DISCRETELY STAR-LINDELOF SPACES

By

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Abstract. A space $X$ is called (discretely) star-Lindelöf if for every open cover $\mathcal{U}$ of $X$, there exists a (discrete closed) countable subset $B$ of $X$ such that $St(B, \mathcal{U}) = X$. We investigate the relationship between these spaces and $\omega_1$-compact spaces, and also study topological properties of discretely star-Lindelöf spaces.

1. Introduction

By a space we mean a topological space. Fleischman [4] defined a space $X$ to be starcompact if for every open cover $\mathcal{U}$ of $X$, there exists a finite subset $B$ of $X$ such that $St(B, \mathcal{U}) = X$, where $St(B, \mathcal{U}) = \bigcup \{ U \in \mathcal{U} : U \cap B \neq \emptyset \}$. He proved that every countably compact space is starcompact, and conversely, van Douwen-Reed-Roscoe-Tree [2] proved that every starcompact $T_2$-space is countably compact. As a generalization of starcompactness, the following class of spaces is also studied by several authors under different names (see [9]):

**Definition 1.1.** A space $X$ is star-Lindelöf if for every open cover $\mathcal{U}$ of $X$, there exists a countable subset $B$ of $X$ such that $St(B, \mathcal{U}) = X$.

Further, Yasui-Gao [13] defined a space in countable discrete web by replacing the word 'countable' by 'countable discrete closed' in the preceding definition. In this paper, we rename a space in countable discrete web as the following definition, which seems to be more natural in the context of the history of star-covering properties:

**Definition 1.2.** A space $X$ is discretely star-Lindelöf if for every open cover $\mathcal{U}$ of $X$, there exists a countable discrete closed subset $B$ of $X$ and $St(B, \mathcal{U}) = X$.

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Recall that a space $X$ is $\omega_1$-compact if there is no uncountable discrete closed subset of $X$. The following diagram illustrates the relationship among spaces we shall consider and more familiar ones:

$$
\begin{array}{cccc}
\text{Lindelöf} & \longrightarrow & \text{$\omega_1$-compact} & \longleftarrow \text{countably compact} \\
& \downarrow^{T_1} & & \\
& \text{discretely star-Lindelöf} & & \\
\downarrow & & & \\
\text{separable} & \longrightarrow & \text{star-Lindelöf} & \longleftarrow \text{starcompact}
\end{array}
$$

The purpose of this paper is to investigate the relationship among spaces on the vertical centerline in the above diagram and to study topological properties of discretely star-Lindelöf spaces. In particular, we give various examples showing the difference between discretely star-Lindelöf spaces and $\omega_1$-compact spaces, and improve some results due to Yasui-Gao [13].

Throughout the paper, the cardinality of a set $A$ is denoted by $|A|$. For a cardinal $\kappa$, $\kappa^+$ denotes the smallest cardinal greater than $\kappa$. In particular, let $\omega$ denote the first infinite cardinal, $\omega_1 = \omega^+$ and $c$ the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Other terms and symbols will be used as in [3].

2. Discretely Star-Lindelöf Spaces and Their Subspaces

The square of the Sorgenfrey line is star-Lindelöf since it is separable, while Yasui-Gao [13] proved that the square is not discretely star-Lindelöf. The following theorem gives an alternative proof of the latter fact.

**Theorem 2.1.** Let $\kappa$ be an infinite cardinal with $\kappa^\omega = \kappa$ and let $X$ be a discretely star-Lindelöf space with $|X| = \kappa$. Then, the cardinality of a discrete closed subset of $X$ is less than $\kappa$.

**Proof.** The proof is based on the idea of that of van Douwen-Reed-Roscoe-Tree [2, Lemma 2.2.4]. Suppose on the contrary that there exists a discrete closed subset $H$ of $X$ with $|H| = \kappa$. Let $\mathcal{F}$ be the set of all countable discrete closed subsets of $X$. Then, $|\mathcal{F}| = \kappa$ since $|X| = \kappa = \kappa^\omega$, and thus, we can enumerate $\mathcal{F}$ as $\mathcal{F} = \{F_\alpha : \alpha < \kappa\}$. By transfinite induction, we can define a subset $H_0 = \{x_\alpha : \alpha < \kappa\}$ of $H$ satisfying that $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$ and $x_\alpha \notin \bigcup_{\beta < \alpha} F_\beta$ for each $\alpha < \kappa$. Define
Discretely star-Lindelöf spaces

\[ U_x = X \setminus (F_x \cup (H_0 \setminus \{x_2\})) \] for each \( x < \kappa \). Then, \( U_x \) is an open neighborhood of \( x_2 \) in \( X \). Let us consider the open cover

\[ \mathcal{U} = \{ U_x : x < \kappa \} \cup \{ X \setminus H_0 \} \]

of \( X \). For each \( x < \kappa \), \( x_2 \notin \text{St}(F_x, \mathcal{U}) \), because \( U_x \) is the only element of \( \mathcal{U} \) containing the point \( x_2 \) and \( U_x \cap F_x = \emptyset \) by the definition. This shows that \( X \) is not discretely star-Lindelöf, which is a contradiction.

**Corollary 2.2.** Let \( X \) be a discretely star-Lindelöf space with \( |X| = \mathfrak{c} \). Then, the cardinality of a discrete closed subset of \( X \) is less than \( \mathfrak{c} \).

This is a special case of Theorem 2.1. The following corollaries are immediate consequences of Corollary 2.2.

**Corollary 2.3.** The square of the Sorgenfrey line, the Niemytzki plane and every Isbell-Mrówka space \( \Psi \) with \( |\Psi| = \mathfrak{c} \) are not discretely star-Lindelöf.

It is worth noting that all of the spaces stated in Corollary 2.3 are star-Lindelöf since they are separable.

**Corollary 2.4.** Under assuming the continuum hypothesis, every discretely star-Lindelöf space with cardinality \( \mathfrak{c} \) is \( \omega_1 \)-compact.

Next, we give a machine which produces discretely star-Lindelöf spaces. For a separable space \( X \) and its countable dense subset \( D \), we define

\[ S(X, D) = X \cup (D \times \kappa^+) \]

where \( \kappa = |X| \), and topologize \( S(X, D) \) as follows: A basic neighborhood of \( x \in X \) in \( S(X, D) \) is a set of the form

\[ G_U(x) = U \cup ((U \cap D) \times \{ \beta : \alpha < \beta < \kappa^+ \}) \]

for a neighborhood \( U \) of \( x \) in \( X \) and for \( \alpha < \kappa^+ \), and a basic neighborhood of \( \langle x, \alpha \rangle \in D \times \kappa^+ \) in \( S(X, D) \) is a set of the form

\[ G_V(\langle x, \alpha \rangle) = \{ x \} \times V \]

for a neighborhood \( V \) of \( \alpha \) in \( \kappa^+ \). When it is not necessary to specify \( D \), we simply write \( S(X) \) instead of \( S(X, D) \). By a Tychonoff space we mean a completely regular \( T_1 \)-space.
Theorem 2.5. Let $X$ be a separable space with a countable dense set $D$. Then, the space $S(X, D)$ is discretely star-Lindelof. Moreover,

1. if $X$ is a Tychonoff space, so is $S(X, D)$;
2. if $X$ is a normal space, so is $S(X, D)$.

Proof. Put $S = S(X, D)$ and let $\mathcal{U}$ be an open cover of $S$. For every $x \in X$, there exist a neighborhood $U$ of $x$ in $X$ and $x(x) < \kappa^+$ such that $G_{U, x(x)}(x)$ is included in some member of $\mathcal{U}$. Since $|X| = \kappa$, we can find $x < \kappa^+$ such that $\forall x \in X$. Then, the set $B_1 = D \times \{x\}$ is countable, discrete closed in $S$ and $\text{St}(B_1, \mathcal{U}) \supseteq X$. For each $x \in D$, there exists a finite set $F_x \subseteq \{x\} \times \kappa^+$ such that $\text{St}(F_x, \mathcal{U}) \supseteq \{x\} \times \kappa^+$, because $\{x\} \times \kappa^+$ is countably compact. Then, the set $B_2 = \bigcup\{F_x : x \in D\}$ is countable, discrete closed in $S$ and $\text{St}(B_2, \mathcal{U}) \supseteq D \times \kappa^+$. If we put $B = B_1 \cup B_2$, then $B$ is a countable discrete closed set in $X$ such that $\text{St}(B, \mathcal{U}) = S$, which proves that $S$ is discretely star-Lindelof. The proof of the statement (1) is left to the reader since it is not difficult.

Finally, to prove the statement (2), assume that $X$ is normal. Let $A_0$ and $A_1$ be disjoint closed subsets of $S(X, D)$. Since $X$ is normal and $\kappa^+ > |X|$, we can find disjoint open subsets $U_0$, $U_1$ of $X$ and $x < \kappa^+$ such that $A_i \cap X \subseteq U_i$ and

$$(U_i \cup ((U_i \cap D) \times (x, \kappa^+))) \cap A_{1-i} = \emptyset$$

for each $i = 0, 1$. Let $X_0 = D \times \kappa^+$ and put

$$B_i = ((U_i \cap D) \times (x, \kappa^+)) \cup (A_i \cap X_0)$$

for $i = 0, 1$. Then, $B_0$ and $B_1$ are disjoint closed in $X_0$. Since $X_0$ is normal, there exist disjoint open sets $V_0$ and $V_1$ in $X_0$ such that $B_i \subseteq V_i$ for each $i = 0, 1$. Let $G_i = U_i \cup V_i$ for $i = 0, 1$. Then, $G_0$ and $G_1$ are disjoint open sets in $S(X, D)$ such that $A_i \subseteq G_i$ for each $i = 0, 1$. The proof is complete. ~\(\square\)

Corollary 2.6. Every Tychonoff space $X$ with $w(X) \leq c$ can be embedded in a discretely star-Lindelof Tychonoff space as a closed subspace.

Proof. Let $X$ be a Tychonoff space $X$ with $w(X) \leq c$. Then, it is known that $X$ can be embedded in a separable Tychonoff space $Y$ as a closed subspace. Indeed, embed $X$ into $[0, 1]^\mathbb{N}$ and take a countable dense subset $D$ of $[0, 1]^\mathbb{N}$. Then, the space $Y$ is obtained from the subspace $X \cup D$ by making each point of $D \setminus X$ isolated. Next, consider the space $S(Y)$ defined above. Then, $S(Y)$ is discretely star-Lindelof by Theorem 2.5 and $X$ is closed in $S(Y)$. ~\(\square\)
Remark 1. If $X$ is one of the spaces stated in Corollary 2.3, then $S(X)$ is discretely star-Lindelöf but not $\omega_1$-compact. Examples of discretely star-Lindelöf spaces with richer properties but not $\omega_1$-compact were also given by Matveev [10].

It is quite interesting to find an example of a normal (discretely) star-Lindelöf space which is not $\omega_1$-compact. Now, we give a consistency example:

Corollary 2.7. Assume Martin's axiom and the negation of the continuum hypothesis and let $\omega_1 \leq \kappa < c$. Then, there exists a normal, discretely star-Lindelöf space $X$ containing a closed discrete subset $B$ with $|B| = \kappa$.

Proof. Under the assumption, it is known ([12]) that there exists a separable normal space $Y$ with a closed discrete subset $B$ with $|B| = \kappa$. Then, the space $X = S(Y)$ is a required one by Theorem 2.5.

Remark 2. Matveev [10] also showed, independently, the existence of a normal discretely star-Lindelöf space which is not $\omega_1$-compact under certain set-theoretic assumption weaker than ours. He also asked if there exists an example within ZFC.

If $X$ is a discretely star-Lindelöf space which is not $\omega_1$-compact, then $X$ contains an uncountable discrete closed subset $B$. Since $B$ is not star-Lindelöf, this shows that a closed subspace of a discretely star-Lindelöf space need not be star-Lindelöf. Yasui-Gao [13] also gave an example showing that a closed subspace of a discretely star-Lindelöf space need not be discretely star-Lindelöf; however, their space is not Hausdorff. Now, we give another stronger example.

Example 2.8. There exists a discretely star-Lindelöf, Tychonoff space $X$ having a regular-closed subspace which is not discretely star-Lindelöf.

Proof. Let $\mathcal{R}$ be a maximal almost disjoint family of infinite subsets of $\omega$ with $|\mathcal{R}| = \omega$, and consider the Isbell-Mrówka space $\Psi = \omega \cup \mathcal{R}$ (see [5, 51, p. 79]). Let $X$ be the space obtained from the space $S(\Psi, \omega) = \Psi \cup (\omega \times \omega^*)$ by making each point of $\omega$ in $\Psi$ isolated. Then, $\Psi$ is a regular-closed subspace of $X$ and is not discretely star-Lindelöf by Corollary 2.3. Hence, it remains to show that $X$ is discretely star-Lindelöf. Let $\mathcal{U}$ be an open cover of $X$. Then, by a similar argument to the proof of Theorem 2.5, we can find countable discrete closed subsets $B_1$ and $B_2$ of $X$ such that $\Psi \setminus \omega \subseteq St(B_1, \mathcal{U})$ and $\omega \times \omega^* \subseteq St(B_2, \mathcal{U})$. Since
no infinite subset of $\omega$ is closed in $\Psi$, the set $B_3 = \omega \setminus St(B_1, \mathcal{U})$ is finite. Hence, if we put $B = B_1 \cup B_2 \cup B_3$, then $B$ is a countable discrete closed set in $X$ such that $St(B, \mathcal{U}) = X$. This proves that $X$ is discretely star-Lindelöf.

\section{Mappings}

In [2, Theorem 2.4.1], van Douwen-Reed-Roscoe-Tree proved that a continuous image of a star-Lindelöf space is star-Lindelöf. First, we give examples showing that a parallel result does not hold for discretely star-Lindelöf spaces.

\textbf{Example 3.1.} There exists a continuous bijection $f : X \to Y$ from a discretely star-Lindelöf, Tychonoff space $X$ to a Tychonoff space $Y$ which is not discretely star-Lindelöf.

\textbf{Proof.} Let $\Psi = \omega \cup \mathcal{R}$ be the same Isbell-Mrówka space as in the proof of Example 2.8. Then, the space $S(\Psi, \omega) = \Psi \cup (\omega \times c^+)$ is discretely star-Lindelöf by Theorem 2.5. Now, we change the topology of $S(\Psi, \omega)$ by declaring that a basic neighborhood of $r \in \mathcal{R}$ is a set of the form

$$G_U(r) = U \cup ((U \cap \omega) \times c^+)$$

for a neighborhood $U$ of $r$ in $\Psi$, and that basic neighborhoods of other points are the same as those in $S(\Psi, \omega)$. We show that the resulting space $Y$ is not discretely star-Lindelöf. For this end, we enumerate the set of all finite subsets of $\omega$ as $\{K_n : n \in \omega\}$. Since $|\mathcal{R}| = c$, we can write $\mathcal{R} = \{r_{n,x} : \langle n, x \rangle \in \omega \times c\}$, where $r_{n,x} \neq r_{n',x'}$ if $\langle n, x \rangle \neq \langle n', x' \rangle$. For each $\langle n, x \rangle \in \omega \times c$, define

$$U_{n,x} = \{r_{n,x} \cup (r_{n,x} \setminus K_n) \cup ((r_{n,x} \setminus K_n) \times c^+)\}.$$  

Then, $U_{n,x}$ is an open neighborhood of $r_{n,x}$ in $Y$. Let us consider the open cover

$$\mathcal{U} = \{U_{n,x} : \langle n, x \rangle \in \omega \times c\} \cup \{\omega \cup (\omega \times c^+)\}.$$  

It remains to show that $St(B, \mathcal{U}) \neq Y$ for every countable discrete closed set $B$ in $Y$. To show this, let $B$ be a countable discrete closed set in $Y$. Since $B \cap \mathcal{R}$ is countable, there exists $\beta < c$ such that

$$B \cap \{r_{n,\beta} : n \in \omega\} = \emptyset.$$  

On the other hand, $B \setminus \mathcal{R}$ is finite since every infinite subset of $\omega \cup (\omega \times c^+)$ has an accumulation point in $Y$. Thus, there exists $m \in \omega$ such that

$$B \setminus \mathcal{R} \subseteq K_m \cup (K_m \times c^+) .$$
Now, $U_{m,\beta}$ is the only element of $\mathcal{U}$ containing the point $r_{m,\beta}$ and $B \cap U_{m,\beta} = \emptyset$ by (1) and (2). Hence, $r_{m,\beta} \notin St(B, \mathcal{U})$, which proves that $Y$ is not discretely star-Lindelöf. Finally, let $X = S(\Psi, \omega)$ and let $f : X \to Y$ be the identity map. Since $f$ is continuous, the proof is complete.

Yasui-Gao [13] proved that the image of a discretely star-Lindelöf space under a closed continuous map is discretely star-Lindelöf. The following example shows that ‘closed map’ cannot be replaced by ‘open map’ in their result.

**Example 3.2.** There exists an open continuous map $f : X \to Y$ from a discretely star-Lindelöf, Tychonoff space $X$ onto a Tychonoff space $Y$ which is not discretely star-Lindelöf.

**Proof.** Let $\Psi$ be the same space as in the proof of Example 2.8 and consider the space $S(\Psi, \omega) = \Psi \cup (\omega \times \varepsilon^+).$ Define a retraction $f : S(\Psi, \omega) \to \Psi$ by $f(p) = p$ for $p \in \Psi$ and $f(\langle n, x \rangle) = n$ for $\langle n, x \rangle \in \omega \times \varepsilon^+$. Then, it is easily checked that $f$ is an open continuous map. The space $S(\Psi, \omega)$ is discretely star-Lindelöf by Theorem 2.5, but the space $\Psi$ is not discretely star-Lindelöf by Corollary 2.3.

Next, we turn to consider preimages. To show that the preimage of a discretely star-Lindelöf space under a closed 2-to-1 continuous map need not be discretely star-Lindelöf, we use the Alexandorff duplicate $A(X)$ of a space $X.$ The underlying set of $A(X)$ is $X \times \{0, 1\};$ each point of $X \times \{1\}$ is isolated and a basic neighbourhood of a point $\langle x, 0 \rangle \in X \times \{0\}$ is a set of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1 \rangle\}),$ where $U$ is a neighborhood of $x$ in $X.$

**Theorem 3.3.** For a space $X$, the following conditions are equivalent:

1. $X$ is $\omega_1$-compact;
2. $A(X)$ is $\omega_1$-compact;
3. $A(X)$ is discretely star-Lindelöf;
4. $A(X)$ is star-Lindelöf.

**Proof.** The implication (1) $\Rightarrow$ (2) follows from the fact that a perfect preimage of an $\omega_1$-compact space is $\omega_1$-compact. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. To show that (4) $\Rightarrow$ (1), suppose that $X$ is not $\omega_1$-compact. Then, there exists an uncountable discrete closed set $D$ in $X$. Since $D \times \{1\}$ is an uncountable, discrete, open and closed set in $A(X)$, $A(X)$ is not star-Lindelöf. \( \square \)
Let $X$ be a discretely star-Lindelöf space which is not $\omega_1$-compact (see Remark 1 above). Then, the space $A(X)$ is not star-Lindelöf by Theorem 3.3. Since the projection $A(X) \to X$ is a closed continuous map, this shows that the preimage of a discretely star-Lindelöf space under a closed 2-to-1 continuous map need not be star-Lindelöf. Now, we give a positive result:

**Theorem 3.4.** Assume that there exists an open and closed, finite-to-one, continuous map $f$ from a space $X$ to a discretely star-Lindelöf space $Y$. Then, $X$ is discretely star-Lindelöf.

**Proof.** Since $f[X]$ is open and closed in $Y$, we may assume that $f[X] = Y$. Let $\mathcal{U}$ be an open cover of $X$ and let $y \in Y$. Since $f^{-1}(y)$ is finite, there exists a finite subcollection $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \bigcup\{U : U \in \mathcal{U}_y\}$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathcal{U}_y$. Since $f$ is closed, there exists an open neighborhood $V_y$ of $y$ in $Y$ such that $f^{-1}[V_y] \subseteq \bigcup\{U : U \in \mathcal{U}_y\}$. Since $f$ is open, we can assume that

$$V_y \subseteq \bigcap\{f[U] : U \in \mathcal{U}_y\}.\tag{3}$$

Taking such open set $V_y$ for each $y \in Y$, we have an open cover $\mathcal{V} = \{V_y : y \in Y\}$ of $Y$. Since $Y$ is discretely star-Lindelöf, there exists a countable discrete closed subset $D$ of $Y$ such that $St(D, \mathcal{V}) = Y$. Since $f$ is finite-to-one and continuous, the set $E = f^{-1}[D]$ is also a countable discrete closed set in $X$. To show that $St(E, \mathcal{U}) = X$, let $x \in X$. Then, there exist $y \in Y$ such that $f(x) \in V_y$ and $V_y \cap D \neq \emptyset$. Since

$$x \in f^{-1}[V_y] \subseteq \bigcup\{U : U \in \mathcal{U}_y\},$$

we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then, $U \cap E \neq \emptyset$, because $V_y \subseteq f[U]$ by (3). Hence, $x \in St(E, \mathcal{U})$, and consequently, we have that $St(E, \mathcal{U}) = X$. $\Box$

As we shall show in Remark 3 below, Theorem 3.4 fails to be true if ‘open and closed, finite-to-one’ is replaced by ‘open perfect’.

4. Products

In [2], van Douwen-Reed-Roscoe-Tree showed that the product of a star-Lindelöf Tychonoff space and a compact Hausdorff space need not be star-Lindelöf. We begin by showing a similar example for discretely star-Lindelöf spaces:
Example 4.1. There exist a discretely star-Lindelöf Tychonoff space $X$ and a compact Hausdorff space $Y$ such that $X \times Y$ is not star-Lindelöf.

Proof. The proof is essentially same as that of [2, Example 3.3.4]. Let $\Psi = \omega \cup \mathcal{A}$ be the same as in the proof of Example 2.8 and define the space $X = S(\Psi, \omega) = \Psi \cup (\omega \times c^+)$. Then, $X$ is discretely star-Lindelöf by Theorem 2.5. Since $|\mathcal{A}| = c$, we can enumerate it as $\mathcal{A} = \{r_x : x < c\}$. On the other hand, let $D = \{y_x : x < c\}$ be the discrete space of cardinality $c$ and let $Y = D \cup \{y_x\}$ be the one-point compactification of $D$. To show that $X \times Y$ is not star-Lindelöf, we consider the open cover

$$\mathcal{U} = \{(\{r_x\} \cup (\omega \times c^+) \times (Y \setminus \{y_x\}) : x < c\}$$

$$\cup \{X \times \{y_x\} : x < c\} \cup \{(\omega \cup (\omega \times c^+) \times Y\}$$

of $X \times Y$. For every countable subset $B$ of $X \times Y$, there exists $x < c$ such that $B \cap (X \times \{y_x\}) = \emptyset$. Then, $\langle r_x, y_x \rangle \notin St(B, \mathcal{U})$ since $X \times \{y_x\}$ is the only element of $\mathcal{U}$ containing $\langle r_x, y_x \rangle$. Hence, $X \times Y$ is not star-Lindelöf.

Remark 3. By Example 4.1, we can see that the preimage of a discretely star-Lindelöf space under an open perfect map need not be star-Lindelöf.

A map $f : X \to Y$ is called an $s$-map if $f^{-1}(y)$ is separable for each $y \in Y$. Ikenaga [6] proved that the preimage of a star-Lindelöf space under an open perfect continuous $s$-map is star-Lindelöf. Hence, the product of a star-Lindelöf space and a separable compact space is star-Lindelöf. Moreover, Ikenaga [6] showed that the product of a Lindelöf space and a separable metric space need not be star-Lindelöf. For products of star-Lindelöf spaces, the reader is also referred to [1]. By contrast, little seems to be known about discretely star-Lindelöf spaces. In fact, the following problem is open:

Problem 4.2. Is the product of a discretely star-Lindelöf Tychonoff space and a separable compact Hausdorff space discretely star-Lindelöf? In particular, is the product of a discretely star-Lindelöf Tychonoff space and a compact metric space discretely star-Lindelöf?

The following theorem is a partial answer to Problem 4.2.

Theorem 4.3. Let $X$ be a discretely star-Lindelöf, countably metacompact space and $Y$ a compact metric space. Then, $X \times Y$ is discretely star-Lindelöf.
Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. Fix a countable base $\mathcal{B}$ of $Y$ and let $\{\mathcal{B}_n : n \in \omega\}$ be the set of all finite covers of $Y$ by members of $\mathcal{B}$. For each $n \in \omega$, we choose a finite set $C_n \subseteq Y$ such that $B \cap C_n \neq \emptyset$ for each $B \in \mathcal{B}_n$. For each $x \in X$, $Y$ being compact, we can find an open neighborhood $G_x$ of $x$ in $X$ and $n(x) \in \omega$ such that $G_x \times B$ is included in some member of $\mathcal{U}$ for each $B \in \mathcal{B}_n(x)$. For each $n \in \omega$, let $U_n = \bigcup\{G_x : n(x) = n\}$. Then, $\{U_n : n \in \omega\}$ is a countable open cover of $X$. Since $X$ is countably metacompact, there exists a point-finite open cover $\mathcal{V} = \{V_n : n \in \omega\}$ of $X$ such that $V_n \subseteq U_n$ for each $n \in \omega$. For each $n \in \omega$, let

$$\mathcal{W}_n = \{G_x \cap V_n : x \in X \text{ and } n(x) = n\}.$$ 

Then, $\mathcal{W} = \bigcup_{n \in \omega} \mathcal{W}_n$ is an open cover of $X$. Since $X$ is discretely star-Lindelöf, there exists a countable discrete closed set $D = \{x_k : k \in \omega\}$ in $X$ such that $St(D, \mathcal{W}) = X$. Since $\mathcal{V}$ is point-finite, the set $M_k = \{n \in \omega : x_k \in V_n\}$ is finite for each $k \in \omega$. Thus, if we put

$$E = \bigcup_{k \in \omega} \left(\{x_k\} \times \bigcup_{n \in M_k} C_n\right),$$

then $E$ is a countable discrete closed subset of $X \times Y$. To show that $St(E, \mathcal{U}) = X \times Y$, let $(s, t) \in X \times Y$ be fixed. Then, there exists $W \in \mathcal{W}$ such that $s \in W$ and $W \cap D \neq \emptyset$, because $St(D, \mathcal{W}) = X$. By the definitions of $D$ and $\mathcal{W}$, there exist $k, n \in \omega$ and $x \in X$ such that $x_k \in D \cap W$, $W = G_x \cap V_n$ and $n(x) = n$. Choose $B \in \mathcal{B}_n$ with $t \in B$. Then, $(s, t) \in G_x \times B$ and $(G_x \times B) \cap (\{x_k\} \times C_n) \neq \emptyset$. This implies that $(s, t) \in St(E, \mathcal{U})$, since $G_x \times B$ is included in some member of $\mathcal{U}$. Hence, $St(E, \mathcal{U}) = X \times Y$. \hfill \Box

The author does not know if the assumption that $X$ is countably metacompact can be removed from Theorem 4.3 in case $X$ is Hausdorff. The following example shows that the assumption is necessary at least in the realm of $T_1$-spaces.

Example 4.4. There exists a discretely star-Lindelöf $T_1$-space $X$ such that the product $X \times (\omega + 1)$ is not discretely star-Lindelöf.

Proof. Let $Y$ be a set with $|Y| = \omega_1$. Define $X = Y \cup \omega_2$, where $\omega_2 = \omega_1^+$, and topologize $X$ as follows: A basic neighborhood of $y \in Y$ is a set of the form

$$G_y(y) = \{y\} \cup \{\beta : \alpha < \beta < \omega_2\}$$

for $\alpha < \omega_2$, and $\omega_2$ is an open subspace of $X$ with the usual order topology. Then,
it is easily checked that $X$ is a discretely star-Lindelöf $T_1$-space. To show that $X \times (\omega + 1)$ is not discretely star-Lindelöf, we write $Y = \bigcup_{n \in \omega} Y_n$, where $Y_m \cap Y_n = \emptyset$ if $m \neq n$ and $|Y_n| = \omega_1$ for each $n \in \omega$. Put $G_x = \{y\} \cup \omega_2$ for each $y \in Y$ and define
\[
\mathcal{U} = \bigcup_{n \in \omega} \bigcup_{y \in Y_n} \{G_x \times \{i\} : i \leq n\} \cup \{G_x \times \{j : n < j \leq \omega\}\}.
\]
Then, $\mathcal{U}$ is an open cover of $X \times (\omega + 1)$. Let $B$ be a countable discrete closed set in $X \times (\omega + 1)$. Then, $B \cap (\omega_2 \times (\omega + 1))$ is finite since $\omega_2 \times (\omega + 1)$ is countably compact. Hence, there exists $n \in \omega$ such that $B \cap (\omega_2 \times \{n\}) = \emptyset$. Since $|Y_n| = \omega_1$, we can find $y \in Y_n$ such that $\langle y, n \rangle \notin B$. Then, $B \cap (G_x \times \{n\}) = \emptyset$, which implies that $\langle y, n \rangle \notin St(B, \mathcal{U})$, because $G_x \times \{n\}$ is the only element of $\mathcal{U}$ containing the point $\langle y, n \rangle$. Hence, $X \times (\omega + 1)$ is not discretely star-Lindelöf.

Let $N$ be the discrete space of non-negative integers. The space $N^{\omega_1}$ is star-Lindelöf since it is separable, but is not $\omega_1$-compact by Mycielski [11]. The following problem, however, still remains open.

**Problem 4.5.** Is the product $N^{\omega_1}$ discretely star-Lindelöf?

There is another star-covering property, called the property $(a)$, which is closely related to discretely star-Lindelöf spaces. Matveev [8] defined a space $X$ to have the property $(a)$ if for every open cover $\mathcal{U}$ of $X$ and for every dense subset $D$ of $X$, there exists a closed (in $X$) discrete subset $F$ of $D$ such that $St(F, \mathcal{U}) = X$. It is obvious that every $T_1$-space $X$ satisfies the following condition: For every open cover $\mathcal{U}$ of $X$, there exists a closed discrete set $F$ in $X$ such that $St(F, \mathcal{U}) = X$. Both discretely star-Lindelöf spaces and spaces with the property $(a)$ were defined by strengthening this condition. Every uncountable discrete space has the property $(a)$, but is not discretely star-Lindelöf. On the other hand, the space $X = \omega_1 \times (\omega_1 + 1)$ is discretely star-Lindelöf since it is countably compact, while it is known ([7, Example 1.5]) that $X$ does not have the property $(a)$. We conclude this paper with the following result due to Ohta under his permission.

**Example 4.6 (Ohta).** The product $N^{\omega_1}$ does not have the property $(a)$.

**Proof.** We consider the open cover $\mathcal{U} = \{U(x, \beta, k) : \{x, \beta\} \subseteq \omega_1, x \neq \beta, k \in N\}$ of $N^{\omega_1}$, where $U(x, \beta, k) = \{x \in N^{\omega_1} : x(\beta) = k\}$. Also, we define the set $D = \{x \in N^{\omega_1} : |\text{supp}(x)| < \omega_1\}$, where $\text{supp}(x) = \{x \in N^{\omega_1} : x(\beta) \neq 0\}$. Then, $D$ is a dense $\sigma$-compact subset of $N^{\omega_1}$. We show that $St(F, \mathcal{U}) \neq N^{\omega_1}$ for every
closed (in \(N^{(\omega)}\)) discrete subset \(F\) of \(D\). For this end, let \(F\) be such a subset of \(D\). Then, \(F\) is at most countable since \(D\) is \(\sigma\)-compact, which implies that the set \(S = \bigcup\{\text{supp}(x): x \in F\}\) is also at most countable. Hence, we can find a point \(y \in N^{(\omega)}\) such that \(y|_S: S \to N\) is one-to-one and \(y(\alpha) = 1\) for each \(\alpha \in \omega_1 \setminus S\). Now, let \(U(\alpha, \beta, k) \in \mathcal{U}\) such that \(U(\alpha, \beta, k) \cap F \neq \emptyset\). It remains to show that \(y \notin U(\alpha, \beta, k)\). We distinguish two cases: If \(\{\alpha, \beta\} \subseteq S\), then \(y \notin U(\alpha, \beta, k)\) since \(y(\alpha) \neq y(\beta)\) by the definition of \(y\). If \(\{\alpha, \beta\} \nsubseteq S\), then we may assume that \(\alpha \notin S\). Since there is a point \(x \in U(\alpha, \beta, k) \cap F\), we have \(k = x(\alpha) = 0\) because \(\alpha \notin \text{supp}(x)\). Hence, if \(y \in U(\alpha, \beta, k)\), then \(y(\alpha) = 0\), which contradicts the definition of \(y\). Consequently, \(y \notin U(\alpha, \beta, k)\). 

\[ \square \]

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References


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