ON ALMOST KÄHLER MANIFOLDS OF CONSTANT CURVATURE

By

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§ 1. Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the corresponding Kähler form is closed (or equivalently $\mathcal{S}_{X,Y,Z}((\nabla X) Y, Z) = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, where $\mathcal{S}$ and $\mathfrak{X}(M)$ denotes the cyclic sum and the Lie algebra of all differentiable vector fields on $M$ respectively). A Kähler manifold, which is defined by $\nabla J = 0$, is necessarily an almost Kähler manifold. It is well-known that an almost Kähler manifold with integrable almost complex structure is a Kähler manifold. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. Concerning the integrability of almost Kähler manifolds, the following conjecture by S. I. Goldberg is known ([2]):

**Conjecture.** A compact almost Kähler Einstein manifold is a Kähler manifold.

K. Sekigawa proved the above conjecture is true for the case where the scalar curvature is nonnegative ([7]). However, the above conjecture is still open in the case where the scalar curvature is negative.

Concerning the above conjecture, Z. Olszak proved that, in dimensions $\geq 8$, an almost Kähler manifold of constant curvature is a flat Kähler manifold ([6]). In dimension 4, D. E. Blair claimed that the same assertion is valid by making use of quaternionic analysis. However, there is a gap in the final step of his proof. The statement “each $a_i = 0$” is not correct ([1], p. 1038). Recently, K. Sekigawa and the author proved that a $2n(\geq 4)$-dimensional complete almost Kähler manifold of constant sectional curvature is a flat Kähler manifold ([5]). The proof in [5] is essentially dependent on the completeness.
The aim of the present paper is to prove that the hypothesis of completeness in the above result is needless, namely we prove the following.

**Theorem.** In dimensions \( \geq 4 \), there are no almost Kähler manifolds of constant curvature unless the constant is 0, in which case the manifold is Kahlerian.

In dimensions \( \geq 8 \), above Theorem is nothing but the result of Z. Olszak, but we shall give a proof which does not depend on the dimension.

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§ 2. Preliminaries

Let \( M = (M, J, g) \) be a 2\( n \)-dimensional almost Kähler manifold. We denote by \( V \) and \( R \) the Riemannian connection and the curvature tensor of \( M \) with respect to the Riemannian metric \( g \). Here, we assume that the curvature tensor \( R \) is defined by \( R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_[X,Y] \) for \( X, Y \in \mathfrak{X}(M) \). Further, we assume that \( M \) is oriented by the volume form \( d\Omega = \Omega^n/n! \), where \( \Omega \) is the Kähler form defined by \( \Omega(X, Y) = g(X, JY) \) for \( X, Y \in \mathfrak{X}(M) \). We recall a curvature identity for almost Kähler manifold due to A. Gray ([3]):

\[
R(w, x, y, z) - R(w, x, Jy, Jz) - R(Jw, Jx, y, z) + R(Jw, Jx, Jy, Jz) \\
+ R(Jw, x, Jy, z) - R(Jw, x, Jz, y) - R(Jx, w, Jy, z) + R(Jx, w, Jz, y) \\
= 2g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y)
\]

for \( w, x, y, z \in T_p M, p \in M \). If \( M \) is also a space of constant curvature \( c \), then the equality (2.1) becomes

\[
2c\{g(x, y)g(w, z) - g(x, z)g(w, y) - g(x, Jy)g(w, Jz) + g(x, Jz)g(w, Jy)\} \\
= g((\nabla_w J)x - (\nabla_x J)w, (\nabla_y J)z - (\nabla_z J)y)
\]

and hence, we have

\[
\|\nabla J\|^2 = -8cn(n - 1).
\]

Since we may assume that \( n \geq 2 \), this implies that \( c \leq 0 \) and that \( c = 0 \) if and only if \( M \) is a flat Kähler manifold.
In the present paper, unless otherwise specified, we assume that all manifolds are connected and of class $C^\infty$ and that all tensor fields are of class $C^\infty$.

§ 3. Proof of the theorem

If there exists a strictly almost Kähler structure on a space of constant curvature, then we have, in the view of the argument in section 2, that locally hyperbolic space must carry such a structure. We denote by $H^{2n}$ the $2n$-dimensional hyperbolic space of constant curvature $-1$. As a model of $H^{2n}$, we take the upper half space $R^{2n}_+ = \{(x_1, \ldots, x_{2n}) \in R^{2n} | x_1 > 0\}$ of $R^{2n}$ and the metric $g$ given by

$$g = \frac{1}{x_1^2} \sum_{i=1}^{2n} dx_i \otimes dx_i.$$ 

Let \( \{X_i = x_i(\partial/\partial x_i)\}_{i=1, \ldots, 2n} \) be a global orthonormal frame field. Then

$$[X_1, X_i] = -[X_i, X_1] = X_i \quad \text{for} \quad i = 2, \ldots, 2n,$$

and are otherwise zero. If we put $\Gamma_{ijk} = g(\nabla X_i X_j, X_k)$, then

$$\Gamma_{i1} = -\Gamma_{ii} = 1 \quad \text{for} \quad i = 2, \ldots, 2n,$$

and are otherwise zero.

Now, we assume that there exists a compatible almost Kähler structure $(J, g)$ on a connected open neighborhood $U$ of a point $p \in H^{2n}$. If we put $J_{ij} = g(JX_i, X_j)$, then

$$J_{ij} = -J_{ji}, \quad \sum_{u=1}^{2n} J_{iu} J_{ju} = \delta_{ij}.$$ 

We can choose isometries $\phi^{(1)}, \ldots, \phi^{(2n)}$ of a neighborhood of $p$ in $U$ such that

1. $\phi^{(a)}(p) = p$ for $a = 1, \ldots, 2n$;
2. $(\phi^{(1)})_p$ is the identity mapping of the tangent space at $p$;
3. $(\phi^{(a)})_p(X_1) = (X_a)_p$, $(\phi^{(a)})_p(X_a) = (X_1)_p$ and $(\phi^{(a)})_p(X_i) = (X_i)_p$ \( (i \neq 1, a) \)

for $a = 2, \ldots, 2n$.

We note that $\phi^{(1)}$ is the identity mapping. For brevity, we shall write $\phi^{(a)}$ instead of $(\phi^{(a)})_*$. We put $\phi^{(a)}(X_i) = \sum_{j=1}^{2n} B^{(a)}_{ij} X_j$. Then, from (2) and (3), we have

$$(B^{(1)}_{ij}(p)) = \begin{pmatrix} 1 & \cdots \\ \cdot & 1 \end{pmatrix}$$.
Thus, it is easy to verify that

\[(3.2)\quad B_{11}^{(a)}(p)B_{ik}^{(a)}(p) = B_{11}^{(a)}(p)B_{k1}^{(a)}(p) = \delta_{i}^{a}\delta_{k}.\]

Since \(\phi^{(a)}(\nabla X_i X_j) = \nabla \phi^{(a)}(X_i)\phi^{(a)}(X_j)\), we have

\[
\sum_{u=1}^{2n} \Gamma_{\dot{u}\dot{v}} B_{\dot{u}\dot{v}}^{(a)} = \sum_{u=1}^{2n} B_{\dot{u}\dot{v}}^{(a)} (X_{\dot{u}} B_{\dot{v}k}^{(a)}) + \sum_{u,v=1}^{2n} B_{\dot{u}\dot{v}}^{(a)} B_{\dot{v}\dot{u}}^{(a)} \Gamma_{\dot{u}\dot{v}k},
\]

and hence

\[(3.3)\quad X_{\dot{u}} B_{\dot{v}k}^{(a)} = \sum_{u,v=1}^{2n} \Gamma_{\dot{u}\dot{v}} B_{\dot{u}\dot{v}}^{(a)} B_{\dot{v}\dot{u}}^{(a)} - \sum_{u=1}^{2n} \Gamma_{\dot{u}\dot{v}} B_{\dot{u}\dot{v}}^{(a)}.
\]

Thus, from (3.1) \(\sim\) (3.3), we have

\[(3.4)\quad (X_{\dot{1}} B_{\dot{1}\dot{k}}^{(a)})(p) = (X_{\dot{1}} B_{\dot{k}\dot{1}}^{(a)})(p) = 0 \quad \text{for} \quad k \geq 2,
\]

\[(X_{\dot{1}} B_{\dot{1}\dot{k}}^{(a)})(p) = (B_{\dot{1}\dot{1}}^{(a)})^{2}(p) - 1 = \delta_{1}^{2} - 1.
\]

We put \(J^{(a)} = \phi^{-1}\circ J \circ \phi^{(a)}\) and \(J_{j}^{(a)} = g(J^{(a)} X_{i}, X_{j})\). We note that each \((J^{(a)}, g)\) is also a compatible almost Kähler structure. It is obvious that

\[J_{j}^{(a)} = \sum_{u,v=1}^{2n} B_{\dot{u}\dot{v}}^{(a)} B_{\dot{v}\dot{u}}^{(a)} J_{uv}.
\]

Thus, we have

\[X_{\dot{1}} J_{\dot{1}}^{(a)} = \sum_{u,v=1}^{2n} (X_{\dot{1}} B_{\dot{u}\dot{v}}^{(a)}) B_{\dot{v}\dot{u}}^{(a)} J_{uv} + \sum_{u,v=1}^{2n} B_{\dot{u}\dot{v}}^{(a)} (X_{\dot{1}} B_{\dot{v}\dot{u}}^{(a)}) J_{uv} + \sum_{u,v=1}^{2n} B_{\dot{u}\dot{v}}^{(a)} B_{\dot{v}\dot{u}}^{(a)} X_{\dot{1}} J_{uv}.
\]
From this equality, by direct calculation, we have

\[
(3.5) \quad \sum_{i,j=1}^{2n} (X_1 J_{ij}^{(a)})^2 = \sum_{i,j=1}^{2n} (X_1 J_{ij})^2 + 2 \sum_{i,u=1}^{2n} (X_1 B_{iu}^{(a)})^2 \\
+ 2 \sum_{i,j,u',v'=1}^{2n} (X_1 B_{iu}^{(a)})(X_1 B_{jv}^{(a)}) B_{iu}^{(a)} B_{jv}^{(a)} J_{uv} J_{u'v'} \\
+ 4 \sum_{j,u,v,u'=1}^{2n} (X_1 B_{ji}^{(a)})(X_1 J_{uv}) B_{ji}^{(a)} J_{uv}.
\]

From (3.4), we have

\[
\sum_{i,u=1}^{2n} (X_1 B_{iu}^{(a)})^2(p) = (X_1 B_{11}^{(a)})^2(p) + \sum_{i,u \geq 2} (X_1 B_{iu}^{(a)})^2(p) = 2(\delta_1^a - 1)^2.
\]

Now, we set

\[
\xi^{(a)} = \sum_{u,v=1}^{2n} B_{iu}^{(a)} J_{iu} B_{1v}^{(a)} J_{1v}, \quad \eta^{(a)} = \sum_{u,v=1}^{2n} (X_1 J_{1u}) B_{1v}^{(a)} J_{uv}.
\]

Then, from (3.2) and (3.4), by direct calculation, we can derive

\[
\sum_{i,j,u',v',u''=1}^{2n} (X_1 B_{iu}^{(a)})(p)(X_1 B_{jv}^{(a)})(p) B_{iu}^{(a)}(p) B_{jv}^{(a)}(p) J_{uv} J_{u'v'}(p) \\
= \sum_{i,u'=1}^{2n} (X_1 B_{11}^{(a)})(p)(X_1 B_{11}^{(a)})(p) B_{1u'}^{(a)}(p) B_{1u'}^{(a)}(p) J_{11} J_{u'1}(p) \\
+ \sum_{i,j,u'=1}^{2n} (X_1 B_{iu}^{(a)})(p)(X_1 B_{jv}^{(a)})(p) B_{1u}^{(a)}(p) B_{1v}^{(a)}(p) J_{1v} J_{u'1}(p) \\
+ \sum_{i,u'=1}^{2n} (X_1 B_{iu}^{(a)})(p)(X_1 B_{jv}^{(a)})(p) B_{iu}^{(a)}(p) B_{jv}^{(a)}(p) J_{uv} J_{j'v}(p) \\
+ \sum_{i,j,u'=1}^{2n} (X_1 B_{iu}^{(a)})(p)(X_1 B_{jv}^{(a)})(p) B_{iu}^{(a)}(p) B_{jv}^{(a)}(p) J_{uv} J_{u'v}(p) \\
= -2(\delta_1^a - 1)^2 \xi^{(a)}(p) + (1 - \delta_1^a)^2 \sum_{u,v' \geq 2} B_{iu}^{(a)}(p) B_{1v'}^{(a)}(p) J_{u1} J_{1v'}(p) \\
= -2(\delta_1^a - 1)^2 \xi^{(a)}(p),
\]
and
\[\begin{align*}
\sum_{j,u,v,v'=1}^{2n} (X_1 B_{jv}^{(a)}(p)(X_1 J_{uv})(p) B_{jv'}^{(a)}(p) J_{uv}(p)) = & \sum_{u,v'=1}^{2n} (X_1 B_{1v}^{(a)}(p)(X_1 J_{uv})(p) B_{1v'}^{(a)}(p) J_{uv}(p)) \\
& + \sum_{u,v'=1}^{2n} \sum_{j,v \geq 2} (X_1 B_{jv}^{(a)}(p)(X_1 J_{uv})(p) B_{jv'}^{(a)}(p) J_{uv}(p)) \\
= & (\delta_1^a - 1) \eta^{(a)}(p) + (1 - \delta_1^a) \sum_{u=1}^{2n} \sum_{v \geq 2} (X_1 J_{uv1})(p) B_{1v}^{(a)}(p) J_{uv}(p) \\
= & 2(\delta_1^a - 1) \eta^{(a)}(p).
\end{align*}\]

Therefore, from (3.5), we obtain
\[\begin{align*}
(3.6) \quad \sum_{i,j=1}^{2n} (\nabla_1 J_{ij}^{(a)})^2(p) = & \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) + 4(\delta_1^a - 1)^2 \\
& - 4(\delta_1^a - 1)^2 \zeta^{(a)}(p) + 8(\delta_1^a - 1) \eta^{(a)}(p).
\end{align*}\]

Thus, taking account of
\[\begin{align*}
\zeta^{(a)}(p) = (J_{1a})^2(p), \quad \eta^{(a)}(p) = - \sum_u (\nabla_1 J_{1u})(p) J_{au}(p),
\end{align*}\]

we have
\[\begin{align*}
(3.7) \quad \sum_{a,i,j=1}^{2n} (\nabla_1 J_{ij}^{(a)})^2(p) = & 2n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) + 8(n - 1) \\
& + 8 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u})(p) J_{au}(p).
\end{align*}\]

Since
\[\begin{align*}
(3.8) \quad g((\nabla X J^{(a)}) Y, Z) = g((\nabla \phi_{(a)}(X) J) \phi_{(a)}(Y), \phi_{(a)}(Z)),
\end{align*}\]

we have
\[\begin{align*}
\sum_{i,j} (\nabla_1 J_{ij}^{(a)})^2(p) = \sum_{i,j} (\nabla_1 J_{ij})^2(p).
\end{align*}\]
Thus, (3.7) becomes

\[ \| \nabla J \|^2(p) = 2n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) + 8(n - 1) + 8 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u})(p)J_{au}(p). \]

Therefore, from (2.2), we have

(3.9) \[ n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij})^2(p) = 4(n - 1)^2 - 4 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u})(p)J_{au}(p). \]

Since the above argument does not depend on the choice of the almost complex structure \(J\), corresponding to (3.9), we can obtain

(3.10) \[ n \sum_{i,j=1}^{2n} (\nabla_1 J_{ij}^{(c)})^2(p) = 4(n - 1)^2 - 4 \sum_{a,u=1}^{2n} (\nabla_1 J_{1u}^{(c)})(p)J_{au}^{(c)}(p). \]

for \(c = 2, \ldots, 2n\). Therefore, from (3.9) and (3.10), we have

\[ n \sum_{c,i,j=1}^{2n} (\nabla_c J_{ij}^{(c)})^2(p) = 8(n - 1)^2 - 4 \sum_{c,a,u=1}^{2n} (\nabla_c J_{cu}^{(c)})(p)J_{au}^{(c)}(p). \]

Again from (3.8), the above equality becomes

\[ n\sum_{c,i,j=1}^{2n} (\nabla_c J_{ij})^2(p) = 8n(n - 1)^2 - 4 \sum_{c,a,u=1}^{2n} (\nabla_c J_{cu})(p)J_{au}(p), \]

namely,

(3.11) \[ n\| \nabla J \|^2(p) = 8n(n - 1)^2 - 4 \sum_{c,a,u} (\nabla_c J_{cu})(p)J_{au}(p). \]

Since an almost Kähler manifold is necessarily a semi-Kähler manifold, the second term in the right-hand-side of (3.11) must vanish. Therefore, (3.11) yields

\[ \| \nabla J \|^2(p) = 8(n - 1)^2. \]

Hence, from (2.2), we have

\[ 8n(n - 1) = 8(n - 1)^2. \]

This implies \(n = 1\).

Therefore, we have finally our Theorem.
Refences


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