, D)]_{i,j=1,...,d} \) be a matrix consisting of pseudo-differential
operators whose symbols \( p_j(t, x, \xi) \) all belong to the class \( C^0([0, T]; S^1) \). Let us consider the following linear Cauchy problem:

\[
\begin{align*}
\frac{d}{dt} U(t) &= P(t)U(t) + F(t), \quad t \in (0, T], \\
U(0) &= U_0,
\end{align*}
\]

(3.2)

where \( U(t) = (U_1(t), \ldots, U_d(t)) \) is an unknown vector valued function, \( F(t) = (F_1(t), \ldots, F_d(t)) \) and \( U_0 = (U_{01}, \ldots, U_{0d}) \) are known vector valued functions. Then the following proposition is concluded.

**Proposition 3.1.** Suppose that \( \det(\lambda I - p(t, x, \xi)) \neq 0 \) for \( \lambda \in C^1(\mathbb{R}^n) \) with \( \Re \lambda > -c_0 \langle \xi \rangle \) for some positive constant \( c_0 \), \( t \in [0, T] \) and \( |\xi| \gg 1 \). Take an arbitrary real number \( s \). Then for any \( U_0 \in (H^{s+1})^d \) and for any \( F(t) \in C^0([0, T]; (H^{s+1})^d) \), there exists a unique solution \( U(t) \in C^1([0, T]; (H^s)^d) \cap C^0([0, T]; (H^{s+1})^d) \) of (3.2).

This proposition was introduced as Proposition 4.5 in [M]. For the proof of the proposition, refer to [M].

Let \( v(t, x) = \langle x \rangle^s e^{\Lambda(t)} u(t, x) \) and transform the equation (3.1) of \( u(t, x) \) to the equation of \( v(t, x) \) such that

\[
\begin{align*}
\langle x \rangle^s \langle \partial_t - \Lambda \rangle^2 \langle x \rangle^s v(t, x) + m(t) \langle x \rangle^s A \langle x \rangle^s v(t, x) &= g(t, x), \\
v(0, x) &= v_0(x), \quad \partial_t v(0, x) = v_1(x),
\end{align*}
\]

(3.3)

where \( \Lambda = \Lambda(t) = \rho(t) \langle D \rangle \), \( \Lambda_t = \Lambda_t(t) = \rho_1(t) \langle D \rangle \), \( \rho(t) = \rho_1 e^{-|\gamma|} \) for \( \rho_1 > 0, \gamma > 0 \) and \( g(t, x) = \langle x \rangle^s e^{\Lambda(t)} f(t, x) \). Then the following lemma is concluded for the Cauchy problem (3.3).

**Lemma 3.2.** Assume that \( u_0 \in H^{s+2}, v_1 \in H^{s+1} \) and \( g(t, x) \in C^0([0, T]; H^{s+1}) \), then there is \( \gamma_0 > 0 \) and the Cauchy problem (3.3) has a unique solution \( v(t, x) \) such that

\[
v(t, x) \in \bigcap_{j=0}^2 C^{2-j}([0, T]; H^{s+1})
\]

for all \( \gamma \geq \gamma_0 \).

**Proof.** Now let us put \( V(t) = (V_1(t), V_2(t)) \), \( V_0 = (V_{01}, V_{02}) \), \( F(t) = \)

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t$(0, g(t))$ and

$$
P(t) = \begin{pmatrix}
(D) & (D)
\end{pmatrix}
- m(t) (D) - 1
\end{pmatrix}.
$$

(3.4)

Where $A_A$ is defined by (2.16). Then we consider the following linear Cauchy problem:

$$
\begin{cases}
\frac{d}{dt} V(t) = P(t) V(t) + F(t), & t \in (0, T], \\
V(0) = V_0.
\end{cases}
$$

(3.5)

At first we show that the symbols of pseudo-differential operator $P(t)$ satisfies the conditions of Proposition 3.1. Clearly $\sigma((D) (x)^{\delta} A_t (x)^{-\delta} (D)^{-1})(t, x, \xi)$, $\sigma((x)^{\delta} A_t (x)^{-\delta})(t, x, \xi)$ and $\sigma((x)^{\delta} A_A (x)^{-\delta} (D)^{-1})(t, x, \xi)$ belong to $C^0([0, T]; S^1)$ by Corollary 2.6.

$$
\det(\lambda I - \sigma(P)(t, x, \xi))
= (\lambda - \sigma((D) (x)^{\delta} A_t (x)^{-\delta} (D)^{-1})(t, x, \xi))(\lambda - \sigma((x)^{\delta} A_t (x)^{-\delta})(t, x, \xi))
+ m(t) \sigma((x)^{\delta} A_A (x)^{-\delta} (D)^{-1})(t, x, \xi)(\xi)
= (\lambda - \rho'(t) (\xi) - \rho'(t) p^0_0 (x, \xi))(\lambda - \rho'(t) (\xi) - \rho'(t) p^0_2 (x, \xi))
+ m(t) (\sigma(A_A)(t, x, \xi) + p^1_1 (t, x, \xi)),
$$

(3.6)

where $\sigma(P) = [\sigma(P_j)]_{i, j = 1, 2}$, $p^0_j (x, \xi) \in S^0 (j = 1, 2)$ and $p^1_j (t, x, \xi) \in ([0, T]; S^1)$, and they satisfy

$$
\sigma((D) (x)^{\delta} A_t (x)^{-\delta} (D)^{-1})(t, x, \xi) = \rho'(t) (\xi) + \rho'(t) p^0_0 (x, \xi)
$$

(3.7)

$$
\sigma((x)^{\delta} A_t (x)^{-\delta})(t, x, \xi) = \rho'(t) (\xi) + \rho'(t) p^0_2 (x, \xi)
$$

(3.8)

$$
\sigma((x)^{\delta} A_A (x)^{-\delta} (D)^{-1})(t, x, \xi) = \sigma(A_A)(t, x, \xi) + p^1_1 (t, x, \xi).
$$

(3.9)

Therefore we have

$$
\det(\lambda I - \sigma(P)(t, x, \xi))
= \lambda^2 - \rho'(t) \lambda (2 (\xi) + p^1_1 (x, \xi) + p^0_2 (x, \xi))
+ \rho'(t)^2 ((\xi) + p^1_1 (x, \xi)) ((\xi) + p^0_2 (x, \xi))
+ m(t) (\sigma(A_A)(t, x, \xi) + p^1_1 (t, x, \xi)).
$$

(3.10)
Let \( \det(\lambda I - \sigma(P)(t, x, \xi)) = 0 \) and solve it in \( \lambda \), then we have

\[
\lambda = \rho'(t) (2\xi) + p_1^0(x, \xi) + p_2^0(x, \xi) \\
\pm \{ \rho'(t)^2 \{ -2\xi (p_1^0(x, \xi) + p_2^0(x, \xi)) - 3p_1^2(x, \xi)p_2^0(x, \xi) + p_1^0(x, \xi)^2 + p_2^0(x, \xi)^2 \} \\
- 4m(t) \sum_{i,j=1}^{n} \{ a(x) + \rho(t)a_1(x, \xi) + \rho(t)^2a_2(\rho(t); x, \xi) + r_2(\rho(t); x, \xi) \} \xi_i \xi_i \\
+ p_1^0(t, x, \xi) \}^{1/2},
\]

where \( a, a_1, a_2 \) and \( r_2 \) are defined in (2.18). Then the order of \( \Re \lambda \) is as follows:

\[
\Re \lambda = -\gamma \rho_1 e^{-\eta t} O(|\xi|) \pm \{ m(t) \rho_1 e^{-\eta t} O(|\xi|) + O(|\xi|^1/2) \}. \quad (3.12)
\]

Hence, obviously there are \( \gamma_0 > 0 \) and \( c_0 > 0 \) such that \( \det(\lambda I - \sigma(P)(t, x, \xi)) > 0 \) for any \( \gamma \) satisfying \( \gamma > \gamma_0 \), \( |\xi| \gg 1 \) and \( \Re \lambda > -c_0(\xi) \). Therefore equation (3.5) has a unique solution \( V(t) = (V_1(t), V_2(t)) \) satisfying

\[
V_1(t), V_2(t) \in C^1([0, T]; H^s) \cap C^0([0, T]; H^{s+1}) \quad (3.13)
\]

for \( V_{01}, V_{02} \in H^{s+1} \). Now, if we let \( v(t) = (D)^{-1}V_1(t) \), then \( v(t) \) satisfies

\[
v(t, x) \in C^1([0, T]; H^{s+1}) \cap C^0([0, T]; H^{s+2}) \quad (3.14)
\]

for \( v(0) = v_0 \in H^{s+2} \). Then we know that \( v(t, x) \) satisfying

\[
\partial_t(D)v(t, x) = \langle D \rangle \langle x \rangle^\delta \Lambda \langle x \rangle^{-\delta} v(t, x) + \langle D \rangle V_2(t), \quad (3.15)
\]

and obviously \( V_2(t) \) is represented by \( v(t, x) \) such that

\[
V_2(t) = \partial_t v(t, x) - \langle x \rangle^\delta \Lambda \langle x \rangle^{-\delta} v(t, x), \quad V_2(0) = V_{02} \in H^{s+1}. \quad (3.16)
\]

Then by (3.5), \( v(t, x) \) satisfies

\[
\langle x \rangle^\delta (\partial_t - \Lambda) \langle x \rangle^{-\delta} + m(t) \langle x \rangle^\delta A \langle x \rangle^{-\delta} v(t, x) = g(t, x). \quad (3.17)
\]

It shows that \( v(t, x) \) is a solution of (3.3) satisfying

\[
v(t, x) \in \bigoplus_{j=0}^{2} C^{2-j}([0, T]; H^{s+j}). \quad \square \quad (3.18)
\]

By Lemma 3.2, obviously we have the following lemma.
Lemma 3.3. For $u_0 \in H^{s+2}_{p_0, \beta, \kappa}$, $u_1 \in H^{s+1}_{p_0, \beta, \kappa}$ and $\langle x \rangle^2 e^{\Lambda(t)} f(t, x) \in C^0([0, T]; H^{s+1})$, there exists a positive constant $\gamma_0$ and the Cauchy problem (3.1) has a unique solution $u(t, x)$ such that

$$\langle x \rangle^2 e^{\Lambda(t)} u(t, x) \in \bigcap_{j=0}^{2} C^{2-j}([0, T]; H^{s+j}).$$

for all $\gamma \geq \gamma_0$.

4. A priori estimate of solution for the linear problem

Let $0 < T < \infty$, $m(t)$ be a non-negative function in $C^0([0, T])$, $\rho(t)$ a positive function in $C^1([0, T]) \cap C^0([0, T])$ such that $\rho(t) < 0$, $\phi(t)$ a positive function in $C^1([0, T])$ satisfying $\phi'(t) \leq 0$ for $t \geq 0$ and $m_0(t) = \int_0^T \chi_\delta(t-\tau) m(\tau) d\tau + \epsilon$, where $\epsilon(\epsilon)$ satisfies $0 < \tilde{\epsilon} < \epsilon$ and $|\int_0^T \chi_\delta(t-\tau) m(\tau) d\tau - m(t)| < \epsilon$, and $\chi_\delta(t) = e^{-1}_\delta(e^{-1}t)$, $\chi(t) \in C^\infty_0((0, 1))$ satisfying $\chi(t) \geq 0$ and $\int_0^1 \chi(t) dt = 1$ for $0 \leq t \leq T$. Then we define $E_n(t)$ as follows:

$$E_n(t) = \frac{1}{2} \left\{ \|\langle x \rangle^2 e^{\Lambda(t)} \rho(t) \|_2^2 + \phi(t) \|v(t)\|_{l^{s+1}}^2 + m(t) (A(D)^4 v(t), (D)^4 v(t)) \right\}. \tag{4.1}$$

for the solution $v(t, x)$ of (3.3).

Lemma 4.1. Assume that $m(t)$ is a non-negative function in $C^0([0, T])$, $\phi(t) = e^{-2\phi t}$, $\rho(t) = \rho_1 e^{-\phi t}$ and $v(t, x)$ is a solution of (3.3) satisfying $v(t, x) \in \bigcap_{j=0}^{2} C^{2-j}([0, T]; H^{s+j})$, then there exist positive constants $\epsilon$, $\gamma_0$, $c$ and $c_0$ such that

$$E_n(t) \leq E_n(0) e^{\int_0^t \phi(\tau) d\tau} + \frac{1}{2} \int_0^t e^{\int_0^\tau \phi(\mu) d\mu} \|g(\tau)\|_d d\tau, \tag{4.2}$$

for $t \in [0, T]$ and for any $\gamma \geq \gamma_0$, where

$$q(t) = \frac{c}{2} \left( \frac{m_0(t)}{m(t)} + \frac{m(t)^2}{\rho(t)} + \frac{m(t)^2 \rho(t)}{\phi(t)} + \frac{m(t)^2 \rho(t)}{\phi(t)} + m_0(t) |\rho_1(t)| + c_0 \right). \tag{4.3}$$

Proof. Note that $m_0(t) \to m(t)$ in $L^1([0, t])$ for arbitrary $t \in [0, T]$.

Differentiating both sides in (4.1), we have
\begin{align}
2E_s(t)E_s(t) &= \frac{d}{dt} \left\{ \frac{1}{2} \| \langle \varphi \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \|_2^2 \right\} \\
&\quad + \frac{d}{dt} \left\{ \frac{1}{2} \varphi(t) \| v(t) \|_{s+1}^2 \right\} \\
&\quad + \frac{d}{dt} \left( \frac{1}{2} m_e(t)(A(D)^2v(t), (D)^2v(t)) \right).
\end{align}

\begin{align}
(4.4) = \Re (\langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t)^2 \langle \cdot \rangle_\kappa^\delta v(t), \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \\
&\quad + \Re (\langle \cdot \rangle_\kappa^\delta \Lambda_t (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t), \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \\
&\quad = \Re (g(t), \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \\
&\quad - m(t) \Re (\langle \cdot \rangle_\kappa^\delta A \langle \cdot \rangle_\kappa^\delta v(t), \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \\
&\quad + \Re (\Lambda_t \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t), \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \\
&\quad + \Re (\rho \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t), \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \\
&\quad \leq \| g(t) \|_s E_s(t) \\
&\quad - m(t) \Re (|\Lambda_t|^{-1/2} \langle \cdot \rangle_\kappa^\delta A \langle \cdot \rangle_\kappa^\delta v(t), |\Lambda_t|^{1/2} \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \\
&\quad - ||| \Lambda_t |^{1/2} \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t) \rangle_2^2 \\
&\quad + C |p| E_s(t)^2,
\end{align}

where \( p_t^0(x,D) \in Op(S^0) \), and we used an equality; \( \| Pu \|_s \leq C_s \| u \|_{s+m} \) for some positive constant \( C_s \) provided \( P \in Op(S^m) \) and \( u \in H^s \) (See [Ku]).

\begin{align}
(4.5) = \frac{1}{2} \varphi'(t) \| v(t) \|_{s+1}^2 \\
&\quad + \varphi(t) \Re ((D) \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_t) \langle \cdot \rangle_\kappa^\delta v(t), (D) v(t) \rangle_s \\
&\quad + \varphi(t) \Re ((D) \langle \cdot \rangle_\kappa^\delta \Lambda_t \langle \cdot \rangle_\kappa^\delta v(t), (D) v(t) \rangle_s
\end{align}
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\[ \frac{\theta'(t)}{\theta(t)} E_s(t)^2 \leq (4.11) \]

\[ + \frac{1}{2} \| \Lambda_i^{1/2} \langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_i \langle \cdot \rangle_\kappa^\delta v(t) \|_r^2 \]  

\[ + \frac{\theta(t)^2}{2 |\rho|} \| v(t) \|_{s+3/2}^2 \]  

\[ + \frac{\theta(t) \rho}{2} \| v(t) \|_{s+3/2}^2 \]  

\[ + C_2 E_s(t)^2, \]  

\[ (4.6) = \frac{1}{2} m_e'(t) (A(D)^s v(t), (D)^s v(t)) + \]  

\[ + m_e(t) \Re (\Lambda_i^{1/2} (D)^{-1} A(D)^s v(t), \Lambda_i^{1/2} (\langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_i \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \rangle_{s+3/2}^2 \]  

\[ + m_e(t) \Re (\Lambda_i^{1/2} (D)^{-1} A(D)^s v(t), \Lambda_i^{1/2} (\langle \cdot \rangle_\kappa^\delta \Lambda_i \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \]  

\[ \leq \frac{|m_e'(t)|}{m_e(t)} E_s(t)^2 \]  

\[ + m_e(t) \Re (\Lambda_i^{1/2} (D)^{-1} A(D)^s v(t), \Lambda_i^{1/2} (\langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_i \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \]  

\[ + m_e(t) \Re (\langle D \rangle^{1/2} A(D)^s v(t), (D)^{s+1/2} v(t) \)  

\[ + m_e(t) \Re (\langle D \rangle^{-1/2} A(D)^s v(t), \rho^0 v(t) \)  

\[ (4.18) + (4.19) \leq m_e(t) \rho \Re (A(D)^{s+1/2} v(t), (D)^{s+1/2} v(t)) \]  

\[ + \frac{C_3 m_e(t) |\rho|}{v(t)} E_s(t)^2, \]  

\[ (4.8) + (4.17) \leq \| \Lambda_i^{1/2} m_e(t) (D)^{-1} A(D)^s v(t), \Lambda_i^{1/2} (\langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_i \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \]  

\[ - (\langle D \rangle^{1/2} m(t) \langle \cdot \rangle_\kappa^\delta A \langle \cdot \rangle_\kappa^\delta v(t), \Lambda_i^{1/2} (\langle \cdot \rangle_\kappa^\delta (\partial_t - \Lambda_i \langle \cdot \rangle_\kappa^\delta v(t) \rangle_s \) \]  

\[ (4.22) \]
Then, using the equality:
\[
\begin{align*}
m_e(t)\langle D \rangle^{-\delta} A(D)^{s} - m(t)\langle x \rangle^{\delta} A\langle x \rangle^{-\delta} \\
= m(t)(A - \langle x \rangle^{\delta} A\langle x \rangle^{-\delta}) + m(t)(A - A_{\lambda}) \\
+ m(t)(\langle D \rangle^{-\delta} A(D)^{s} - A) + \{m_e(t) - m(t)\}\langle D \rangle^{-\delta} A(D)^{s},
\end{align*}
\]
(4.23)
we obtain the estimate;
\[
\begin{align*}
\|\Lambda|^{-1/2}\{m_e(t)\langle D \rangle^{-\delta} A(D)^{s} - m(t)\langle \cdot \rangle^{\delta} A\langle \cdot \rangle^{-\delta}\}v(t)\|_s \\
\leq |m_e(t) - m(t)|\|\Lambda|^{-1/2}\langle D \rangle^{-\delta} A(D)^{s}v(t)\|_s \\
+ m(t)\{\|\Lambda|^{-1/2}(\langle D \rangle^{-\delta} A(D)^{s} - A)v(t)\|_s \\
+ \|\Lambda|^{-1/2}(A - A_{\lambda})v(t)\|_s \\
+ \|\Lambda|^{-1/2}(A - A_{\lambda})v(t)\|_s\}
\leq C_2|m_e(t) - m(t)|\rho_i^{-1/2}\|v(t)\|_{s+3/2} \\
+ m(t)(\|\Lambda|^{-1/2}p^{1}(\cdot, D)v(t)\|_s\rho(t)\|\Lambda|^{-1/2}\tilde{a}_1(\cdot, D)v(t)\|_s \\
+ \rho(t)^2\|\Lambda|^{-1/2}\tilde{a}_2(\rho; \cdot, D)v(t)\|_s + \|\Lambda|^{-1/2}\tilde{r}(\rho; \cdot, D)v(t)\|_s
\leq (C_2|m_e(t) - m(t)| + C_3m(t)\rho(t)^2)|\rho_i|^{-1/2}\|v(t)\|_{s+3/2} \quad (4.24) \\
+ C_4m(t)|\rho_i|^{-1/2}\|v(t)\|_{s+1} \quad (4.25) \\
+ m(t)\rho(t)|\rho_i|^{-1/2}\|\tilde{a}_1(\cdot, D)v(t)\|_{s-1/2}, \quad (4.26)
\end{align*}
\]
where \( p^{1}(x, \xi) \in S^{1}, \quad \tilde{a}_1(x, \xi) = \sum_{i,j=1}^{n} a_1 \xi_i \xi_j, \quad \tilde{a}_2(\rho, x, \xi) = \sum_{i,j=1}^{n} a_2 \xi_i \xi_j \) and \( \tilde{r}(\rho; x, \xi) = \sum_{i,j=1}^{n} r_i \xi_i \xi_j, \) and \( a_1, a_2 \) and \( r_i \) defined in (2.17). Besides, by Proposition 2.1, (4.26) is estimated in the following:
\[
\begin{align*}
\|\tilde{a}_1(\cdot, D)v(t)\|_{s-1/2} = \left\| \sum_{|\alpha|=1} \tilde{a}_1(\cdot, D)^{s}(D)^{-1}v(t) \right\|_{s-1/2}^2 \\
\leq C_5 \sum_{|\alpha|=1} \|\tilde{a}_1(\cdot, D)v(t)\|_{s-1/2}^2 + C_6\|v(t)\|_{s+1/2}^2 \\
\leq C_7\mathcal{R}(\tilde{a}_1(\cdot, D)v(t), v(t))_{s+1/2} + C_8\|v(t)\|_{s+1/2}^2 \\
\leq C_7\mathcal{R}(A\langle D \rangle^{s+1/2}v(t), (D)^{s+1/2}v(t)) + C_9\|v(t)\|_{s+1/2}^2 \quad (4.27)
\end{align*}
\]
(4.28)
where $\tilde{a}(x, \xi) = \sigma(A)(x, \xi)$. Therefore (4.8) + (4.17) is estimated as below

\begin{equation}
(4.8) + (4.17) \leq 2\{C_1^2 m_e(t) - m(t)|^2 + C_2^2 m(t)^2 \rho A|^2 \rho |^1 \|v(t)\|_{L^3}^2 \end{equation}

\begin{equation}
+ \{4C_2^2 m(t)^2 \rho |^1 + C_{10} m(t)^2 \rho |^1 \rho |^1 E(t)^2 \end{equation}

\begin{equation}
+ C_7 m(t)^2 \rho |^1 \Re(A(D)^{s+1/2}v(t), (\langle D \rangle)^{s+1/2}v(t)) \end{equation}

\begin{equation}
+ \frac{1}{2} \|\Lambda_1\|^{1/2} (\langle \gamma \rangle (\partial_t - \Lambda_1) \langle \gamma \rangle \gamma |^1 v(t)\|_{L^2}^2. \end{equation}

Note that $C_j (j = 1, \ldots, 10)$ are positive constants independent of $t$ and $\gamma$. Hence combing the preceding estimates, we have the following estimate for (4.1);

\begin{equation}
2E'(t)E(t) \leq \|g(t)\|_E(t)
\end{equation}

\begin{equation}
+ c \left( |\rho(t)| + \frac{|m(t)|}{\rho(t)} \right) + \frac{m(t)^2 \rho(t)^2}{\rho(t)} + \frac{m(t)^2 \rho(t)^2}{\rho(t)} + \frac{m(t)^2 \rho(t)^2}{\rho(t)} + c_0 \right) E(t)^2
\end{equation}

\begin{equation}
+ c^2 \left( \frac{m(t)^2 \rho(t)^2}{\rho(t)} + \frac{m(t)^2 \rho(t)^2}{\rho(t)} + \frac{m(t)^2 \rho(t)^2}{\rho(t)} + \frac{m(t)^2 \rho(t)^2}{\rho(t)} \right) \|v(t)\|_{L^3}^2
\end{equation}

\begin{equation}
+ c^2 (m(t)^2 \rho(t)^2 + m(t)^2 \rho(t)^2 |\rho(t)|^{-1}) \Re(A(D)^{s+1/2}v(t), (\langle D \rangle)^{s+1/2}v(t)).
\end{equation}

Thus, if we let $\gamma > 0$ and $\varepsilon > 0$ satisfying

\begin{equation}
\varepsilon \leq e^{-2\gamma T}, \quad \gamma^2 \geq \max \left\{ \sup_{0 \leq t \leq T} \left\{ \frac{|\rho(t)|^1 \|m(t)\|_{L^2}^2}{\rho(t)} \right\} , \frac{2}{\rho(t)} + M_0^2 \rho(t)^2 \right\},
\end{equation}

where $M_0 = \max_{0 \leq t \leq T} m(t)$, then the third and the fourth terms are non-positive.

**Lemma 4.2.** Assume that $m(t)$ is a non-negative function satisfying $m(t) \in C^0([0, T]) \cap L^1([0, T])$ and $v(t, x) \in \mathbb{C}^{2-j}([0, T]; H^{s+j})$. Then there are $\rho(t)$ and $\varphi(t)$ in $C^1([0, T])$ with $\rho_1(t) \in L^1([0, T])$, $\rho(0) = \rho_1$ and $\varepsilon > 0$ such that the estimate (4.2) is established for (4.3).

**Proof.** If we choose $\rho(t)$ and $\varepsilon > 0$ suitably, we can prove that (4.35) and (4.35) are non-positive. Indeed, put $\varphi(t)$ and $\rho(t)$;

\begin{equation}
\varphi(t) = \rho_1^2 e^{-2c t} \left\{ 1 + \int_0^t \left| m(t) \right| (1 + 1/\sqrt{m(t)}) dt \right\},
\end{equation}

\begin{equation}
\rho(t) = \left( \rho_1 e^{-ct} - c \int_0^t \varphi(t) \left| m(t) - m(t) \right| dt \right) e^{-c \int_0^t \left| m(t) \right| (1 + 1/\sqrt{m(t)}) dt},
\end{equation}
then \( \varphi(t) \) and \( \rho(t) \) belong to \( C^1([0, T]) \) with \( \rho, \in L^1([0, T]) \) and \( \rho(t) > 0 \) for sufficiently small \( \varepsilon > 0 \), and they satisfy

\[
\begin{align*}
\begin{cases} 
\rho(0) = \rho_1, \\
\rho_\varepsilon(t) \leq -c \bigg( \frac{m_\varepsilon(t) - m(t)}{\sqrt{\varphi(t)}} + \frac{m(t)\rho(t)^2}{\sqrt{\varphi(t)}} + \frac{m(t)\rho(t)}{\sqrt{m_\varepsilon(t)}} + \sqrt{\varphi(t)} \bigg) 
\end{cases}
\end{align*}
\]

(4.40)

for \( t \in (0, T) \). Hence we obtain (4.2). \( \square \)

**Lemma 4.3.** Assume that \( m(t), \varphi(t) \) and \( \rho(t) \) satisfy the conditions of Lemma 4.1 and that \( u(t, x) \) is a solution of the Cauchy problem (3.1) satisfying (3.19), then \( u(t, x) \) has the inequality as

\[
(e^{-2s}) \| \langle \gamma \rangle^{\delta} e^{\rho(t)(D)} u(t) \|_{L^1}^2 + \| \langle \gamma \rangle^{\delta} e^{\rho(t)(D)} \rho_\varepsilon u(t) \|_{L^2}^2 \leq c e^\int_0^t q(\tau) d\tau \left( \| \langle \gamma \rangle^{\delta} e^{\rho_1(D)} u_0 \|_{L^1} + \| \langle \gamma \rangle^{\delta} e^{\rho_1(D)} u_1 \|_{L^1} \right) + \int_0^t \| \langle \gamma \rangle^{\delta} e^{\rho(t)(D)} f(t) \|_{L^1} d\tau,
\]

(4.41)

for \( t \in [0, T] \), where \( q(\tau), \gamma \) and \( \varepsilon \) are given by Proposition 4.1, and the positive constant \( c \) is independent of \( \gamma \).

**Proof.** It is obvious by Lemma 4.1.

**5. Local existence of solutions for the nonlinear problem**

Let \( 0 \leq \tau < T_1 \). For \( T \in (\tau, T_1] \) we consider the Cauchy problem:

\[
\begin{align*}
\begin{cases} 
\partial_\tau^2 u(t, x) + M((Au(t), u(t))) Au(t, x) = f(t, x), & \tau < t < T, \\
u(t, x) = u_0(x), & \partial_\tau u(t, x) = u_1(x).
\end{cases}
\end{align*}
\]

(5.1)

**Theorem 5.1.** Assume that (1.4), (1.5) and (1.6) are valid. Let \( 0 < \rho_1 < \rho_0 / \sqrt{n} \). Then for any \( u_0(x) \in H^{s+2}_{\rho_1, \delta} \), \( u_1(x) \in H^{s+1}_{\rho_1, \delta} \) and \( \langle x \rangle^{\delta} e^{\rho(t)(D)} f(t, x) \in C^0([0, T_1]; H^{s+1}) \) with \( \rho(t) = \rho_1 e^{-\gamma(t)} \), there exist \( T \in (\tau, T_1] \) and \( \gamma_0 > 0 \) such that the Cauchy problem (5.1) has a solution satisfying

\[
\langle x \rangle^{\delta} e^{\rho(t)(D)} u(x, t) \in \bigcap_{j=0}^2 C^{2-j}([\tau, T]; H^{s+j})
\]

(5.2)

for any \( \gamma \geq \gamma_0 \).
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Proof. We may assume $t = 0$ without loss of generality. We shall prove the existence of the solution of (5.1) by Schauder's fixed point theorem. For $T > 0$ and $s \in \mathbb{R}$, we introduce a space of functions:

$$X_{T, \delta, \kappa} = \{ w(t, x); \langle x \rangle^\delta e^{\rho(t)D}w(t, x) \in C^0([0, T]; H^{s+1}) \cap C^1([0, T]; H^s) \}$$

(5.3)

equipped with its norm $\| \cdot \|_{X_{T, \delta, \kappa}}$ as

$$\| w \|_{X_{T, \delta, \kappa}} = \sup_{0 \leq t \leq T} \left\{ \frac{1}{2} \left( \| \langle \rangle^\delta e^{\rho(t)D}w(t) \|_{H^{s+1}}^2 + \| \langle \rangle^\delta e^{\rho(t)D} \partial_t w(t) \|_2^2 \right) \right\}^{1/2}$$

(5.4)

for every $w \in X_{T, \delta, \kappa}$. Let $B_{T, \delta, \kappa}(R)$ be a convex subspace of $X_{T, \delta, \kappa}$ such that

$$B_{T, \delta, \kappa}(R) = \left\{ u \in X_{T, \delta, \kappa}; \langle x \rangle^\delta e^{\rho(t)D}u(t, x) \in \bigoplus_{j=0}^2 C^{2-j}([0, T]; H^{s+j}), \| u \|_{X_{T, \delta, \kappa}} \leq R \right\},$$

(5.5)

for $R \gg 1$. We now define the two functions

$$m(t) = m(t; w) = M(\eta(t; w)), \quad \eta(t; w) = \sum_{i,j=1}^n (a_{ij}D_i w(t), D_j w(t)),$$

(5.6)

for each $w \in X_{T, 0, \kappa}^{s+1}$, where $s' < s$. Note that $m(t) = M(\eta(t; w)) \in C^0([0, T])$, and if $w \in B_{T, 0, \kappa}(R)$ for $R > 0$, then for arbitrary fixed $v > 0$, there exists a positive constant $\varepsilon$ independent of $w$ such that

$$\int_0^T |m_\varepsilon(t; w) - m(t; w)|dt < v,$$

(5.7)

where $m_\varepsilon(t; w) = \int_0^T \chi_\varepsilon(t - \tau)m(\tau; w)d\tau + \varepsilon$ and $\chi_\varepsilon(t)$ is defined in section 4. Then we define the mapping $\Phi$ from $w \in X_{T, 0, \kappa}^{s+1}$ into $u \in X_{T, 0, \kappa}^{s+1}$ such that

$$\partial_t^2 u(t, x) + M(\eta(t; w))A u(t, x) = f(t, x).$$

(5.8)

We shall prove that $\Psi$ is a compact mapping from $B_{T, 0, \kappa}(R)$ into itself for $s' < s$ and sufficiently small $T$. By Lemma 3.3, $u(t, x)$ in (5.8) satisfies

$$\langle x \rangle^\delta e^{\rho(t)D}u(t, x) \in \bigoplus_{j=0}^2 C^{2-j}([0, T]; H^{s+j})$$

(5.9)

for $u_0 \in H^{s+2}_{\rho_1, \delta, \kappa}$, $u_1 \in H^{s+1}_{\rho_1, \delta, \kappa}$ and every fixed $w \in B_{T, 0, \kappa}(R)$. Then by Lemma 4.1,
we have
\[
\left\{ \frac{1}{2} \left( \| \gamma \delta e^{\rho(t)D} u(t) \|_{s+1}^2 + \| \gamma \delta e^{\rho(t)D} \partial_t u(t) \|_s^2 \right) \right\}^{1/2}
\]
\[
\leq e^{\gamma T} \left\{ \frac{1}{2} \left( e^{-2\eta T} \| \gamma \delta e^{\rho(t)D} u(t) \|_{s+1}^2 + \| \gamma \delta e^{\rho(t)D} \partial_t u(t) \|_s^2 \right) \right\}^{1/2}
\]
\[
\leq e^{\gamma T} \left\{ c \varepsilon \int_0^T q(\tau) d\tau \left( \| \gamma \delta e^{\rho(D)} u_0 \|_{s+1} + \| \gamma \delta e^{\rho(D)} u_1 \|_s + \int_0^t \| \gamma \delta e^{\rho(D)} f(\tau) \|_s d\tau \right) \right\}
\]
\[
\leq c' \varepsilon \int_0^T (q(\tau)+\gamma) d\tau,
\]
(5.10)
where \( c' \) is independent of \( T \) and \( R \). Therefore for sufficiently large \( R \), we can find \( T(R) = T > 0 \) such that
\[
c' \varepsilon \int_0^T (q(\tau)+\gamma) d\tau = R.
\]
(5.11)

On the other hand, by Proposition 2.2, we have obviously that the embedding \( B_{T,\delta,\kappa}(R) \hookrightarrow B_{T,0,\kappa}(R) \) is compact for \( s' < s \) and \( \delta > 0 \). Hence the mapping \( \Psi \) defined (5.8) is a compact mapping from \( B_{T,0,\kappa}(R) \) into itself. Then by Schauder's fixed point theorem, \( \Psi \) has a fixed point \( u(t,x) \) in \( B_{T,0,\kappa} \). Further by Lemma 3.3, the fixed point is a solution of (5.1) satisfying
\[
\langle x \rangle \delta e^{\rho(D)} u(t,x) \in \bigcap_{j=0}^2 C^{2-j}([0,T]; H^{s+j})
\]
(5.12)
for \( u_0 \in H^{s+2} \) and \( u_1 \in H^{s+1} \).

6. Global existence of solution for the non-linear problem

In this section we shall prove our main theorem. Now we introduce the following energy:
\[
E(t) = \frac{1}{2} (\| \partial_t u(t) + u(t) \|^2 + \| u(t) \|^2 + F(\eta(t)))
\]
(6.1)
where \( F(\eta) = \int_0^\eta M(\lambda) d\lambda \) and \( \eta(t) = (Au(t), u(t)) \). Then for the energy \( E(t) \), according to [DS] and [KY], the following energy estimate is concluded.

**Proposition 6.1.** Assume that \( M(\eta) \) is a non-negative continuous function in \([0, \infty)\) and \( f(t,x) \in C^0([0,T]; L^2) \). If \( u(t,x) \) is a solution of the Cauchy problem
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(1.3) in \((0, T)\) such that \(u(t, x) \in \bigcap_{j=0}^{2} C^{2-j}([0, T); H^j)\), then we have the energy estimate:

\[
E(t)^2 + \int_0^t e^{3(t-\tau)} M(\eta(\tau))\eta(\tau) d\tau \leq E(0)^2 e^{3t} + \frac{1}{2} \int_0^t e^{3(t-\tau)} \|f(\tau)\|^2 d\tau \tag{6.2}
\]

for \(t \in [0, T)\).

**Proof.** Differentiating (6.1), from the equation (1.3) we get,

\[
2E'(t)E(t) = \Re(f(t) + \partial_t u(t), \partial_t u(t) + u(t)) + \Re(\partial_t u(t), u(t)) - M(\eta(t))\eta(t)
\]

\[
\leq \frac{1}{2} \|f(t)\|^2 + 3E(t)^2 - M(\eta(t))\eta(t)
\]

(6.3)

for \(t \in [0, T)\), which yields (6.2). \(\square\)

**Corollary 6.2.** If (6.2) holds and \(T < \infty\), then \(M(\eta(t)) \in L^1([0, T])\).

**Proof.** From (6.2), it is evident that \(M(\eta(t))\eta(t) \in L^1([0, T])\). On the other hand

\[
\int_0^t M(\eta(\tau)) d\tau = \int_{[0, t] \cap \{\eta(\tau) > 1\}} M(\eta(\tau)) d\tau + \int_{[0, t] \cap \{\eta(\tau) \leq 1\}} M(\eta(\tau)) d\tau
\]

\[
\leq \int_0^t M(\eta(\tau))\eta(\tau) d\tau + \sup_{0 \leq \tau \leq 1} M(\eta(t))
\]

(6.4)

for all \(t \in [0, T)\), which implies that \(M(\eta(t)) \in L^1([0, T])\). \(\square\)

Now we can prove our main theorem. Let \(\Lambda(t, \gamma) = \rho_1 e^{-\gamma t}(D)\) and \(T^*\) the real number defined by

\[
T^* = \max \left\{ T > 0; \text{there exist } \gamma > 0 \text{ and a solution } u(t, x) \text{ satisfying (1.3)} \right\}
\]

in \((0, T)\) such that \(\langle x \rangle^2 e^{\Lambda(t, \gamma)} u(t, x) \in \bigcap_{j=1}^{2} C^{2-j}([0, T); H^j)\}.

Theorem 5.1 ensures \(T^* > 0\). We shall claim \(T^* = \infty\). Suppose that \(T^* < 0\). Then it follows from Proposition 6.2 that \(m(t) = M(Au(t), u(t)) \in L^1([0, T^*])\). Hence, Proposition 3.2 and the fact that \(m(t) \in C^0([0, T^*]) \cap L^1([0, T^*])\) yield that \(v(t, x) = \langle x \rangle^2 e^{\Lambda(t)} u(t, x)\) which satisfies (3.19) with \(s = 0, 1\) and \(T = T^*\), where \(\Lambda(t) = \rho(t)(D)\) and \(\rho(t)\) is introduced in (4.39). Let us take \(\gamma > 0\) such that
\( \rho_1 e^{-\gamma t} \leq \rho(t) \) for \( t \in [0, T^*) \). Then the definition of \( T^* \) and (4.2) imply

\[
\langle x \rangle_e^{\delta} e^{A(t,t)} u(t, x) \in \bigcap_{j=0}^{2} C^{2-j}([0, T^*]; H^j),
\]

where \( \Lambda(t, \gamma) = \rho_1 e^{-\gamma t} \langle D \rangle \). Hence we have the limits \( u(T^* - 0) \in H^2 \) and \( \partial_t u(T^* - 0) \) which satisfy

\[
\langle x \rangle_e^{\delta} e^{A(T^*, \gamma)} u(T^* - 0) \in H^1.
\]

Therefore, applying Theorem 5.1 with \( \rho_2 = \rho_1 e^{\gamma T^*} \), we have a solution \( \bar{u}(t, x) \) of the Cauchy problem (5.1) in \((T^*, T), \ T > T^* \) with initial data \( \bar{u}(T^*) = u(T^* - 0) \) and \( \partial_t \bar{u}(T^*) = \partial_t u(T^* - 0) \), which satisfies

\[
\langle x \rangle_e^{\delta} \exp(\rho_2 e^{-\gamma(t-T^*)} \langle D \rangle) \bar{u}(t, x) \in \bigcap_{j=0}^{2} C^{2-j}([T^*, T]; H^j).
\]

(6.5)

Then \( \Lambda(t, \gamma) = \rho_2 e^{-\gamma(T-T^*)} \langle D \rangle \) implies that

\[
\langle x \rangle_e^{\delta} e^{A(t, \gamma)} \bar{u}(t, x) \in \bigcap_{j=0}^{2} C^{2-j}([T^*, T]; H^j).
\]

(6.6)

Now let us define

\[
w(t, x) = \begin{cases} u(t, x), & t \in (0, T^*) \\ \bar{u}(t, x), & t \in [T^*, T). \end{cases}
\]

(6.7)

Then \( w(t, x) \) has to satisfy (1.3) in \((0, T) \) and

\[
\langle x \rangle_e^{\delta} e^{A(t, \gamma)} w(t, x) \in \bigcap_{j=0}^{2} C^{2-j}([0, T]; H^j).
\]

(6.8)

This result contradicts the definition of \( T^* \). Thus, we have proved that \( T^* = \infty \). 

\[\square\]

References


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