REALIZATIONS OF SUBGROUPS OF TYPE $D_8$ OF CONNECTED EXCEPTIONAL SIMPLE LIE GROUPS OF TYPE $E_8$

By
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Introduction

In [4], we realized subgroups of type $A_8$, $A_4 \times A_4$ and $A_2 \times E_6$ of the compact simple Lie group of type $E_8$. In this paper, we shall realize subgroups of type $D_8$ of the compact and non-compact simple Lie groups of type $E_8$.

In [5], [6], [10] and [11], Yokota and some members of his school found all involutive automorphisms $\sigma$ and realized subgroups $G^\sigma$ of fixed points of connected exceptional simple Lie groups $G$ explicitly, which correspond to Berger’s result of simple Lie algebras [2]. But in their results concerning subgroups of type $D_8$ of Lie groups of type $E_8$, the definition of subgroups are not clear and proof of isomorphism is very difficult in comparison with their other results.

We improve those results in this paper. Our improvement make results that are more simple and intelligible. Hence they are of widely applicable to symmetric spaces. Our results are as follows.

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In §2, §3 and §4, we make a new realization of exceptional Lie algebras.

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of type $E_8$. Since this new realization starts from the definition of spinor groups directly, the final results will be made clearly understandable.

In §5, we define connected exceptional simple Lie groups $G$ of type $E_8$.

In §6, we find involutive automorphisms $\sigma$ and realize subgroups $G^\sigma$ which are isomorphic to semi-spinor groups.

Finally the author wishes to express his sincere thanks to Professor Ichiro Yokota, who motivated the author to study this subject and to Professor Hiroshi Asano and Professor Ryosuke Ichida for their valuable suggestions and constant encouragement.

§ 1. Notations and Preliminaries

$V^C := V \oplus iV$ the complexification of a real vector space $V$, $C := \mathbb{R}^C$.

$\tau$: the complex conjugation of $V^C$ (resp. $C^m$, $M(n,C)$) with respect to $V$ (resp. $\mathbb{R}^m$, $M(n,\mathbb{R})$).

$I_m$: the $m \times m$ unit matrix.

$I_{p,q} := \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$, $J_p := \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$.

$SO(n, C) := \{ A \in M(n, C) \mid \tau A = A \}$,

$SO(n) := \{ A \in SO(n, C) \mid \tau A = A \}$,

$SO(p, q) := \{ A \in SO(p + q, C) \mid \tau(I_{p,q}AI_{p,q}) = A \}$,

$SO^*(2p) := \{ A \in SO(2p, C) \mid \tau(J_pAJ_p^{-1}) = A \}$,

$so(n, C) := \{ X \in M(n, C) \mid \tau X = X \}$,

$so(n) := \{ X \in so(n, C) \mid \tau X = X \}$,

$so(p, q) := \{ X \in so(p + q, C) \mid \tau(I_{p,q}XI_{p,q}) = X \}$,

$so^*(2p) := \{ X \in so(2p, C) \mid \tau(J_pXJ_p^{-1}) = X \}$.

§ 1.1. Spinor Groups and Semi-spinor Groups ([1])

Let $K = \mathbb{R}$ or $C$ and $\{e_1, e_2, \ldots, e_n\}$ be the canonical basis of $K^n$. Let $T$ be the tensor algebra of $K^n$ and $U$ the two-sided ideal of $T$ generated as

$$x \otimes x + (x, x)1 \quad (x \in K^n)$$

where $(, )$ is the symmetric bilinear form of $K^n$ satisfying $(e_i, e_j) = \delta_{ij}$ (Kro-
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necker's delta). Define the Clifford algebra $Cl(K^n)$ by

$$Cl(K^n) := T/U.$$  

We denote the multiplication of $\alpha, \beta \in Cl(K^n)$ by $\alpha \cdot \beta$. It is clear that $K$ and $K^n$ can be naturally considered as subspaces of $Cl(K^n)$ and for $x, y \in K^n$ we have

$$x \cdot y + y \cdot x = -2(x, y).$$

It is known that spinor groups $Spin(n) = Spin(n, \mathbb{R})$ and $Spin(n, \mathbb{C})$ are defined by

$$Spin(n, K) := \left\{ \alpha = a_1 \cdot a_2 \cdots a_{2q} \in Cl(K^n) \mid a_i \in K^n, q = 1, 2, 3 \ldots, \prod_{i=1}^{2q} (a_i, a_i) = 1 \right\}.$$  

The unit element of $Spin(n, K)$ is $1 = -a \cdot a$ ($a \in K^n, (a, a) = \pm 1$) and the inverse element of

$$\alpha = a_1 \cdot a_2 \cdots a_{2q-1} \cdot a_{2q} \in Spin(n, K)$$

is

$$\alpha^{-1} = a_{2q} \cdot a_{2q-1} \cdots a_2 \cdot a_1 \in Spin(n, K).$$

The vector representation $p : Spin(n, K) \rightarrow SO(n, K)$ is given by

$$p(\alpha)x = \alpha \cdot x \cdot \alpha^{-1} \ (x \in K^n).$$  

It is known that $Spin(n, K)$ is a covering group of $SO(n, K)$ (double covering), and $Spin(n, K)$ ($n \geq 3$) is simply connected. Let

$$\omega = e_1 \cdot e_2 \cdot e_3 \cdots e_{2n-1} \cdot e_{2n} \in Spin(2n, K).$$

It is known that the centers of spinor groups are

$$z(Spin(2n+1, K)) = \{1, -1\} \cong \mathbb{Z}_2,$$

$$z(Spin(2, K)) = Spin(2, K),$$

$$z(Spin(4n, K)) = \{1, -1\} \times \{1, \omega\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \ (n \geq 1),$$

$$z(Spin(4n+2, K)) = \{1, \omega, -1, -\omega\} \cong \mathbb{Z}_4 \ (n \geq 1)$$

and

$$Spin(n, K)/\{1, -1\} \cong SO(n, K) \ (n \geq 1).$$
Define semi-spinor groups \( Ss(4n) = Ss(4n, \mathbb{R}) \) and \( Ss(4n, \mathbb{C}) \) by

\[
Ss(4n) := \text{Spin}(4n)/\{1, \omega\}, \quad Ss(4n, \mathbb{C}) := \text{Spin}(4n, \mathbb{C})/\{1, \omega\}.
\]

It is known that

\[
Ss(4n, K) \cong SO(4n, K) \quad (n \geq 3),
\]

and

\[
Ss(4n, K) \cong \text{Spin}(4n, K)/\{1, -\omega\}.
\]

Let \( l, m \) be non-negative integers and \( l + m = n \). Define the symmetric bilinear form \((\cdot, \cdot)_{l,m}\) of \( \mathbb{R}^n \) satisfying

\[
(e_i, e_j)_{l,m} := \begin{cases} 
-1 & (1 \leq i = j \leq l), \\
1 & (l + 1 \leq i = j \leq l + m = n), \\
0 & (i \neq j).
\end{cases}
\]

Let \( U_{l,m} \) be the two-sided ideal of the tensor algebra \( T \) of \( \mathbb{R}^n \) generated by

\[
x \otimes x + (x, x)_{l,m}1 \quad (x \in \mathbb{R}^n).
\]

Define the Clifford algebra \( Cl(\mathbb{R}^n)_{l,m} \) as

\[
Cl(\mathbb{R}^n)_{l,m} := T/U_{l,m}.
\]

We denote the multiplication of \( \alpha, \beta \in Cl(\mathbb{R}^n)_{l,m} \) by \( \alpha \cdot \beta \). The spinor group \( \text{Spin}(l, m) \) defined as

\[
\text{Spin}(l, m) := \left\{ \alpha = a_1 \cdots a_{2q} \in Cl(\mathbb{R}^n)_{l,m} \mid a_i \in \mathbb{R}^n, \quad q = 1, 2, 3 \ldots, \prod_{i=1}^{2q} (a_i, a_i)_{l,m} = 1 \right\}.
\]

Clearly \( Cl(\mathbb{R}^n)_{0,n} = Cl(\mathbb{R}^n) \) and \( \text{Spin}(0, n) = \text{Spin}(n) \).

For any \( C \)-linear transformation \( K : C^n \to C^n \) and any element \( \alpha = a_1 \cdot a_2 \cdots a_{2m} \in \text{Spin}(n, C) \), we define \( K(\alpha) \in \text{Spin}(n, C) \) as

\[
K(\alpha) = K(a_1) \cdot K(a_2) \cdots K(a_{2m}) \in \text{Spin}(n, C).
\]

Now, we identify an element

\[
\alpha = \left( \begin{array}{c} b_1 \\ c_1 \end{array} \right) \cdot \left( \begin{array}{c} b_2 \\ c_2 \end{array} \right) \in \text{Spin}(l, m), \quad (b_i \in \mathbb{R}^l, c_i \in \mathbb{R}^m)\]
with
\[ x' = \begin{pmatrix} ib_1 \\ c_1 \end{pmatrix} \cdot \begin{pmatrix} ib_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ -ic_1 \end{pmatrix} \cdot \begin{pmatrix} b_2 \\ ic_2 \end{pmatrix} \in \text{Spin}(l + m, C). \]

Using this identification, we can consider the following
\[ \text{Spin}(l, m) = \{ x \in \text{Spin}(l + m, C) \mid \tau_I l_m(x) = x \}. \]

Then \( \text{Spin}(l, m) \) is a real subgroup of \( \text{Spin}(l + m, C) \).

Let \( l + m \geq 3 \) and \( lm \neq 0 \). It is known that \( \text{SO}(l, m) \) is not connected (it has two connected components) and \( \text{Spin}(l, m) \) is connected. As same as in the case of \( \text{Spin}(n) \), \( \text{Spin}(l, m) \) is a double covering group of \( \text{SO}(l, m)_0 \) (the connected component of the identity of \( \text{SO}(l, m) \)). Furthermore \( \text{Spin}(1, 2) \) and \( \text{Spin}(l, m) \) \( (l, m \geq 2) \) are not simply connected, while \( \text{Spin}(1, m) \) \( (m \geq 3) \) is simply connected.

Let \( l \equiv m \equiv 0 \mod 2 \). Since \( \tau_I l_m \omega = (-1)^l \omega = \omega \), we see \( \omega \in \text{Spin}(l, m) \). If \( l + m \equiv 0 \mod 4 \), we define a real subgroup \( \text{Ss}(l, m) \) of \( \text{Spin}(l + m, C) \) by
\[ \text{Ss}(l, m) := \text{Spin}(l, m)/\{1, \omega\}. \]

Furthermore, consider a double covering group of \( \text{SO}^*(2m) \). It is known that \( \text{SO}^*(2m) \) is connected and not simply connected. Define
\[ \tilde{G} := p^{-1}(\text{SO}^*(2m)) = \{ x \in \text{Spin}(2m, C) \mid p(x) \in \text{SO}^*(2m) \} \]
\[ = \{ x \in \text{Spin}(2m, C) \mid \tau(J_m p(x) J_m^{-1}) = p(x) \}. \]

Since \( \tau(J_m p(x) J_m^{-1}) = p(J_m \tau x) \) and \( \text{Ker} p = \{ \pm 1 \} \), we see
\[ \tilde{G} = \{ x \in \text{Spin}(2m, C) \mid \tau J_m(x) = \pm x \} = \tilde{G}_+ \cup \tilde{G}_-, \]
where \( \tilde{G}_\pm = \{ x \in \tilde{G} \mid J_m \tau x = \pm x \} \). It is clear that \( \tilde{G}_+ \cap \tilde{G}_- = \phi \). Since \( \tau J_m \) is an involutive automorphism of \( \text{Spin}(2m, C) \), \( \tilde{G}_+ \) is connected ([7]). It is clear that \( \tilde{G}_+ \) is closed, because \( \tau J_m \) is continuous.

Now, we assume \( \tilde{G}_- \neq \phi \). For any \( x \in \tilde{G}_+ \) and \( \beta, \beta' \in \tilde{G}_- \), we have \( x \cdot \beta \in \tilde{G}_- \) and \( \beta \cdot \beta' \in \tilde{G}_+ \). This shows \( \tilde{G}_- = \tilde{G}_+ \cdot \beta \) \( (\beta \in \tilde{G}_- \) and \( \tilde{G}_- \) is closed and connected. Moreover, we can prove that \( \tilde{G} \) is connected as follows. Let \( x \in \tilde{G}_+ \), \( \beta \in \tilde{G}_- \) and \( A = p(x), B = p(\beta) \in \text{SO}^*(2m) \). There exists a continuous curve \( \gamma : I \rightarrow \text{SO}^*(2m) \subset \text{SO}(2m, C) \) \( (I = [0, 1]) \) such that \( \gamma(0) = A \) and \( \gamma(1) = B \). Hence, we can choose a continuous curve \( \tilde{\gamma} : I \rightarrow \text{Spin}(2m, C) \) such that \( p(\tilde{\gamma}(t)) = \gamma(t) \) and \( \tilde{\gamma}(0) = x \). From the definition of \( \tilde{G} \), \( \tilde{\gamma} \subset \tilde{G} \) is clear. Since \( p(\tilde{\gamma}(1)) = \gamma(1) = B \), we see \( \tilde{\gamma}(1) = \beta \) or \( -\beta \). This shows \( \tilde{G} = \tilde{G}_+ \cup \tilde{G}_- \) is connected. This contradicts \( \tilde{G}_+ \cap \tilde{G}_- = \phi \).
Hence we see $\tilde{G}_+ = \phi$ and $\tilde{G} = \tilde{G}_+$ is a connected simple Lie group. Thus we can define a double covering group of $SO^*(2m)$ as follows:

$$Spin^*(2m) := \{x \in Spin(2m, C) \mid \tau J_m(x) = x\}.$$  

Clearly, $Spin^*(2m)$ is a real subgroup of $Spin(2m, C)$. From the fact above, $Spin^*(2m)$ is a connected simple Lie group. On the other hand, since $\pi_1(SO^*(2m)) = Z \ (m : \text{odd})$ or $Z \times Z_2 \ (m : \text{even})$ ([9]), $Spin^*(2m)$ is not simply connected.

Since $\tau J_m \omega = (-1)^m e_{m+1} \cdots e_{2m} \cdot e_1 \cdots e_m = (-1)^m (-1)^m \omega = \omega$, we see $\omega \in Spin^*(2m)$. Hence we define a real subgroup $Ss^*(4n)$ of $Ss(4n, C)$ as

$$Ss^*(4n) := Spin^*(4n) / \{1, \omega\}.$$  

§ 1.2. Vector Representation of $Spin(n, K)$

For $1 \leq i \neq j \leq n$, define an element $\alpha_{ij}(t)$ of $Spin(n, K)$ as

$$\alpha_{ij}(t) := -e_i \cdot \left(\cos \frac{t}{2} e_i + \sin \frac{t}{2} e_j\right).$$

It is clear $\alpha_{ij}(0) = 1$. Since $\alpha_{ij}(t_1) \cdot \alpha_{ij}(t_2) = \alpha_{ij}(t_1 + t_2)$, we see

$$\{\alpha_{ij}(t) \in Spin(n, K) \mid t \in \mathbb{R}\}$$

is a 1-parameter subgroup of $Spin(n, K)$. Since

$$x \cdot y \cdot x = -2(x, y)x + (x, x)y, \quad (x, y \in K^n \subset Cl(K^n)),$$

we see

$$p(\alpha_{ij}(t)) e_i = \cos te_i - \sin te_j, \quad p(\alpha_{ij}(t)) e_j = \sin te_i + \cos te_j,$$

$$p(\alpha_{ij}(t)) e_k = e_k \quad (k \neq i, j).$$

§ 1.3. Cayley Algebra and Half-spinor Representations of $so(8, C)$ ([3], [8])

Let $\mathbb{C}$ be the division Cayley algebra over $\mathbb{R}$ and $\mathbb{C}^C$ its complexification. We denote the multiplication and the canonical conjugation of $\mathbb{C}$ (resp. $\mathbb{C}^C$) by $xy$ and $\bar{x}$ ($x, y \in \mathbb{C}$ (resp. $\mathbb{C}^C$)). The symmetric bilinear form of $\mathbb{C}$ (resp. $\mathbb{C}^C$) is defined by

$$(x, y) := \frac{1}{2} (xy + y\bar{x}).$$

Let $\{e_0, e_1, \ldots, e_7\}$ be the $\mathbb{R}$-basis (resp. $\mathbb{C}$-basis) of $\mathbb{C}$ (resp. $\mathbb{C}^C$) with the fol-
lowing relation:
\[ e_0 e_k = e_k e_0 = e_k \quad (0 \leq k \leq 7), \quad (e_0 = 1, \text{ the unit element}), \]
\[ e_k^2 = -e_0 \quad (1 \leq k \leq 7), \quad e_k e_l = -e_l e_k \quad (1 \leq k \neq l \leq 7), \]
\[ e_1 e_2 = e_3, \quad e_1 e_4 = e_5, \quad e_2 e_5 = e_7, \quad e_3 e_4 = e_7, \]
\[ e_3 e_5 = e_6, \quad e_6 e_4 = e_2, \quad e_6 e_7 = e_1, \]
\[ e_0 \in \mathbb{C}, \quad e_k = -e_k \quad (k \neq 0), \]
\[ (e_i, e_j) = \delta_{ij} \quad \text{(Kronecker's delta)}. \]

Let \( \gamma \) be the automorphism of \( \mathbb{C} \) satisfying
\[ \gamma(e_i) = \begin{cases} e_i, & i = 0, 1, 2, 3, \\ -e_i, & i = 4, 5, 6, 7. \end{cases} \]

Define the split Cayley algebra \( \mathbb{C}' \) by
\[ \mathbb{C}' := \{ x \in \mathbb{C} \mid \gamma x = x \} \]
\[ = \{ e_0, e_1, e_2, e_3, i e_4, i e_5, i e_6, i e_7 \}_{R\text{-span}}. \]

We identify \( \mathbb{C} \) with \( C^8 \) by
\[ \sum_{i=0}^{7} x_i e_i = \gamma(x_0, x_1, \ldots, x_7). \]

Then we see
\[ \text{so}(8, C) = \text{so}(\mathbb{C}) = \{ X \in \text{gl}(\mathbb{C}) \mid (X x, y) + (x, X y) = 0 \text{ for } x, y \in \mathbb{C} \}, \]
\[ \text{so}(8) = \text{so}(\mathbb{C}) = \{ X \in \text{gl}(\mathbb{C}) \mid (X x, y) + (x, X y) = 0 \text{ for } x, y \in \mathbb{C} \}, \]
\[ \text{so}(4, 4) = \text{so}(\mathbb{C}') = \{ X \in \text{gl}(\mathbb{C}') \mid (X x, y) + (x, X y) = 0 \text{ for } x, y \in \mathbb{C}' \}. \]

Define element \( G_{ij} \) \((0 \leq i \neq j \leq 7)\) of \( \text{so}(8, C) \) and \( \text{so}(8) \) by
\[ G_{ij} e_k = \delta_{jk} e_i - \delta_{ik} e_j. \]

For an element \( x \in \mathbb{C} \) or \( \mathbb{C}', \) we denote the left (resp. right) multiplication by \( L_x \)
(resp. \( R_x \)). Define element \( F_{ij} \) \((0 \leq i \neq j \leq 7)\) of \( \text{so}(8, C) \) or \( \text{so}(8) \) as
\[ F_{ij} = \frac{1}{2} L_{e_i} L_{e_j}. \]
It is known that both
\[ \{ G_{ij} | 0 \leq i < j \leq 7 \} \quad \text{and} \quad \{ F_{ij} | 0 \leq i < j \leq 7 \} \]
are $C$- (resp. $R$-) bases of $\mathfrak{so}(8, C)$ (resp. $\mathfrak{so}(8)$). Furthermore, both
\[ \{ G_{ij}, G_{4+i4+j}, iG_{k4+l} | 0 \leq i < j \leq 3, 0 \leq k, l \leq 3 \} \]
and
\[ \{ F_{ij}, F_{4+i4+j}, iF_{k4+l} | 0 \leq i < j \leq 3, 0 \leq k, l \leq 3 \} \]
are $R$-bases of $\mathfrak{so}(4,4)$.

In the following statements of this subsection, we can replace $\mathfrak{so}(8, C)$ with $\mathfrak{so}(8)$ (resp. $\mathfrak{so}(4,4)$) and $\mathbb{C}^C$ with $\mathbb{C}$ (resp. $\mathbb{C}'$).

Define $C$-linear transformations $\pi$, $\kappa$ and $v$ of $\mathfrak{so}(8, C)$ as
\[ \pi G_{ij} = F_{ij}, \quad \kappa Xx = \overline{Xx} \quad (x \in \mathbb{C}^C), \quad v = \pi \kappa. \]

It is known that $\pi$, $\kappa$ and $v$ are outer automorphisms of the Lie algebra $\mathfrak{so}(8, C)$ and
\[ \pi^2 = \kappa^2 = v^3 = (v\pi)^2 = \text{id}, \quad \pi = \kappa = \kappa v, \quad \kappa = \pi v = v^2 \pi, \]
\[ v^2 = \kappa \pi, \quad v\pi = \pi v^2 = \kappa v = v^2 \kappa = \kappa \pi \kappa = \pi \kappa \pi, \]
where id denotes the identity map. Hence we have

**Lemma 1.1.**
\[ vG_{ij} = \frac{1}{2} L_{\alpha_i} L_{\alpha_j}, \quad v^2 G_{ij} = \frac{1}{2} R_{\alpha_i} R_{\alpha_j}, \quad v\pi G_{ij} = \frac{1}{2} R_{\alpha_i} R_{\alpha_j}. \]

The following lemma is well known.

**Lemma 1.2.** For $X \in \mathfrak{so}(8, C)$ and $x, y \in \mathbb{C}^C$, we have
\[ X(xy) = (\pi Xx)y + x(\pi y) z \]
\[ x \times y = (y, z)x - (x, z)y \quad (z \in \mathbb{C}^C). \]

For $x, y \in \mathbb{C}^C$, we define a $C$-linear transformation $x \times y$ of $\mathbb{C}^C$ as
\[ (x \times y)z = (y, z)x - (x, z)y \quad (z \in \mathbb{C}^C). \]

Clearly, we see $x \times y = -y \times x$, $x \times y \in \mathfrak{so}(8, C)$ for any $x, y \in \mathbb{C}^C$ and $e_i \times e_i = 0$, $e_i \times e_j = G_{ij}$ ($i \neq j$). Hence we have the following
**Lemma 1.3.** For $x, y, z \in \mathbb{C}^C$, the following relations are valid.

1. \[ \kappa(x \times y) = \bar{x} \times \bar{y}, \]
2. \[ \pi(x \times y) = \frac{1}{2} L_x L_y - \frac{1}{2} (x, y)\text{id} = \frac{1}{4} (L_x L_y - L_y L_x), \]
3. \[ \nu(x \times y) = \frac{1}{2} L_x L_y - \frac{1}{2} (x, y)\text{id} = \frac{1}{4} (L_x L_y - L_y L_x), \]
4. \[ \nu^2(x \times y) = \frac{1}{2} R_x R_y - \frac{1}{2} (x, y)\text{id} = \frac{1}{4} (R_x R_y - R_y R_x), \]
5. \[ \nu\pi(x \times y) = \frac{1}{2} R_x R_y - \frac{1}{2} (x, y)\text{id} = \frac{1}{4} (R_x R_y - R_y R_x), \]
6. \[ x\bar{y} \times z = \pi(x \times zy) - \nu^2(\bar{x} \times y). \]

**Proof.** (1) It is easily obtained. (2) Since $\pi G_{ij} = F_{ij}$ and $L_x L_{\bar{y}} = \text{id}$, the first equality is clear. Using this, we have

\[ 4\pi(x \times y) = 2\pi(x \times y) - 2\pi(y \times x) \]
\[ = \{L_x L_y - (x, y)\text{id}\} - \{L_y L_{\bar{x}} - (y, x)\text{id}\} = L_x L_y - L_y L_{\bar{x}}. \]

The relations (3), (4) and (5) are obtained in a way similar to (2). (6) Let $p \in \mathbb{C}^C$. Using (2) and (4), we have

\[ 2\{\pi(x \times zy) - \nu^2(\bar{x} \times y)\}p \]
\[ = x\{(\bar{y}z)p\} - (py)(\bar{z}x) = x\{(\bar{y}\bar{z})p\} - (py)(\bar{x}z) \]
\[ = x\{2(z, p)\bar{y} - (\bar{y}\bar{p})z\} - (py)(\bar{x}z) \]
\[ = 2(z, p)xy - x\{(py)z\} - (py)(\bar{x}z) = 2(z, p)xy - 2(x, py)z \]
\[ = 2(z, p)xy - 2(xy, p)z = 2(xy \times z)p. \]

The following is known.

**Lemma 1.4.** The representation $\pi$ is an even half-spinor representation of $\mathfrak{so}(8, \mathbb{C})$ and the representation $\nu$ is an odd half-spinor representation of $\mathfrak{so}(8, \mathbb{C})$.

**§ 2. Spinor Group Spin(16, K)**

We identify $(\mathbb{C}^C)^2$ (resp. $\mathbb{C}^2$) with $C^{16}$ (resp. $R^{16}$) as

\[ t(x, y) = t(x_0, x_1, \ldots, x_7, y_0, y_1, \ldots, y_7) \]
where \( x = \sum_{i=0}^{7} x_i e_i, \ y = \sum_{i=0}^{7} y_i e_i \). Using this identification, we can consider that groups \( \text{Spin}(16, C) \) and \( \text{Spin}(16) \) are

\[
\text{Spin}(16, C) = \left\{ x = \bar{a}_1 \cdots \bar{a}_{2q} \in Cl((C^2)^2) \mid \begin{align*}
q &= 1, 2, 3, \ldots, \\
\prod_{i=1}^{2q} (\bar{a}_i, \bar{a}_i) &= 1
\end{align*} \right\},
\]

\[
\text{Spin}(16) = \left\{ x = \bar{a}_1 \cdots \bar{a}_{2q} \in Cl(C^2) \mid \begin{align*}
q &= 1, 2, 3, \ldots \\
\bar{a}_i &\in C^2, (\bar{a}_i, \bar{a}_i) = 1
\end{align*} \right\},
\]

where \( (\bar{a}_1, \bar{a}_2) = (a_1, a_2) + (b_1, b_2), \ \bar{a}_i = \iota(a_i, b_i) \in C^2. \)

Let

\[
V = C \otimes C \otimes C \otimes C.
\]

\( V \) is a 128 dimensional real vector space.

In the following statements of this section, the replacement \( \text{Spin}(16, C) \) (resp. \( \text{so}(16, C), C^C, V^C, \ldots \) etc.) with \( \text{Spin}(16) \) (resp. \( \text{so}(16), C, V, \ldots \) etc.) is possible.

\section*{§ 2.1. Even Half-spinor Representations of \( \text{Spin}(16, C) \) and \( \text{so}(16, C) \) ([8])}

\textbf{Definition 2.1.} We define a representation \( \rho \) of \( \text{Spin}(16, C) \) on \( V^C \) as

\[
\rho \left( \begin{pmatrix} a_1 \\ b_2 \\ b_2 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \\ b_2 \end{pmatrix} \right)(x \otimes y, 0) = \begin{pmatrix} a_1 \end{pmatrix} (\bar{a}_2 x) \otimes y - x \otimes b_1 (\bar{b}_2 y), \bar{a}_1 x \otimes \bar{b}_2 y - \bar{a}_2 x \otimes \bar{b}_1 y),
\]

\[
\rho \left( \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right)(0, z \otimes w) = (-a_1 z \otimes b_2 w + a_2 z \otimes b_1 w, -\bar{a}_1 (a_2 z) \otimes w - z \otimes \bar{b}_1 (b_2 w)),
\]

\[
\rho(\bar{a}_1 \cdot \bar{a}_2 \cdots \bar{a}_{2m-1} \cdot \bar{a}_{2m}) = \rho(\bar{a}_1 \cdot \bar{a}_2) \cdots \rho(\bar{a}_{2m-1} \cdot \bar{a}_{2m}).
\]

Since \( \rho(1) = \rho((-\bar{a}) \cdot \bar{a}) = \rho \left( \begin{pmatrix} -a \\ -b \\ b \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} \right) = 1, \ \rho \) is well-defined.
For $i, j = 0, 1, \ldots, 7$, define elements $\alpha_k^i(t)$ ($k = 1, 2, 3$) of $\text{Spin}(16, C)$ as

$$\alpha_k^i(t) = \begin{pmatrix} -e_i & \cos \frac{t}{2} e_i + \sin \frac{t}{2} e_i \\ 0 & 0 \end{pmatrix}, \quad (i \neq j)$$

$$\alpha_k^i(t) = \begin{pmatrix} 0 & 0 \\ -e_i & \cos \frac{t}{2} e_i + \sin \frac{t}{2} e_i \end{pmatrix}, \quad (i \neq j)$$

$$\alpha_k^i(t) = \begin{pmatrix} -e_i & \cos \frac{t}{2} e_i \\ 0 & \sin \frac{t}{2} e_i \end{pmatrix}.$$  

An element of $\mathfrak{so}(16, C)$ can be written by the sum of the following elements

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \begin{pmatrix} 0 & uv \\ -v'u & 0 \end{pmatrix}$$

where $A, B \in \mathfrak{so}(8, C)$ and $u, v \in \mathbb{C}^C = C^8$. From §1.2, we have

$$\frac{d}{dt} p(\alpha_k^i(t)) \bigg|_{t=0} = \begin{pmatrix} G_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{d}{dt} p(\alpha_k^i(t)) \bigg|_{t=0} = \begin{pmatrix} 0 & 0 \\ 0 & G_{ij} \end{pmatrix},$$

$$\frac{d}{dt} p(\alpha_k^i(t)) \bigg|_{t=0} = \begin{pmatrix} 0 & e_i' e_j \\ -e_j' e_i & 0 \end{pmatrix}.$$  

**Proposition 2.2.** The representation $dp$ of $\mathfrak{so}(16, C)$ on $V^C$ is given as follows.

$$dp \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) (x \otimes y, z \otimes w)$$

$$= (\pi Ax \otimes y + x \otimes \pi By, vA z \otimes w + z \otimes vBw),$$

$$dp \left( \begin{pmatrix} 0 & uv \\ -v'u & 0 \end{pmatrix} \right) (x \otimes y, z \otimes w) = \left( \frac{1}{2} uz \otimes vw, -\frac{1}{2} \bar{u}x \otimes \bar{v}y \right).$$  

**Proof.** From Lemma 1.1, we have

$$dp \left( \begin{pmatrix} G_{ij} & 0 \\ 0 & 0 \end{pmatrix} \right) (x \otimes y, z \otimes w)$$

$$= \frac{d}{dt} p(\alpha_k^i(t))(x \otimes y, z \otimes w) \bigg|_{t=0}.$$
Similarly we have
\[ d\rho\left(\begin{pmatrix} 0 & 0 \\ 0 & G_{ij} \end{pmatrix}\right)(x \otimes y, z \otimes w) = (x \otimes \pi G_{ij} y, z \otimes vG_{ij} w). \]

Furthermore
\[ d\rho\left(\begin{pmatrix} 0 & e_i^j \\ -e_j^i & 0 \end{pmatrix}\right)(x \otimes y, z \otimes w) \]
\[ = \frac{d}{dt} \rho(x^j(t)) (x \otimes y, z \otimes w) \bigg|_{t=0} \]
\[ = \frac{d}{dt} \left( e_i \left( \cos \frac{t}{2} \tilde{e}_i x \right) \otimes y + e_i z \otimes \sin \frac{t}{2} e_j w, \right. \]
\[ - \tilde{e}_i x \otimes \sin \frac{t}{2} \tilde{e}_j y + \tilde{e}_i \left( \cos \frac{t}{2} e_j z \right) \otimes w \bigg|_{t=0} \]
\[ = \left( \frac{1}{2} e_i z \otimes e_j w, - \frac{1}{2} \tilde{e}_i x \otimes \tilde{e}_j y \right). \]

It is clear that the representation \((\text{Spin}(16, C), \rho, V^C)\) is irreducible. Since \(\dim V^C = 128\), we see the following

**Proposition 2.3.** The representation \(\rho\) (resp. \(d\rho\)) of the Lie group \(\text{Spin}(16, C)\) (resp. the Lie algebra \(\text{so}(16, C)\)) is an even half-spinor representation.

§ 2.2. **Bilinear Map**

**Definition 2.4.** Define an anti-symmetric bilinear map
\[ \times : V^C \times V^C \to \text{so}(16, C) \quad (V \times V \to \text{so}(16)) \]
by
Subgroups $D_8$ of $E_8$

$$(x_1 \otimes y_1, 0) \times (x_2 \otimes y_2, 0) = \begin{pmatrix} (y_1, y_2)\pi(x_1 \times x_2) & 0 \\ 0 & (x_1, x_2)\pi(y_1 \times y_2) \end{pmatrix},$$

$$(0, z_1 \otimes w_1) \times (0, z_2 \otimes w_2) = \begin{pmatrix} (w_1, w_2)v^2(z_1 \times z_2) & 0 \\ 0 & (z_1, z_2)v^2(w_1 \times w_2) \end{pmatrix},$$

$$(x \otimes y, 0) \times (0, z \otimes w) = \begin{pmatrix} 0 & \frac{1}{2}(x^2)R(x) \\ -\frac{1}{2}(y\bar{w})^t(y\bar{w}) & 0 \end{pmatrix}.$$

**Lemma 2.5.** For $\alpha \in \text{Spin}(16, C)$ and $P, Q \in V^C$, we have

$$\text{Ad}(\alpha)(P \times Q) = p(\alpha)(P \times Q)p(\alpha)^{-1} = \rho(\alpha)P \times \rho(\alpha)Q.$$ 

**Proof.** It is sufficient to prove for

$$\alpha = \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix}, \quad (a_1, a_i) + (b_1, b_i) = 1$$

in the following 3 cases. Case 1. $P = (x_1 \otimes y_1, 0), \ Q = (x_2 \otimes y_2, 0)$. Case 2. $P = (0, z_1 \otimes w_1), \ Q = (0, z_2 \otimes w_2)$. Case 3. $P = (x \otimes y, 0), \ Q = (0, z \otimes w)$.

**Case 1.** Let

$$p(\alpha)(P \times Q)p(\alpha)^{-1} = \begin{pmatrix} A_1 & C_1 \\ -\overline{C}_1 & B_1 \end{pmatrix}, \quad \rho(\alpha)P \times \rho(\alpha)Q = \begin{pmatrix} A_2 & C_2 \\ -\overline{C}_2 & B_2 \end{pmatrix}.$$

Then we have

$$A_1 = 4\{(y_1, y_2)(a_1, \pi(x_1 \times x_2)a_2) + (x_1, x_2)(b_1, \pi(y_1 \times y_2)b_2)\}a_1 \times a_2 + 4\{(a_1, a_2) + (b_1, b_2)\}(y_1, y_2)\pi(x_1 \times x_2)a_2 \times a_1 - 2(y_1, y_2)\pi(x_1 \times x_2)a_1 \times a_1 - 2(y_1, y_2)\pi(x_1 \times x_2)a_2 \times a_2 + (y_1, y_2)\pi(x_1 \times x_2)$$

and

$$A_2 = (y_1, y_2)\pi(a_1(\bar{a}_2x_1) \times a_1(\bar{a}_2x_2)) + (b_1(\bar{b}_2y_1), b_1(\bar{b}_2y_2))\pi(x_1 \times x_2) + (y_1, b_1(\bar{b}_2y_2))\pi(a_1(\bar{a}_2x_1) \times x_2) + (b_1(\bar{b}_2y_1), y_2)\pi(x_1 \times a_1(\bar{a}_2x_2)) + (\bar{b}_2y_1, \bar{b}_2y_2)v^2(\bar{a}_1x_1 \times \bar{a}_1x_2) + (\bar{b}_1y_1, \bar{b}_1y_2)v^2(\bar{a}_2x_1 \times \bar{a}_2x_2) - (\bar{b}_2y_1, \bar{b}_1y_2)v^2(\bar{a}_1x_1 \times \bar{a}_2x_2) - (\bar{b}_1y_1, \bar{b}_2y_2)v^2(\bar{a}_2x_1 \times \bar{a}_1x_2).$$
From Lemma 1.3, we have

\[(y_1, b_1(\bar{b}_2 y_2)) = (\bar{b}_1 y_1, \bar{b}_1 y_2) = 2(b_1, \pi(y_1 \times y_2)b_2) + (y_1, y_2)(b_1, b_2),\]

\[\pi(a_1(\bar{a}_2 x_1) \times x_2) - v^2(\bar{a}_2 x_1 \times \bar{a}_1 x_2) = a_1 \times x_2(\bar{a}_2 x_1)\]

\[= (x_1, x_2)a_1 \times a_2 + 2\pi(x_1 \times x_2)a_2 \times a_1,\]

\[v^2(\bar{a}_1 \times a_2) = \pi(x_1 \times a(\bar{a}_2 x_2)) - x_1(\bar{a}x_2) \times a\]

\[= (a, a)\pi(x_1 \times x_2) - 2\pi(x_1 \times x_2)a \times a,\]

\[\pi(a_1(\bar{a}_2 x_1) \times a_1(\bar{2} x_2)) = (a_1, a_1)(a_2, a_2)\pi(x_1 \times x_2) - 2(a_2, a_2)\pi(x_1 \times x_2)a_1 \times a_1\]

\[- 2(a_1, a_1)\pi(x_1 \times x_2)a_2 \times a_2 + 4(a_1, a_2)\pi(x_1 \times x_2)a_2 \times a_1\]

\[+ 4(a_1, a_1)\pi(x_1 \times x_2)a_1 \times a_2.\]

Using these, we see \(A_2 = A_1\). We can obtain \(B_1 = B_2\) and \(C_1 = C_2\) similarly. We can prove Cases 2 and 3 in a way similar to Case 1.

This lemma implies the following

**Lemma 2.6.** For \(X \in \mathfrak{so}(16, C)\) and \(P, Q \in V^C\), we have

\[\[X, P \times Q\] = d\rho(X)P \times Q + P \times d\rho(X)Q.\]

**Lemma 2.7.** For \(P_i \in V^C\) \((i = 1, 2, 3)\), we have

\[d\rho(P_1 \times P_2)P_3 + d\rho(P_2 \times P_3)P_1 + d\rho(P_3 \times P_1)P_2 = 0.\]

**Proof.** It is sufficient to prove the following 4 cases. Case 1. \(P_i = (x_i \otimes y_i, 0)\), \((i = 1, 2, 3)\). Case 2. \(P_i = (x_i \otimes y_i, 0), (i = 1, 2), P_3 = (0, z \otimes w)\). Case 3. \(P_i = (0, z_i \otimes w_i), (i = 1, 2), P_3 = (x \otimes y, 0)\). Case 4. \(P_i = (0, z_i \otimes w_i), (i = 1, 2, 3)\).

**Case 1.**

\[d\rho(P_1 \times P_2)P_3 = ((y_1, y_2)\{(x_1 \times x_2)x_3\} \otimes y_3 + (x_1, x_2)x_3 \otimes \{(y_1 \times y_2)y_3\}, 0)\]

\[= ((x_2, x_3)x_1 \otimes (y_1, y_2)y_3 - (x_3, x_1)x_2 \otimes (y_1, y_2)y_3\]

\[+ (x_1, x_2)x_3 \otimes (y_2, y_3)y_1 - (x_1, x_2)x_3 \otimes (y_3, y_1)y_2, 0)\]
Subgroups $D_8$ of $E_8$

\begin{align*}
= -((x_3, x_1)x_2 \otimes (y_2, y_3)y_1 - (x_1, x_2)x_3 \otimes (y_2, y_3)y_1 \\
+ (x_2, x_3)x_1 \otimes (y_3, y_1)y_2 - (x_2, x_3)x_1 \otimes (y_1, y_2)y_3, 0) \\
- ((x_1, x_2)x_3 \otimes (y_3, y_1)y_2 - (x_2, x_3)x_1 \otimes (y_3, y_1)y_2 \\
+ (x_3, x_1)x_2 \otimes (y_1, y_2)y_3 - (x_3, x_1)x_2 \otimes (y_2, y_3)y_1, 0) \\
= -d\rho(P_2 \times P_3)P_1 - d\rho(P_3 \times P_1)P_2
\end{align*}

Hence the formula can be proved in this case. The other cases can be probed in a way similar to Case 1.

\[\square\]

\section*{§ 2.3. Symmetric Bilinear Form in $V^C$}

Define a symmetric bilinear form $(,)$ in $V^C$ as

\[((x_1 \otimes y_1, z_1 \otimes w_1), (x_2 \otimes y_2, z_2 \otimes w_2)) = (x_1, x_2)(y_1, y_2) + (z_1, z_2)(w_1, w_2)\]

Then we have the following

**Lemma 2.8.** For $x \in \text{Spin}(16, C)$, $X \in \text{so}(16, C)$ and $P, Q \in V^C$, we have

1. $(\rho(x)P, \rho(x)Q) = (P, Q)$,
2. $(d\rho(X)P, Q) + (P, d\rho(X)Q) = 0$,
3. $\text{tr} X(P \times Q) = 2(d\rho(X)P, Q)$.

**Proof.** (1) and (2) are clear. In order to prove (3), we consider the following lemma.

**Lemma 2.9.** For $A \in \text{so}(8, C)$ and $x, y \in \mathbb{C}^C$, we have

4. $\text{tr} A\pi(x \times y) = 2(\pi Ax, y)$,
5. $\text{tr} Av^2(x \times y) = 2(v^2 Ax, y)$.

**Proof.** Since $\text{so}(8, C)$ is spanned by $a \times b$ ($a, b \in \mathbb{C}^C$), it is sufficient to prove for $A = a \times b$. From Lemma 1.3, we have

\[
\text{tr} A\pi(x \times y) = \sum_i (\pi(x \times y)A e_i, e_i)
\]

\[
= \sum_i \frac{1}{2}(x(yA e_i), e_i) - \sum_i \frac{1}{2}(x, y)(A e_i, e_i)
\]
Thus (4) is proved. (5) can be proved similarly. □

PROOF OF LEMMA 2.8 (3). Let \( P = (x_1 \otimes y_1, z_1 \otimes w_1) \) and \( Q = (x_2 \otimes y_2, z_2 \otimes w_2) \). For \( X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \) \((A, B) \in \text{so}(8, G))\), we have

\[
\text{tr} \, X(P \times Q) = (y_1, y_2) \text{tr} \, A \pi(x_1 \times x_2) + (w_1, w_2) \text{tr} \, A \nu^2(z_1 \times z_2) \\
+ (x_1, x_2) \text{tr} \, B \pi(y_1 \times y_2) + (z_1, z_2) \text{tr} \, B \nu^2(w_1 \times w_2) \\
= 2(y_1, y_2)(\pi A x_1, x_2) + 2(w_1, w_2)(\nu^2 A z_1, z_2) \\
+ 2(x_1, x_2)(\pi B y_1, y_2) + 2(z_1, z_2)(\nu^2 B w_1, w_2) \\
= 2(dp(X)P, Q).
\]

For \( X = \begin{pmatrix} 0 & u'v' \\ -v'u & 0 \end{pmatrix} \), we have

\[
2\text{tr} \, X(P \times Q) = -\text{tr} \, u'v' \{ (x_1 \bar{z}_2)'(y_1 \bar{w}_2) - (x_2 \bar{z}_1)'(y_2 \bar{w}_1) \} \\
- \text{tr} \, v'u \{ (x_1 \bar{z}_2)'(y_1 \bar{w}_2) - (x_2 \bar{z}_1)'(y_2 \bar{w}_1) \} \\
= -(v, y_1 \bar{w}_2)(u, x_1 \bar{z}_2) + (v, y_2 \bar{w}_1)(u, x_2 \bar{z}_1) \\
- (u, x_1 \bar{z}_2)(v, y_1 \bar{w}_2) + (u, x_2 \bar{z}_1)(v, y_2 \bar{w}_1) \\
= 2(ux_1, x_2)(vw_1, y_2) - 2(\bar{u}x_1, z_2)(\bar{v}y_1, w_2) \\
= 4(dp(X)P, Q). \square
\]

§ 3. Complex Exceptional Lie Algebra \( g^C \) of Type \( E_8 \)

§ 3.1. Lie Algebra \( g^C \)

Let

\[ g = \text{so}(16) \oplus V. \]

\( g \) is a 248 dimensional real vector space.
Subgroups $D_8$ of $E_8$

**Definition 3.1.** We define an anti-symmetric bilinear multiplication $[,]$ in $g$ (resp. $g^C$) as

$$[(X, P), (Y, Q)] = ([X, Y] - P \times Q, dp(X)Q - dp(Y)P)$$

where $X, Y \in \mathfrak{so}(16)$ (resp. $\mathfrak{so}(16, C)$) and $P, Q \in V$ (resp. $V^C$).

**Lemma 3.2.** $g$ and $g^C$ are Lie algebras with the multiplication $[,]$.

**Proof.** We can prove the Jacobi identity using Lemmas 2.6 and 2.7. □

§ 3.2. Simplicity of $g^C$ and Type of $g^C$

**Lemma 3.3.** $g^C$ is a simple Lie algebra.

**Proof.** Let $a$ be a non-zero ideal of $g^C$. There are three cases to be considered: Case 1. $\mathfrak{so}(16, C) \cap a \neq \{0\}$. Case 2. $V^C \cap a \neq \{0\}$. Case 3. $\mathfrak{so}(16, C) \cap a = \{0\}$ and $V^C \cap a = \{0\}$.

**Case 1.** Since $\mathfrak{so}(16, C) \cap a$ is a non-zero ideal of $\mathfrak{so}(16, C)$, we see $\mathfrak{so}(16, C) \subset a$. Moreover we have

$$V^C = dp(\mathfrak{so}(16, C)) V^C = [\mathfrak{so}(16, C), V^C] \subset [a, V^C] \subset a.$$ 

Then we have $a = g^C$.

**Case 2.** For any non-zero element $P \in V^C \cap a$, we can choose an element $Q \in V^C$ such that $P \times Q \neq 0$. Hence we can reduce this case to the Case 1.

**Case 3.** Let $q : g^C \to \mathfrak{so}(16, C)$ denote the projection. Since $q(a)$ is a non-zero ideal of $\mathfrak{so}(16, C)$, we see $q(a) = \mathfrak{so}(16, C)$. Let $V^C = \sum_{\alpha_i} (V^C)_{\alpha_i}$ be a weight decomposition of the representation $(\mathfrak{so}(16, C), dp, V^C)$ with respect to a Cartan subalgebra $\mathfrak{b}_{\mathfrak{so}(16, C)} \subset \mathfrak{so}(16, C)$. Choose an element $H \in \mathfrak{b}_{\mathfrak{so}(16, C)}$ such that $\alpha_i(H) \neq 0$ for any weight $\alpha_i$. Since $H \in \mathfrak{b}_{\mathfrak{so}(16, C)} \subset q(a)$, there exist a non-zero element

$$P = \sum_{\alpha_i} b_{\alpha_i} P_{\alpha_i} \in V^C$$

where $P_{\alpha_i} \in (V^C)_{\alpha_i}$ is a weight vector and $b_{\alpha_i} \in C$, such that $(H, P) \in a$. Then we
have

\[ [(H, 0), (H, P)] = (0, d\rho(H)P) = \left(0, \sum_{x_i} \alpha_i(H)b_{x_i}P_{x_i} \right) \in V^C \cap a = \{0\}. \]

This shows \( \alpha_i(H)b_{x_i}P_{x_i} = 0 \). Since \( \alpha_i(H) \neq 0 \), we see \( b_{x_i} = 0 \), i.e., \( P = \sum_{x_i} b_{x_i}P_{x_i} = 0 \). This is a contradiction.

Since \( \dim_C g^C = 248 \), we have the following

**Theorem 3.4.** \( g^C \) is a complex exceptional Lie algebra of type \( E_8 \).

§ 3.3. Killing Form of \( g^C \)

**Proposition 3.5.** The Killing form \( B \) of \( g^C \) is given by

\[ B((X, P), (Y, Q)) = 30\text{tr}XY - 60(P, Q). \]

**Proof.** Let us define a symmetric bilinear form as

\[ B_1((X, P), (Y, Q)) = \text{tr}XY - 2(P, Q). \]

Using Lemma 2.8, we see that \( B_1 \) is \( g^C \)-invariant. Since \( g^C \) is simple, there exists some \( \alpha \in C \) such that \( B = \alpha B_1 \). Since \( B(R, R) = 60 \) and \( B_1(R, R) = 2 \) for \( R = \left( \begin{pmatrix} iG_{ij} & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) \in g^C \), we see \( B = 30B_1 \).

§ 4. Real Forms of \( g^C \)

§ 4.1. Involutions of \( g^C \)

Let \( \gamma \) be the automorphism of \( \mathbb{C}^C \) defined in § 1.2. Using this, we define a \( C \)-linear transformation \( \gamma_1 \) on \( g^C \) as

\[ \gamma_1 \left( \begin{pmatrix} A & u'v \\ -v'u & B \end{pmatrix}, (x \otimes y, z \otimes w) \right) \]

\[ = \left( \begin{pmatrix} \gamma A\gamma^{-1} & (\gamma u)'u \\ -v'((\gamma u) \otimes B) \end{pmatrix}, (\gamma x \otimes y, \gamma z \otimes w) \right). \]

It is clear \( \gamma_1^2 = \text{id} \). Furthermore, define \( C \)-linear transformations \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon \) of
Subgroups $D_8$ of $E_8$

$\mathfrak{g}^C$ as
\[
\begin{align*}
\varepsilon_1(X, (x \otimes y, z \otimes w)) &= (I_{8,8}X I_{8,8}, (-x \otimes y, z \otimes w)), \\
\varepsilon_2(X, (x \otimes y, z \otimes w)) &= (I_{8,8}X I_{8,8}, (x \otimes y, -z \otimes w)), \\
\varepsilon(X, (x \otimes y, z \otimes w)) &= (X, (-x \otimes y, -z \otimes w)).
\end{align*}
\]
It is clear that $\varepsilon = \varepsilon_1 \varepsilon_2 = \varepsilon_2 \varepsilon_1$ and $\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon^2 = \text{id}$. From the definition, it is clear that $\gamma_1$, $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon$ are commutable with each other. Furthermore we have the following Lemma through straightforward calculations.

**Lemma 4.1.** $\gamma_1$, $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon$ are involutive automorphisms of $\mathfrak{g}^C$.

Let $J$ be a $C$-linear transformation of $\mathfrak{g}^C$ defined by
\[
J(X, (x \otimes y, z \otimes w)) = (J_8X I_8^{-1}, (y \otimes x, w \otimes z))
\]
where $J_8 = \begin{pmatrix} -I_8 & 0 \\ I_8 & 0 \end{pmatrix}$. It is clear that $J^2 = \text{id}$ and $\varepsilon J = J \varepsilon$. Furthermore we have the following Lemma through straightforward calculations.

**Lemma 4.2.** $J$ is an involutive automorphism of $\mathfrak{g}^C$.

Since $\tau$ is the complex conjugation, $\tau \gamma_1$, $\tau \varepsilon_j$, $\tau \varepsilon$, $\tau \gamma_1 \varepsilon$, $\tau J$ and $\tau \varepsilon J$ ($j = 1, 2$) are complex conjugate linear involutions of the complex Lie algebra $\mathfrak{g}^C$.

### §4.2. Real Forms of $\mathfrak{g}^C$

Since the Killing form $B$ of $\mathfrak{g}^C$ is negative definite on $\mathfrak{g}$,
\[
\mathfrak{g} = (\mathfrak{g}^C)^\tau = \{ R \in \mathfrak{g}^C | \tau R = R \} = \text{so}(16) \oplus V
\]
is a compact real form of $\mathfrak{g}^C$.

Let us consider the $\mathcal{R}$-subalgebra $(\mathfrak{g}^C)^{\tau_1}$ of $\mathfrak{g}^C$ defined as
\[
(\mathfrak{g}^C)^{\tau_1} := \{ R \in \mathfrak{g}^C | \tau_1 R = R \} = \text{so}(8, 8) \oplus (V^C)^{\tau_1},
\]
where
\[
(V^C)^{\tau_1} = i\mathbb{C} \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathbb{C}.
\]
It is clear that $(\mathfrak{g}^C)^{\tau_1}$ is a real form of $\mathfrak{g}^C$, i.e., $(\mathfrak{g}^C)^{\tau_1} = \mathfrak{g}^C$. Let
\[
\text{so}(8, 8) = \text{t}_{\text{so}(8, 8)} \oplus \text{p}_{\text{so}(8, 8)}
\]
be a Cartan decomposition of so(8, 8). The Cartan decomposition of \((g^C)^{\gamma_1}\) is given by

\[
(g^C)^{\gamma_1} = \mathfrak{f} \oplus \mathfrak{p}, \quad \mathfrak{f} = \mathfrak{f}_{so(8,8)} \oplus \mathfrak{f}(V^C)^{\gamma_1},
\]

\[
\mathfrak{p} = \mathfrak{p}_{so(8,8)} \oplus \mathfrak{p}(V^C)^{\gamma_1},
\]

where

\[
\mathfrak{f}_{(V^C)^{\gamma_1}} = \{(0, z \otimes w) \mid z, w \in \mathbb{C}\},
\]

\[
\mathfrak{p}_{(V^C)^{\gamma_1}} = \{(ix \otimes y, 0) \mid x, y \in \mathbb{C}\}.
\]

The Cartan involution is \(e_1\) and the Cartan index of \((g^C)^{\gamma_1}\) is \(\dim \mathfrak{p} - \dim \mathfrak{f} = 128 - 120 = 8\). Hence \((g^C)^{\gamma_1}\) is an exceptional non-compact real simple Lie algebra of type \(E_{6(8)}\). Similarly, let us consider the other real forms of \(g^C\) as follows:

\[
(g^C)^{\gamma} := \{R \in g^C \mid \tau e R = R\} = so(16) \oplus iV,
\]

(Cartan involution \(e\), Cartan index 8),

\[
(g^C)^{\gamma_1} := \{R \in g^C \mid \tau e_1 \gamma_1 R = R\} = so(4, 12) \oplus (V^C)^{\gamma_1},
\]

(Cartan involution \(e_1 \gamma_1\), Cartan index \(-24\)),

\[
(g^C)^{\gamma J} := \{R \in g^C \mid \tau J R = R\} = so^*(16) \oplus (V^C)^{\gamma J},
\]

(Cartan involution \(J\), Cartan index \(-24\)),

\[
(g^C)^{\gamma J} := \{R \in g^C \mid \tau e J R = R\} = so^*(16) \oplus i(V^C)^{\gamma J},
\]

(Cartan involution \(e J\), Cartan index 8),

where

\[
(V^C)^{\gamma_1} = i\mathbb{C}^I \otimes \mathbb{C} \oplus \mathbb{C}^I \otimes \mathbb{C},
\]

\[
(V^C)^{\gamma J} = (\mathbb{C}^C \otimes \mathbb{C}^C)^{\gamma J} \oplus (\mathbb{C}^C \otimes \mathbb{C}^C)^{\gamma J},
\]

\[
(\mathbb{C}^C \otimes \mathbb{C}^C)^{\gamma J} = \left\{ \sum_k x_k \otimes y_k \mid \sum_k x_k \otimes y_k = \sum_k \tau y_k \otimes \tau x_k \right\}.
\]

Thus we have the following
THEOREM 4.3. (1) $\mathfrak{g}$ is a compact exceptional real Lie algebra of type $E_8$.
(2) $(g^C)_{\text{re}} \cong (g^C)_{\text{rl}} \cong (g^C)_{\text{rd}}$ is a non-compact exceptional real Lie algebra of type $E_{8(8)}$.
(3) $(g^C)_{\text{re}l} \cong (g^C)_{\text{rd}}$ is a non-compact exceptional real Lie algebra of type $E_{8(-24)}$.

§ 5. Exceptional Simple Lie Groups of Type $E_8$ ([5], [6], [10], [11])

Define a positive definite Hermitian inner product of $g^C$ as

$$\langle R_1, R_2 \rangle := -B(R_1, \tau R_2), \quad (R_i \in g^C).$$

We define a complex exceptional simple Lie group and a compact exceptional simple Lie group of type $E_8$ as

$$E_8^C := \text{Aut}_C g^C,$$
$$E_8 := \{ \alpha \in E_8^C | \langle \alpha R_1, \alpha R_2 \rangle = \langle R_1, R_2 \rangle \}.$$

For a non-compact exceptional simple Lie algebra $e_8(k)$ of type $E_{8(k)}$ ($k = 8$ or $-24$), a non-compact connected exceptional simple Lie group $E_{8(k)}$ can be defined as

$$E_{8(k)} := \text{Aut}_R e_8(k)$$

$$:= ((E_8)^C)_{\sigma_k} = \{ \alpha \in E_8^C | \sigma_k \alpha = \alpha \sigma_k \},$$

where $\sigma_k$ is the complex conjugation of $e_8^C = g^C$ with respect to $e_8(k)$. Hence from Theorem 4.3 we have the following

THEOREM 5.1.

(1) $E_8 \cong (E_8^C)^\tau := \{ \alpha \in E_8^C | \tau \alpha = \alpha \tau \},$
(2) $E_{8(8)} \cong (E_8^C)_{\text{re}} \cong (E_8^C)_{\text{rl}} \cong (E_8^C)_{\text{rd}},$
(3) $E_{8(-24)} \cong (E_8^C)_{\text{re}l} \cong (E_8^C)_{\text{rd}}.$

From definitions of the transformation $\varepsilon$ and the real forms of $g^C$, $\varepsilon$ can be considered as an element of the groups $E_8^C$, $E_8$, $E_{8(8)}$ and $E_{8(-24)}$.

§ 6. Subgroups of Type $D_8$

Clearly, the complex exceptional Lie algebra $g^C$ of type $E_8$ has a classical subalgebra $\mathfrak{so}(16, C)$ of type $D_8$. For real exceptional Lie algebras of type $E_8$, from §4.2 we see the following (1), (2) and (3).
(1) The compact exceptional Lie algebra of type $E_8$ has a classical subalgebra $so(16)$ of type $D_8$.

(2) The non-compact exceptional Lie algebra of type $E_{8(8)}$ has classical subalgebras $so(16)$, $so(8,8)$ and $so^*(16)$ of type $D_8$.

(3) The non-compact exceptional Lie algebra of type $E_{8(-24)}$ has classical subalgebras $so(4,12)$ and $so^*(16)$ of type $D_8$.

In this section, we consider subgroups of type $D_8$ of complex and real exceptional Lie groups of type $E_8$.

§ 6.1. Subgroups of Type $D_8$ of $E_8^C$

In this subsection, consider subgroup of $E_8^C$;

$$(E_8^C)_e = \{ x \in E_8^C \mid ex = x \} \subset E_8^C.$$

We define a mapping $\varphi : Spin(16, C) \to (E_8^C)_e$ as

$$\varphi(x)(X, P) = (\text{Ad}(x)X, \rho(x)P).$$

The Lie algebra of $(E_8^C)_e$ is isomorphic to $(E_8^C)_e = so(16, C) = \text{LieSpin}(16, C)$. Thus the differential of $\varphi$ is surjective. Since $(E_8^C)_e$ is connected ([7]), $\varphi$ is surjective. Since

$$z(\text{Spin}(16, C)) = \{1, -1\} \times \{1, \omega\}$$

where

$$\omega = \begin{pmatrix} e_0 \\ 0 \\ e_1 \\ 0 \\ \vdots \\ e_7 \\ 0 \\ e_0 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ 0 \\ 0 \\ e_0 \end{pmatrix} \cdots \begin{pmatrix} e_7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_1 \\ \cdots \\ e_7 \end{pmatrix},$$

and $\rho(\pm 1) = \rho(\pm \omega) = \pm \text{id}$, we see $\text{Ker} \varphi = \{1, \omega\}$. Hence we have the following

**Theorem 6.1.** The complex exceptional Lie group $E_8^C$ of type $E_8$ has the following subgroup of type $D_8$.

$$(E_8^C)_e \cong Ss(16, C) := \text{Spin}(16, C)/\{1, \omega\}.$$

§ 6.2. Preliminaries for Non-compact Case

Define elements $\omega_1, \omega_2 \in \text{Spin}(16, C)$ as

$$\omega_1 = \begin{pmatrix} e_0 \\ 0 \\ e_1 \\ 0 \\ \cdots \\ e_7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ 0 \\ 0 \\ e_0 \end{pmatrix} \cdots \begin{pmatrix} e_7 \\ 0 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 0 \\ e_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ e_7 \end{pmatrix}.$$
It is clear that $\omega_1^2 = \omega_2^2 = 1$ and $\omega = \omega_1 \cdot \omega_2 = \omega_2 \cdot \omega_1$. Since

$$p(\omega_1) = I_{8,8}, \quad p(\omega_2) = -I_{8,8} \in SO(16, C)$$

and

$$L_{e_0}L_{e_1}L_{e_2}L_{e_3}L_{e_4}L_{e_5}L_{e_6}L_{e_7} = \text{id}_C,$$

we see the following

**Lemma 6.2.**

$$\varphi(\omega_1) = \varphi(\omega_2) = e_2, \quad \varphi(-\omega_1) = \varphi(-\omega_2) = e_1,$$

$$\varphi(1) = \varphi(\omega) = 1, \quad \varphi(-1) = \varphi(-\omega) = e.$$

Furthermore we define elements $\omega_3, \omega_4 \in Spin(16, C)$ as

$$\omega_3 = \begin{pmatrix} e_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_3 \\ 0 \end{pmatrix},$$

$$\omega_4 = \begin{pmatrix} e_4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_6 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_7 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_3 \end{pmatrix}.$$

It is clear that $\omega_3^2 = \omega_4^2 = 1$ and $\omega = \omega_3 \cdot \omega_4 = \omega_4 \cdot \omega_3$. Since

$$p(\omega_3) = I_{4,12}, \quad p(\omega_4) = -I_{4,12} \in SO(16, C)$$

and

$$L_{e_0}L_{e_1}L_{e_2}L_{e_3} = L_{e_4}L_{e_5}L_{e_6}L_{e_7} = -\gamma,$$

we have the following

**Lemma 6.3.**

$$\varphi(\omega_3) = \varphi(\omega_4) = e_1\gamma_1, \quad \varphi(-\omega_3) = \varphi(-\omega_4) = e_2\gamma_1.$$

Define an element $j \in Spin(16, C)$ as

$$j = \begin{pmatrix} 1 \\ e_0 \\ \sqrt{2}e_0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ e_1 \\ \sqrt{2}e_1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ e_2 \\ \sqrt{2}e_2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ e_3 \\ \sqrt{2}e_3 \\ 1 \end{pmatrix}.$$

It is clear that $j^2 = 1.$
Lemma 6.4.
\[ j = \omega_1 \cdot j \cdot \omega_2 = \omega_2 \cdot j \cdot \omega_1, \quad \omega_1 \cdot j = j \cdot \omega_2, \quad \omega_2 \cdot j = j \cdot \omega_1. \]

Proof. Since \( \begin{pmatrix} e_k \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_k \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ e_k \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_k \end{pmatrix} = -1 \), we see
\[ \begin{pmatrix} e_k \\ 0 \end{pmatrix} \cdot \begin{pmatrix} e_k \\ 0 \end{pmatrix} = \begin{pmatrix} e_k \\ 0 \end{pmatrix} \cdot \left\{ \begin{pmatrix} e_k \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ e_k \end{pmatrix} \right\} = -\begin{pmatrix} e_k \\ 0 \end{pmatrix}. \]

Hence
\[ \omega_1 \cdot j \cdot \omega_2 = \begin{pmatrix} e_0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} e_0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} \frac{1}{\sqrt{2}} e_7 \\ 0 \end{pmatrix} = (-1)^8 \begin{pmatrix} e_0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} e_7 \\ 0 \end{pmatrix} = j. \]

Furthermore we see \( \omega_1 \cdot j = \omega_1 \cdot j \cdot \omega_2 = j \cdot \omega_2. \]

Lemma 6.5.
\( \varphi(j \cdot \omega_1) = \varphi(j \cdot \omega_2) = J, \quad \varphi(-j \cdot \omega_1) = \varphi(-j \cdot \omega_2) = \varepsilon J. \)

Proof. From §1.2, we have \( \rho(j \cdot \omega_1) = J_8. \) Hence
\[ \varphi(j \cdot \omega_1)(X, P) = (J_8 X J_8^{-1}, \rho(j \cdot \omega_1) P). \]

Let \( j_k = \begin{pmatrix} \frac{1}{\sqrt{2}} e_k \\ 0 \\ \frac{1}{\sqrt{2}} e_k \end{pmatrix} \cdot \begin{pmatrix} 0 \\ e_k \end{pmatrix}. \) Then \( j \cdot \omega_1 = j_0 \cdot j_1 \cdots j_7, \) and we have
\[ \rho(j_k)(x \otimes y, z \otimes w) = \frac{1}{\sqrt{2}} (-x \otimes y + E_k(z \otimes w), -E_k(x \otimes y) - z \otimes w), \]
where \( E_k(z \otimes w) = e_k z \otimes e_k w. \) Using this, we have
\[ \rho(j \cdot \omega_1)(x \otimes y, z \otimes w) = \frac{1}{8} (A(x \otimes y), A(z \otimes w)), \]
where

\[ A = 1 - \sum_{k<l} E_k E_l + \sum_{1 \leq k<j < m} E_k E_l E_m. \]

Through the straightforward calculations, we see \( A(e_i \times e_j) = 8e_j \times e_i \) for any \( 0 \leq i, j \leq 7 \). Then we have \( \varphi(j \cdot \omega) = J \). Since \( \varphi(\omega) = 1 \) and \( \varphi(-\omega) = e \), others are clear.

\[ \square \]

§6.3. Subgroup \( S_{S}(16) \) of \( E_8 \) and \( E_{8}(8) \)

From §6.1, we have the following

**Theorem 6.6.** (1) The compact exceptional Lie group \( (E_8^C)^* \) of type \( E_8 \) has the following subgroup of type \( D_8 \).

\[ ((E_8^C)^*)^e \cong S_{S}(16) := \text{Spin}(16)/\{1, \omega\}. \]

(2) The non-compact exceptional Lie group \( (E_8^C)^{\text{te}} \) of type \( E_{8}(8) \) has the following subgroup of type \( D_8 \).

\[ ((E_8^C)^{\text{te}})^e \cong S_{S}(16) := \text{Spin}(16)/\{1, \omega\}. \]

**Proof.** (1)

\[ ((E_8^C)^*)^e = ((E_8^C)^*)^e = \{ \varphi(\alpha) \mid \alpha \in \text{Spin}(16, C), \tau \varphi(\alpha) \tau = \varphi(\alpha) \} \]
\[ = \{ \varphi(\alpha) \mid \alpha \in \text{Spin}(16) \} \]
\[ \cong S_{S}(16) := \text{Spin}(16)/\{1, \omega\}. \]

(2) \( ((E_8^C)^*)^e = ((E_8^C)^{\text{te}})^e \) is clear. \[ \square \]

§6.4. Subgroup \( S_{S}(8, 8) \times 2 \) of \( E_{8}(8) \)

In this subsection, consider subgroup of \( (E_8^C)^{\text{te}} \);

\[ ((E_8^C)^{\text{te}})^e = \{ \alpha \in (E_8^C)^{\text{te}} \mid \text{exc} = \alpha \} \subset (E_8^C)^{\text{te}} \cong E_{8}(8). \]

Since

\[ ((E_8^C)^{\text{te}})^e = ((E_8^C)^{\text{te}})^{\text{te}} = \{ \varphi(\alpha) \mid \alpha \in \text{Spin}(16, C), e_1 \tau \varphi(\alpha) e_1 \tau = \varphi(\alpha) \} \]
\[ = \{ \varphi(\alpha) \mid \alpha \in \text{Spin}(16, C), e_1 \varphi(e_1) e_1 = \varphi(\alpha) \}, \]
we consider
\[ G = \{ x \in \text{Spin}(16, C) \mid \epsilon_1 \varphi(\tau x) e_1 = \varphi(x) \}. \]

Clearly, \( ((E^C_8)^{\epsilon_1})^e = \varphi(G) \). Let \( x \in G \). From Lemma 6.2, we have
\[ \varphi(x) = \epsilon_1 \varphi(\tau \varphi) e_1 = \varphi(-\omega_i) \varphi(\tau x) \varphi(-\omega_j) = \varphi(\omega_i \cdot \tau x \cdot \omega_j), \quad (i, j = 1, 2). \]

Since \( \text{Ker} \varphi = \{ 1, \omega \} \) and \( \omega \cdot \omega_1 = \omega_2 \), we see
\[ x = \omega_1 \cdot \tau x \cdot \omega_1 \quad \text{or} \quad x = \omega_1 \cdot \tau x \cdot \omega_2. \]

Hence we have
\[ G = G_1 \cup G_2, \quad \text{where} \quad G_1 = \{ x \in \text{Spin}(16, C) \mid \omega_1 \cdot \tau x \cdot \omega_1 = x \}, \]
\[ G_2 = \{ \beta \in \text{Spin}(16, C) \mid \omega_1 \cdot \tau \beta \cdot \omega_2 = \beta \}. \]

**Lemma 6.7.** For \( x \in G_1 \) and \( \beta, \beta' \in G_2 \), we have
\[ x \cdot \beta \in G_2, \quad \beta \cdot \beta' \in G_1. \]

**Proof.**
\[ \omega_1 \cdot \tau(x \cdot \beta) \cdot \omega_2 = \omega_1 \cdot \tau x \cdot \tau \beta \cdot \omega_2 = \omega_1 \cdot \tau x \cdot \omega_1 \cdot \tau \beta \cdot \omega_2 = x \cdot \beta, \]
\[ \omega_1 \cdot \tau(\beta \cdot \beta') \cdot \omega_1 = \omega_1 \cdot \tau \beta \cdot \tau \beta' \cdot \omega_1 = \omega_1 \cdot \tau \beta \cdot \omega_2 \cdot \omega_2 \cdot \tau \beta' \cdot \omega_1 = \beta \cdot \beta'. \]

From Lemma 6.4, we have \( j \in G_2 \). Clearly, \( G_1 \cap G_2 = \phi \). Then we have
\[ G_2 = G_1 \cdot j \quad \text{and} \quad G = G_1 \cup G_2 = G_1 \times \{ 1, j \} = G_1 \times 2. \]

For \( \chi = a_1 \cdot a_2 \cdots a_{2m} \in \text{Spin}(16, C) \), \( a_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \),

we see
\[ \omega_1 \cdot \tau a_i \cdot \omega_1 = \omega_1 \cdot \tau a_1 \cdot \omega_1 \cdot \omega_1 \cdot \tau a_2 \cdot \omega_1 \cdots \omega_1 \cdot \tau a_{2m} \cdot \omega_1 \]

and
\[ \omega_1 \cdot \tau a_i \cdot \omega_1 = I_{8,8} \tau a_i = \begin{pmatrix} -\tau a_i \\ \tau b_i \end{pmatrix}. \]
Then, from §1.1, we see

\[ G_1 = \text{Spin}(8, 8) = \{ x \in \text{Spin}(16, C) \mid I_{8, 8} \tau x = x \}. \]

From Lemmas 6.2 and 6.5, we see \( \phi(j) = J_{E_2} \). Since \( \omega \in \text{Spin}(8, 8) \) and 
\( \text{Ker} \phi|_{\text{Spin}(8, 8)} = \{ 1, \omega \} \), we see

\[ \phi(G) \cong \text{Spin}(8, 8)/\{ 1, \omega \} \times \{ 1, \phi(j) \} = Ss(8, 8) \times \{ 1, J_{E_2} \}. \]

Thus we have the following

**Theorem 6.8.** The non-compact exceptional Lie group \((E_8^C)^{\gamma_1}\) of type \(E_{8(8)}\) has the following subgroup of type \(D_8\).

\[ ((E_8^C)^{\gamma_1})^G \cong Ss(8, 8) \times \{ 1, J_{E_2} \} = Ss(8, 8) \times 2. \]

**§ 6.5. Subgroup \(S_5(4, 12)\) of \(E_{8(-24)}\)**

In this subsection, consider subgroup of \((E_8^C)^{\gamma_1}\);

\[ ((E_8^C)^{\gamma_1})^G = \{ x \in (E_8^C)^{\gamma_1} \mid e \varepsilon e = x \} \subset (E_8^C)^{\gamma_1} \cong E_{8(-24)}. \]

Since

\[ ((E_8^C)^{\gamma_1})^G = ((E_8^C)^{\gamma_1})^G = \{ \phi(x) \mid x \in \text{Spin}(16, C), \gamma_1 e_1 \phi(x) e_1 \gamma_1 = \phi(x) \} = \{ \phi(x) \mid x \in \text{Spin}(16, C), \gamma_1 e_1 \phi(e \varepsilon e) e_1 \gamma_1 = \phi(x) \}, \]

we consider

\[ G' = \{ x \in \text{Spin}(16, C) \mid \gamma_1 e_1 \phi(e \varepsilon e) e_1 \gamma_1 = \phi(x) \}. \]

From Lemma 6.3, the same as §6.4 we have the following

\[ G' = G'_1 \cup G'_2, \text{ where } G'_1 = \{ x \in \text{Spin}(16, C) \mid \omega_3 \cdot \tau x \cdot \omega_3 = x \}, \]

\[ G'_2 = \{ \beta \in \text{Spin}(16, C) \mid \omega_3 \cdot \tau \beta \cdot \omega_4 = \beta \}. \]

For

\[ x = \tilde{x}_1 \cdot \tilde{x}_2 \cdots \tilde{x}_{2m} \in \text{Spin}(16, C), \quad \tilde{x}_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \]

we see

\[ \omega_3 \cdot \tau x \cdot \omega_3 = \omega_3 \cdot \tau \tilde{x}_1 \cdot \omega_3 \cdot \omega_3 \cdot \tau \tilde{x}_2 \cdot \omega_3 \cdots \omega_3 \cdot \tau \tilde{x}_{2m} \cdot \omega_3. \]
and

$$\omega_3 \cdot \tau \tilde{a}_i \cdot \omega_3 = I_{4,12} \tau \tilde{a}_i = \left( \begin{array}{c} -\gamma \tau a_i \\ \tau b_i \end{array} \right).$$

Then, from §1.1, we see

$$G'_1 = \text{Spin}(4,12) = \{ x \in \text{Spin}(16, C) \mid I_{4,12} \tau x = x \}.$$

It is known that the symmetric space $$(E_8^C)^{\tau_1\gamma_1}/((E_8^C)^{\tau_1\gamma_1})^\epsilon$$ is simply connected ([9]). Then we have the following exact sequence.

$$\begin{array}{cccc}
\pi_1(E/E^\epsilon) & \longrightarrow & \pi_0(E^\epsilon) & \longrightarrow \\
 & \longrightarrow & \pi_0(E) & \longrightarrow \\
 & \longrightarrow & 0 & 0
\end{array}$$

where $$E = (E_8^C)^{\tau_1\gamma_1}$$. Then $$E^\epsilon = ((E_8^C)^{\tau_1\gamma_1})^\epsilon$$ is connected. On the other side we see

$$((E_8^C)^{\tau_1\gamma_1})^\epsilon = \varphi(G') = \varphi(G'_1) \cup \varphi(G'_2)$$

and $$\varphi(G'_1) \cap (G'_2) = \phi$$. Hence we have $$G'_2 = \phi$$, $$G' = G'_1 = \text{Spin}(4,12)$$ and

$$((E_8^C)^{\tau_1\gamma_1})^\epsilon = \varphi(G'_1).$$

Since $$\omega \in \text{Spin}(4,12)$$ and $$\text{Ker} \varphi|_{\text{Spin}(4,12)} = \{1, \omega \}$$, we see

$$\varphi(G') = \varphi(G'_1) \cong \text{Spin}(4,12)/\{1, \omega \} = Ss(4,12).$$

Thus we have the following

**Theorem 6.9.** The non-compact exceptional Lie group $$(E_8^C)^{\tau_1\gamma_1}$$ of type $$E_8(-24)$$ has the following subgroup of type $$D_8$$.

$$(E_8^C)^{\tau_1\gamma_1})^\epsilon \cong \text{Ss}(4,12).$$

**§ 6.6. Subgroup Ss*(16) × 2 of $$E_8(8)$$ and $$E_8(-24)$$**

In this subsection, consider subgroups of $$(E_8^C)^{\tau J}$$ and $$(E_8^C)^{\tau dJ}$$;

$$((E_8^C)^{\tau J})^\epsilon = \{ x \in (E_8^C)^{\tau J} \mid \text{ex} = \text{x} \} \subset (E_8^C)^{\tau J} \cong E_8(-24),$$

$$((E_8^C)^{\tau dJ})^\epsilon = \{ x \in (E_8^C)^{\tau dJ} \mid \text{ex} = \text{x} \} \subset (E_8^C)^{\tau dJ} \cong E_8(8).$$
It is clear that
\[
((E_8^C)^{\tau J})^e = (E_8^C)^{\tau J})^e.
\]
Since
\[
((E_8^C)^{\tau J})^e = ((E_8^C)^{\tau J})^e = \{\varphi(x) \mid x \in \text{Spin}(16, C), J\tau\varphi(x)\tau J = \varphi(x)\}
\]
\[
= \{\varphi(x) \mid x \in \text{Spin}(16, C), J\varphi(\tau x)J = \varphi(x)\},
\]
we consider
\[
G'' = \{x \in \text{Spin}(16, C) \mid J\varphi(\tau x)J = \varphi(x)\}.
\]
From Lemmas 6.4 and 6.5, the same as §6.4 we have the following
\[
G'' = G''_1 \cup G''_2,
\]
where \(G''_1 = \{x \in \text{Spin}(16, C) \mid j \cdot \omega_1 \cdot \tau x \cdot \omega_1 \cdot j = x\},\)
\[
G''_2 = \{\beta \in \text{Spin}(16, C) \mid j \cdot \omega_1 \cdot \tau \beta \cdot \omega_2 \cdot j = \beta\}.
\]
The same as Lemma 6.7, we have the following

**Lemma 6.10.** For \(x \in G''_1\) and \(\beta, \beta' \in G''_2\), we have
\[
x \cdot \beta \in G''_2, \quad \beta \cdot \beta' \in G''_2.
\]
From Lemma 6.4, we see
\[
j \cdot \omega_1 \cdot \tau \omega_1 \cdot \omega_2 \cdot j = j \cdot \omega_2 \cdot j = \omega_1 \cdot j^2 = \omega_1,
\]
i.e., \(\omega_1 \in G''_2\). Clearly, \(G''_1 \cap G''_2 = \emptyset\). Then we have
\[
G''_2 = G''_1 \cdot \omega_1 \quad \text{and} \quad G'' = G''_1 \cup G''_2 = G''_1 \times \{1, \omega_1\} = G''_1 \times 2.
\]
For
\[
x = a_1 \cdot a_2 \cdots a_{2m} \in \text{Spin}(16, C), \quad a_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix},
\]
we see
\[
j \cdot \omega_1 \cdot \tau \omega_1 \cdot j = j \cdot \omega_1 \cdot \tau a_1 \cdot \omega_1 \cdot j \cdot j \cdot \omega_1 \cdot \tau a_2 \cdot \omega_1 \cdot j \cdots j \cdot \omega_1 \cdot \tau a_{2m} \cdot \omega_1 \cdot j
\]
and
\[
j \cdot \omega_1 \cdot \tau a_i \cdot \omega_1 \cdot j = J_{8\tau a_i} = \begin{pmatrix} -tb_i \\ \tau a_i \end{pmatrix}.
\]
Then, from §1.1, we see
\[
G''_1 = \text{Spin}^*(16) = \{x \in \text{Spin}(16, C) \mid J_8 \tau x = x\}.
\]
From Lemma 6.2, we have $\varphi(\omega_1) = e_2$. Since $\omega \in Spin^*(16)$ and $\text{Ker} \varphi|_{Spin^*(16)} = \{1, \omega\}$, we see

$$\varphi(G^\prime) \cong Spin^*(16)/\{1, \omega\} \times \{1, \varphi(\omega_1)\} = Ss^*(16) \times \{1, e_2\}.$$  

Thus we have the following

**Theorem 6.11.** (1) The non-compact exceptional Lie group $(E_8^C)^{\tau_J}$ of type $E_8(-24)$ has the following subgroup of type $D_8$.

$$((E_8^C)^{\tau_J})^c \cong Ss^*(16) \times \{1, e_2\} = Ss^*(16) \times 2.$$  

(2) The non-compact exceptional Lie group $(E_8^C)^{\tau_J}$ of type $E_8(8)$ has the following subgroup of type $D_8$.

$$((E_8^C)^{\tau_J})^c \cong Ss^*(16) \times \{1, e_2\} = Ss^*(16) \times 2.$$  

**References**


