CLASSIFICATION OF PROJECTIVE SURFACES AND PROJECTIVE NORMALITY

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§ 0. Introduction

For a very ample line bundle $L$ on a smooth irreducible projective curve $C$ of genus $g$, $\phi_L : C \to P(H^0(L))$ is the projective embedding by the vector space $H^0(L)$. One says that $L$ is normally generated if $\phi_L(C)$ is projectively normal. Equivalently, $L$ is normally generated if and only if the natural maps $S^mH^0(L) \to H^0(L^m)$ are surjective for all $m \geq 0$. Noether's theorem is that the canonical bundle is normally generated unless $C$ is hyperelliptic (see [18]). It is well-known that if $\deg(L) \geq 2g + 1$, then $L$ is normally generated (see [4], [15]). Furthermore several authors have reported on the normal generation of non-special low degree line bundle. Homma have proved that for a nonhyperelliptic curve $C$ of genus 3 every very ample line bundle of degree 6 on $C$ is normally generated ([8]). Lange and Martens have showed that a general line bundle of degree $2g$ on a nonhyperelliptic curve $C$ of genus $g$ is normally generated ([11]). Recently Green and Lazarsfeld have showed the sufficient condition for $L$ to be normally generated (see (1.1) or [6], Theorem 1). In this paper we shall study on the normal generation of special low degree line bundle (i.e. $\deg(L) \leq 2g - 3$). Our result is as follows.

**Theorem 1.** The following very ample line bundles $L$ on a nonsingular projective curve $C$ of genus $g$ is normally generated.

1. $\deg(L) = 2g - 3$ ($g \geq 5$) and $h^1(L) = 1$
2. $\deg(L) = 2g - 4$ ($g \geq 11$) and $h^1(L) = 1$
3. $\deg(L) = 2g - 5$ ($g \geq 9$) and $h^1(L) = 2$
4. $\deg(L) = 2g - 6$ ($g \geq 15$) and $h^1(L) = 2$

Next we shall use the result above for the classification of projective surfaces.

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The projective normality of a nonsingular curve \( C \) is useful tool for calculating some invariants of projective surfaces.

Several authors have classified projective surfaces according to some invariants. Main invariants are \( \Delta \)-genus \( \Delta \) and sectional genus \( g_H \). The classification of nondegenerate surfaces of \( \Delta \)-genus \( \Delta = 0, 1 \) is due to Del Pezzo. The classification of nondegenerate surfaces of \( \Delta = 2, 3 \) is essentially due to Castelnuovo. Recently Ein has classified nondegenerate surfaces of \( \Delta = 4 \) ([3]). On the other hand, the classification of nondegenerate surfaces of \( g_H \leq 3 \) is essentially due to Castelnuovo ([1], [2], or [9]). Roth has classified projective surfaces of \( g_H = 4 \) ([17]). Recently Ein has given a mordern proof to Roth's results ([3]). Furthermore the classification of nondegenerate surfaces may be given for \( \Delta \leq 5 \) or \( g_H \leq 7 \) ([10], [12]). In this paper we shall classify nondegenerate surfaces with large \( \Delta \)-genus and large sectional genus. We shall pay attention to the relation between \( \Delta \) and \( g_H \). If \( \deg(S) \geq 2\Delta + 1 \), then \( g_H \leq \Delta \) (see (2.2), (a)). Therefore our main purpose is to classify nondegenerate surfaces of degree \( \leq 2\Delta \) with \( g_H > \Delta \). In this paper we shall assume that such a surface exists. Our result is as follows.

**Theorem 2.** Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate linearly normal smooth surface with \( \Delta \)-genus \( \Delta \) and sectional genus \( g_H \). Our classification of projective surfaces is as follows.

1. If \( \deg(S) \geq \Delta - 1 \) with \( g_H = (\frac{1}{2})\Delta \geq 1 \) (\( \Delta \) is even), then \( S \) is a scroll over a curve of genus \( g_H \).
2. If \( \deg(S) \geq 2\Delta \) with \( g_H = \Delta \geq 2 \), then \( S \) is a rational surface.
3. If \( \deg(S) = 2\Delta \) with \( g_H \geq \Delta + 1 \geq 4 \), then \( S \) is a K3 surface and \( g_H = \Delta + 1 \).
4. If \( \deg(S) = 2\Delta - 1 \) with \( g_H = \Delta + 1 \geq 5 \), then \( S \) is the projection of a K3 surface from a point in the surface.
5. If \( \deg(S) = 2\Delta - 1 \) with \( g_H = \Delta + 2 \geq 9 \), then \( S \) is a minimal elliptic surface of Kodaira dimension 1.
6. If \( \deg(S) = 2\Delta - 2 \) with \( g_H = \Delta + 1 \geq 11 \), then \( S \) is a K3 surface blown up at one point or two points.
7. If \( \deg(S) = 2\Delta - 2 \) with \( g_H = \Delta + 2 \geq 15 \), then \( S \) is a minimal elliptic surface of Kodaira dimension 1.

The organization of the paper is as follows. First we shall discuss the normal generation of special very ample line bundle on \( C \) in \( \S 1 \). Secondly we shall classify projective surfaces in \( \S 2 \). Lastly we shall discuss the projective normality of projective surfaces of degree \( 2\Delta \) with \( g_H \geq \Delta \geq 4 \) in \( \S 3 \).
Classification of projective surfaces

Notation. We work throughout over the complex numbers.

(1) $C$ is a smooth irreducible projective curve of genus $g \geq 2$ and $L$ is a very ample line bundle on $C$. We denote by $K_C$ (or $\omega_C$) the canonical bundle on $C$.

(2) The Clifford index of a line bundle $A$ on $C$ is defined by
\[
\text{Cliff}(A) = \deg(A) - 2(h^0(A) - 1).
\]

The Clifford index of $C$ is taken to be
\[
\text{Cliff}(C) = \min\{\text{Cliff}(A) : h^0(A) \geq 2, h^1(A) \geq 2\}
\]
(c.f. [13], [14]). We remark that $\text{Cliff}(C) \geq 0$ with equality if and only if $C$ is hyperelliptic by Clifford's theorem, and that $\text{Cliff}(C) = 1$ if and only if $C$ is either trigonal or a smooth plane quintic.

(3) For a divisor $D$ on a nonsingular variety $V$, we denote by $\mathcal{O}_V(D)$ the line bundle associated to $D$. By abuse of notation, we sometimes use $D$ itself instead of $\mathcal{O}_V(D)$. We denote by $h^i(D)$ the dimension of $i$-th cohomology $H^i(V, \mathcal{O}_V(D))$.

(4) $S$ is a nondegenerated smooth projective surface and $H \in |\mathcal{O}_S(1)|$ is a smooth hyperplane section of $S$, and $g_H$ is its genus (sectional genus). We denote by $K_S$ the canonical bundle on $S$. Let $\mathcal{L}$ be an very ample line bundle on $S$. We define a $\Delta$-genus for $S$ and $\mathcal{L}$ by
\[
\Delta = (\mathcal{L})^2 + 2 - h^0(S, \mathcal{L}).
\]

We denote by $p_g = h^0(K_S)$ the geometric genus of $S$ and by $q = h^1(\mathcal{O}_S)$ the irregularity of $S$.

§ 1. Normal generation of very ample line bundles on curves

Our main tool is the following results of [6].

**Lemma 1.1** ([6], Theorem 1). Let $L$ be a very ample line bundle on $C$, with
\[
\deg(L) \geq 2g + 1 - 2 \cdot h^1(L) - \text{Cliff}(C)
\]
(and hence $h^1(L) \leq 1$). Then $L$ is normally generated.

**Lemma 1.2** ([6], Theorem 3 and Remark 1.3). Let $L$ be a very ample line bundle on $C$ with $\deg(L) = 2g + 1 - k$. Assume that $2k + 4e + 1 \leq g$ and $e \geq -1$, and consider the embedding $C \subseteq P(H^0(L)) = P'$ defined by $L$. Then $L$ fails to be normally generated if and only if there exists an integer $1 \leq n \leq r - 2 - e - h^1(L)$, and an effective divisor $D$ on $C$ of degree at least $2n + 2$ such that

(a) $H^1(C, L^2(-D)) = 0$ and
(b) \( D \) spans an \( n \)-plane \( \Lambda \subseteq \mathbb{P}^r \) in which \( D \) fails to impose independent conditions on quadrics.

\textbf{Remark 1.3.} The conclusion of the theorem in [6] holds with \( 1 \leq n \leq r - 2 - e \). \( \text{(see [6], page 80, line 8.}) \) But since we have

\[ k + 2e - 2r(L(-D)) \leq \text{Cliff}(L(-D)) \leq \text{Cliff}(C) \leq k - 1 - 2h^1(L), \]

it follows that \( h^0(L(-D)) \geq e + 2 + h^1(L) \). \( \text{(c.f. [6], page 80, line 12.)} \) Therefore we can obtain somewhat stronger bound that \( 1 \leq n \leq r - 2 - e - h^1(L) \) in (1.2).

The following two lemmas are well-known.

\textbf{Lemma 1.4.} Let \( L \) be a very ample line bundle on a hyperelliptic curve of genus \( g \geq 2 \). Then \( L \) is nonspecial \( \text{(i.e. } h^1(L) = 0) \).

\textbf{Lemma 1.5.} Let \( L \) be a very ample line bundle of degree \( 2g - 4 \) on an elliptic-hyperelliptic curve of genus \( g \geq 4 \). Then \( L \) is nonspecial \( \text{(i.e. } h^1(L) = 0) \).

\textbf{Lemma 1.6.} Let \( L \) be a very ample line bundle of degree \( 2g - 2 - k \) \( (k \geq 1) \) with \( h^1(L) = 1 \) on \( C \) of genus \( g \geq 5 \). Then \( C \) is not trigonal.

\textbf{Proof.} Since \( L \) is special, we can write \( L = K - D \) for some effective divisor of degree \( k \). By the very ampleness of \( L \), we have

\[ \dim |K - Q - R| = \dim |K - D| - 2 \]

for all \( Q, R \) in \( C \). Using Riemann-Roch, this say that

\[ \dim |D + Q + R| = \dim |D| = h^1(L) - 1 = 0 \]

for all \( Q, R \) in \( C \). Since \( \dim |P + Q + R| \leq \dim |D + Q + R| \) for some point \( P \) in \( C \), we have \( \dim |P + Q + R| = 0 \). Assume that \( C \) is trigonal. Then for any given \( P \), there exist \( Q, R \) such that \( \dim |P + Q + R| = 1 \). It is a contradiction. \( \square \)

\textbf{Proposition 1.7.} Let \( L \) be a very ample line bundle of degree \( 2g - 3 \) with \( h^1(L) = 1 \) on \( C \) of genus \( g \geq 5 \). Then \( L \) is normally generated.

\textbf{Proof.} By virtue of (1.6), we have \( \text{Cliff}(C) \geq 2 \). Hence we get the inequation:

\[ 2g + 1 - 2 \cdot h^1(L) - \text{Cliff}(C) \leq 2g - 3. \]

Using (1.1), we can prove the Proposition. \( \square \)
Proposition 1.8. Let $L$ be a very ample line bundle of degree $2g - 4$ with $h^1(L) = 1$ on $C$ of genus $g \geq 11$. Then $L$ is normally generated.

Proof. It is immediate from (1.1) that if $\text{Cliff}(C) \geq 3$, then $L$ is normally generated. On the other hand, we have $\text{Cliff}(C) \geq 2$ by virtue of (1.6). Therefore it remains only to prove in the case of $\text{Cliff}(C) = 2$. Assume that $L$ is not normally generated. If $e = 0$ and $k = 5$, then $2k + 4e + 1 = 11 \leq g$. Hence Lemma 1.1 gives the existence of an integer

(*) \[ 1 \leq n \leq r(L) - 2 - e - h^1(L) = g - 6, \]

and an effective divisor $D$ on $C$ of degree $\geq 2n + 2$ which spans an $n$-plane in $P(H^0(L))$. Since $n = r(L) - h^0(L(-D))$, we have

\[ \text{Cliff}(L(-D)) \leq \text{Cliff}(L) = \text{Cliff}(C) = 2. \]

Moreover $h^0(L(-D)) \geq 2$ and $h^1(L(-D)) \geq 2$ thanks to (*). In view of the definition of $\text{Cliff}(C)$ we get $\text{Cliff}(L(-D)) = 2$ and $\deg(D) = 2n + 2$. Hence we have $6 \leq \deg(L(-D)) \leq 2g - 8$ and $6 \leq \deg(K - (L - D)) \leq 2g - 8$. By using ([14], Beispiel 8), $C$ must be an elliptic-hyperelliptic curve. This contradicts with (1.5).

Proposition 1.9. Let $L$ be a very ample line bundle of degree $2g - 5$ with $h^1(L) = 2$ on $C$ of genus $g \geq 9$. Then $L$ is normally generated.

Proof. Since $h^1(L) = 2$ and $h^0(L) \geq 2$, we have $\text{Cliff}(C) \leq \text{Cliff}(L) = 1$. Moreover $\text{Cliff}(C) \neq 0$ by (1.4), and consequently $\text{Cliff}(C) = 1$. Assume that $L$ is not normally generated. If $e = -1$ and $k = 6$, then $2k + 4e + 1 = 9 \leq g$. So thanks to (1.1), there exists an effective divisor $D$ on $C$ of degree $\geq 2n + 2$ ($1 \leq n \leq g - 6$) such that

\[ \text{Cliff}(L(-D)) \leq \text{Cliff}(L) = \text{Cliff}(C) = 1. \]

From the definition of $\text{Cliff}(C)$ we get $\text{Cliff}(L(-D)) = 1$, $\deg D = 2n + 2$,

(#) \[ 5 \leq \deg(L(-D)) \leq 2g - 9, \quad \text{and} \quad 7 \leq \deg(K - (L - D)) \leq 2g - 7. \]

On the other hand, by using ([13], 2.51) we have $h^0(L(-D)) = 2$ or $h^0(K - (L - D)) = 2$. Since $\text{Cliff}(L(-D)) = 1$, we get $\deg(L(-D)) = 3$ or $\deg(K - (L - D)) = 3$. This contradicts with (#).
PROPOSITION 1.10. Let $L$ be a very ample line bundle of degree $2g - 6$ with $h^1(L) = 2$ on an elliptic-hyperelliptic curve of genus $g \geq 15$. Then $L$ is normally generated.

PROOF. First we claim that $\text{Cliff}(C) = 2$ because $C$ is an elliptic-hyperelliptic curve of genus $g \geq 15$. Assume that $L$ is not normally generated. If $e = 0$ and $k = 7$, then $2k + 4e + 1 = 15 \leq g$. By applying (1.2) there exist an integer

$$1 \leq n \leq r(L) - 2 - e - h^1(L) = g - 8$$

and an effective divisor $D$ on $C$ of degree $\geq 2n + 2$ such that

$$\text{Cliff}(L(-D)) \leq \text{Cliff}(L) = \text{Cliff}(C) = 2.$$ 

By the definition of $\text{Cliff}(C)$ we must have $\deg(D) = 2n + 2$. If we consider $B = K - (L - D)$, then $\deg(B) = 2n + 6$ and $r(B) = n + 2$. Moreover $\text{Cliff}(B) = \text{Cliff}(C)$, and $B$ is generated by its global section.

Consider the map $\phi_B : C \to \mathbb{P}^{n+2}$. Assume that $\phi_B$ is birational onto its image. Since $g \geq 15$, we get $r(B) \geq g - 5$ i.e. $n \geq g - 7$ by ([6], (2.3)). This contradicts with $(\diamond)$. Therefore $\phi_B$ factors through a branched covering $\pi : C \to Y$ of degree $m \geq 2$, where $Y$ is a smooth curve mapped birationally onto its image in $\mathbb{P}^{n+2}$ by a line bundle $B_0$ with $r(B_0) = r(B) = n + 2$, and $B = \pi^*(B_0)$:

\[ C \xrightarrow{\pi} Y \xrightarrow{\phi_B} \mathbb{P}^{n+2}. \]

If $m \geq 3$, then $\text{Cliff}(\pi^*(B_0(-y))) \leq \text{Cliff}(B) = \text{Cliff}(C)$ for any $y \in Y$. Since $h^0(\pi^*(B_0(-y))) \geq 2$ and $h^1(\pi^*(B_0(-y))) \geq 2$, it is impossible. Hence $m = 2$ and $\deg(B_0) = n + 3$. Furthermore $B_0$ embeds $Y$ in $\mathbb{P}^{n+2}$ as an elliptic normal curve. Since $h^0(B(-D)) = h^1(L) = 2$, there is an effective divisor $D_0$ (on $Y$) which spans $n$-plane in $\mathbb{P}^{n+2}$ and $D \leq \pi^*(D_0)$. We claim that an elliptic normal curve $Y \subset \mathbb{P}^{n+2}$ has no $(n + 2)$-secant $n$-planes. Hence we have $\deg(D_0) = n + 1$ and $D = \pi^*(D_0)$. Since $L = K + \pi^*(D_0 - B_0)$, we have

$$h^1(L(-\pi^*y)) = h^0(\pi^*(B_0 - D_0 + y)) = h^0(B_0 - D_0) + 1$$

for any $y \in Y$ and $h^1(L) = h^0(B_0 - D_0)$. So $h^1(L(-\pi^*y)) = h^1(L) + 1$. This means that $L$ is not very ample. It is a contradiction. $\square$
Proposition 1.11. Let \( L \) be a very ample line bundle of degree \( 2g - 6 \) with \( h^1(L) = 2 \) on \( C \) of genus \( g \geq 15 \). Then \( L \) is normally generated.

Proof. Thanks to (1.10), we may assume that \( C \) is not an elliptic-hyperelliptic curve.

Assume that \( L \) is not normally generated. If \( e = 0 \) and \( k = 7 \), then \( 2k + 4e + 1 = 15 \leq g \). By virtue of (1.2) there exists an effective divisor \( D \) on \( C \) of degree \( \geq 2n + 2 \) \( (1 \leq n \leq g - 8) \) such that

\[
\text{Cliff}(L(-D)) \leq \text{Cliff}(L) = 2.
\]

If \( \text{Cliff}(L(-D)) = 0 \), then \( C \) must be a hyperelliptic curve. This contradicts with (1.4). If \( \text{Cliff}(L(-D)) = 1 \), then we have \( \text{deg}(D) = 2n + 3 \),

\[
(\clubsuit) \quad 7 \leq \text{deg}(L(-D)) \leq 2g - 11, \quad \text{and} \quad 9 \leq \text{deg}(K - (L - D)) \leq 2g - 9.
\]

On the other hand, as in the proof of (1.2) we get \( \text{deg}(L(-D)) = 3 \) or \( \text{deg}(K - (L - D)) = 3 \) by ([13], 2.51). This contradicts with \((\clubsuit)\). If \( \text{Cliff}(L(-D)) = 2 \), then we have \( \text{deg}(D) = 2n + 2 \),

\[
(\heartsuit) \quad 8 \leq \text{deg}(L(-D)) \leq 2g - 10, \quad \text{and} \quad 8 \leq \text{deg}(K - (L - D)) \leq 2g - 10.
\]

We claim that \( \text{Cliff}(C) \geq 1 \) by (1.4). Assume that \( \text{Cliff}(C) = 1 \). From \((\heartsuit)\) there exists the line bundle of degree \( \geq 8 \) such that Clifford index is 2. But it is a contradiction by the proof of ([13], 2.57). Next we assume that \( \text{Cliff}(C) = 2 \). Applying ([14], Beispiel 8), \( C \) must be an elliptic-hyperelliptic curve. This is in contradiction with the assumption. \( \square \)

§ 2 Classification of projective surfaces

Lemma 2.1. Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate linearly normal smooth surface, and let \( H \in |\mathcal{O}_S(1)| \) be a nonsingular hyperplane section of \( S \). Then the following holds:

(a) \( h^0(\mathcal{O}_H(1)) \geq n \).

(b) If \( \mathcal{O}_H(1) \) is special, then \( h^0(\mathcal{O}_H(1)) \leq (\frac{1}{2}) \text{deg}(S) + 1 \).

(c) If \( \mathcal{O}_H(1) \) is normally generated and \( H^1(\mathcal{O}_H(2)) = 0 \), then \( q = h^1(\mathcal{O}_s) = 0 \) and \( p_g = h^2(\mathcal{O}_s) = h^1(\mathcal{O}_H(1)) \).

Proof. (a) There is a long exact sequence

\[
0 \rightarrow H^0(\mathcal{O}_s) \rightarrow H^0(\mathcal{O}_s(1)) \rightarrow H^0(\mathcal{O}_H(1)) \rightarrow H^1(\mathcal{O}_s) \rightarrow H^1(\mathcal{O}_s(1)).
\]

So \( h^0(\mathcal{O}_H(1)) \geq n \).
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(b) That inequality comes from Clifford's theorem ([7], IV, 5.4).
(c) Consider the exact sequences

\[ 0 \rightarrow H^0(\mathcal{O}_s(t-1)) \rightarrow H^0(\mathcal{O}_s(t)) \rightarrow H^0(\mathcal{O}_H(t)) \rightarrow 0, \quad \text{for } t \geq 1. \]

Since \( \mathcal{O}_H(1) \) is normally generated, the natural restriction map \( H^0(\mathcal{O}_s(t)) \rightarrow H^0(\mathcal{O}_H(t)) \) is surjective for all \( t \geq 1 \). Thus \( H^1(\mathcal{O}_s(t-1)) \cong H^1(\mathcal{O}_s(t)) = 0 \) for all \( t \geq 2 \) by Serre's vanishing theorem. Furthermore \( q = h^1(\mathcal{O}_s) = 0 \) since \( h^1(\mathcal{O}_s) \leq h^1(\mathcal{O}_s(1)) \).

We remark that \( H^2(\mathcal{O}_s(t)) \cong H^0(\mathcal{O}_s(K_s-tH)) = 0 \) for a large enough \( t \). Since \( H^1(\mathcal{O}_H(t)) = 0 \) for all \( t \geq 2 \), we have \( H^2(\mathcal{O}_s(t-1)) \cong H^2(\mathcal{O}_s(t)) = 0 \) for all \( t \geq 2 \).

Moreover \( \deg(S) \geq 2\Delta + 1 \) by the lemma above. Theorem 2.3. Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate linearly normal smooth surface of degree \( 2\Delta \) with \( g_H \geq \Delta + 1 \geq 4 \). Then \( S \) is a K3 surface and \( g_H = \Delta + 1 \).

Proof. We have \( h^0(\mathcal{O}_H(1)) \geq n \) by (2.1.(a)). If \( \deg(\mathcal{O}_H(1)) \geq 2\Delta + 1 \), we have \( \deg(\mathcal{O}_H(1)) < 2(h^0(\mathcal{O}_H(1)) - 1) \). Using Clifford's theorem ([7], IV, 5.4), we get \( h^1(\mathcal{O}_H(1)) = 0 \). Therefore \( \chi(\mathcal{O}_H(1)) = h^0(\mathcal{O}_H(1)) = \deg(\mathcal{O}_H(1)) + 1 - g_H \geq n \), i.e. \( g_H \leq \Delta \).

(b) Since \( h^1(\mathcal{O}_H(1)) = 0 \), we have \( h^1(\mathcal{O}_s(1)) = \chi(\mathcal{O}_H(1)) = n + \Delta - g_H \leq h^0(\mathcal{O}_s(1)) + h^1(\mathcal{O}_s) - h^0(\mathcal{O}_s) \leq n + h^1(\mathcal{O}_s) \leq n + g_H \). So \( g_H \geq \frac{1}{2}\Delta \) and \( h^1(\mathcal{O}_s) \geq \Delta - g_H \).

First we treat the case of \( g_H > \Delta \) (and hence \( \deg(S) \leq 2\Delta \) by the lemma above).

In (2.3) the case of \( \Delta = 3,4 \) is well-known (see [3], [9]).
Classification of projective surfaces

**Theorem 2.4.** Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate linearly normal smooth surface of degree \( 2\Delta - 1 \) with \( g_H = \Delta + 1 \geq 5 \). Then \( S \) is the projection of a K3 surface from a point in the surface.

**Proof.** As in the proof of (2.3) we get \( h^0(\mathcal{O}_H(1)) = n = \Delta \) and \( h^1(\mathcal{O}_H(1)) = 1 \). Hence \( \mathcal{O}_H(1) \) is normally generated by virtue of (1.7). Therefore, by applying (2.1, (c)), we get \( p_g = 1 \) and \( q = 0 \). Since \( H.K_s = 1 \) by adjunction formula, \( [K_s] \) is the line, and \( (K_s)^2 = -1 \). Let

\[
\rho : S \to \mathbb{P}(H^0(\mathcal{O}_s(H + K_s)))
\]

be the adjunction mapping given by \( |H + K_s| \) and let \( Y \) be \( \rho(S) \). We have that \( (H + K_s)^2 = 2g_H - 2 \) and \( h^0(\mathcal{O}_s(H + K_s)) = g_H + 1 \). Furthermore \( K_s = \rho^*(K_Y) + K_s \), so \( K_Y = 0 \). We denote the stein factorization of \( \rho \) by

\[
S \xrightarrow{f} S' \xrightarrow{h} Y \subseteq \mathbb{P}(H^0(\mathcal{O}_s(H + K_s))),
\]

where \( S' \) is a K3 surface and \( h \) is an isomorphism by ([20], (2.4)). So \( S \) is the projection of a degree \( 2g_H - 2 \) K3 surface from a point in the surface. \( \square \)

In (2.4) the case of \( \Delta = 4 \) is known (see [3], [9]).

**Theorem 2.5.** Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate linearly normal smooth surface of degree \( 2\Delta - 1 \) with \( g_H = \Delta + 2 \geq 9 \). Then \( S \) is a minimal elliptic surface of Kodaira dimension 1.

**Proof.** As in the proof of (2.3), we have \( h^0(\mathcal{O}_H(1)) = n = \Delta \) and \( h^1(\mathcal{O}_H(1)) = 2 \). So \( \mathcal{O}_H(1) \) is normally generated by (1.9). Hence we obtain \( p_g = 2 \) and \( q = 0 \) by (2.1, (c)). We claim that \( |K_s| \) is without fixed components. Let \( C \) be a member in the variable part of \( |K_s| \). Since \( p_g = 2 \), we have \( C^2 \geq 0 \) ([5], p. 536), and \( C.K_s \geq 0 \). From adjunction formula we have \( H.K_s = 3 \). If \( |K_s| \) has fixed components, \( C.H \leq 2 \). But it is impossible by genus formula for \( C \). Hence \( (K_s)^2 = 0 \) and the virtual genus of \( K_s \) \( g(K_s) = 1 \). So \( |K_s| \) is base point free and \( S \) is minimal. \( \square \)

**Theorem 2.6.** Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate linearly normal smooth surface of degree \( 2\Delta - 2 \) with \( g_H = \Delta + 1 \geq 11 \). Then \( S \) is a K3 surface blown up at one point or two points.
Proof. Using (2.1, (a), (b)) we have $h^0(\mathcal{O}_H(1)) = g_H - 1$ or $g_H - 2$. If $h^0(\mathcal{O}_H(1)) = g_H - 1$, then $\text{Cliff}(\mathcal{O}_H(1)) = 0$ and $H$ is a hyperelliptic curve. This contradicts with (1.4). Hence $h^0(\mathcal{O}_H(1)) = g_H - 2$ and $h^1(\mathcal{O}_H(1)) = 1$. Therefore $\mathcal{O}_H(1)$ is normally generated by (1.8). By applying (2.1(c)) we get $p_g = 1$ and $q = 0$. From adjunction formula we have $H.K_s = 2$. Let $D$ be an effective number of $|K_s|$. Since $D^2 \geq -2$ by (12), (0.6), (iii), $D$ is reduced. If $D$ is irreducible, then $D^2 = -1$ and $S$ is a K3 surface at one point. If $D$ is reducible, then $D^2 = 2$ and $(H + K_s)^2 = 2g_H - 2$. We denote the adjunction map by $p$. Let

$$S \rightarrow S^+ \xrightarrow{h} Y \in \mathbb{P}(H^0(\mathcal{O}_s(H + K_s)))$$

be the Stein factorization of $p$, where $h$ is isomorphism by ([20], 2.4). By using ([12], (0.6), (ii), (iv)) we have that $K_s$ is trivial and $S$ is a K3 surface blown up at two points.

**Theorem 2.7.** Let $S \subseteq \mathbb{P}^n$ be a nondegenerate linearly normal smooth surface of degree $2\Delta - 2$ with $g_H = \Delta + 2 \geq 15$. Then $S$ is a minimal elliptic surface of Kodaira dimension 1.

Proof. As in the proof of (2.6), we have $h^0(\mathcal{O}_H(1)) = g_H - 3$ and $h^1(\mathcal{O}_H(1)) = 2$. So $\mathcal{O}_H(1)$ is normally generated by (1.11). By applying (2.1(c)) we get $p_g = 2$ and $q = 0$. As in the proof of (2.5), $|K_s|$ is base point free pencil. Therefore $S$ is a minimal elliptic surface of Kodaira dimension 1.

Next we treat of the case of $g_H = \Delta$.

For next lemma we shall use the following convention. Let $E$ be a normalized rank 2 vector bundle on a nonsingular curve $C$. Let $S = \mathbb{P}(E)$, and let $C_0$ be the section determined by the natural embedding $\mathbb{P}(\wedge^2 E) \rightarrow \mathbb{P}(E)$. Then $C_0 \in |\mathcal{O}_s(1)|$, and $(C_0)^2 = \deg(E)$.

**Lemma 2.8.** Let $\pi : S \cong \mathbb{P}(E) \rightarrow C$ be a scroll mapping, with $E$ a normalized rank 2 vector bundle on a hyperelliptic curve $C$ of genus $g$. Suppose $\mathcal{O}_s(C_0) \otimes \pi^*D$ is very ample, and the linear system $|\mathcal{O}_s(C_0) \otimes \pi^*D|$ embeds $S$ in $\mathbb{P}^n$. Then $\Delta - 2 \geq g$ $(\geq 2)$.

Proof. Since $\mathcal{O}_s(C_0) \otimes \pi^*D|_{C_0} = \wedge^2 E \otimes D$ is very ample line bundle on hyperelliptic curve $C$, $h^1(\wedge^2 E \otimes D) = 0$ by (1.4). Moreover $\deg(\wedge^2 E \otimes D) =$
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deg(E) + deg(D) ≥ g + 3 by Halphen's theorem ([7], IV, 6.1). By Nagata's theorem ([16], Theorem 2), we have deg(E) ≤ g. So deg(D) ≥ 3. Since h₁(D) ≤ g - (1/2)deg(D) by Clifford's theorem ([7], IV, 5.4) and the Riemann Roch theorem, we get h₁(D) ≤ g - 2. Consider the following exact sequence:

\[ 0 \to D \to E \otimes D \to \wedge^2 E \otimes D \to 0. \]

From the exact sequence, we have h₁(D) ≥ h₁(E ⊗ D), and \( \chi(E \otimes D) = \deg(S) + 2(1 - g) \) i.e. \( \Delta = 2g - h₁(E \otimes D) ≥ 2g - h₁(D). \) Therefore \( \Delta ≥ 2g - (g - 2) = g + 2. \)

**Theorem 2.9.** Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate linearly normal smooth surface of degree \( \geq 2\Delta \) with \( h_{\mathcal{H}} = \Delta \geq 2. \) Then \( S \) is a rational surface.

**Proof.** Let \( H \in |\mathcal{O}_S(1)| \) be a nonsingular hyperplane section of \( S. \) By adjunction formula we have \( H.K_S < 0 \) under our condition. So the Kodaira dimension is \( -\infty. \) Hence we have only to show that \( q = h₁(\mathcal{O}_S) = 0. \) If \( \deg(S) \geq 2\Delta + 1, \mathcal{O}_H(1) \) is normally generated by ([15], Theorem 6 or [4]). By applying (2.1(c)) we get \( q = 0. \) If \( \deg(S) = 2\Delta, \) \( H \) is not hyperelliptic curve by ((2.8) and [19], Theorem (5.10)). Therefore \( \mathcal{O}_H(1) \) is normally generated by (1.1). So \( q = 0 \) by same way.

Finally we consider the case of \( g_H < \Delta. \) If \( h₁(\mathcal{O}_H(1)) = 0, \) we may assume that \( g_H \geq (1/2)\Delta \) by (2.2, (b)).

**Theorem 2.10.** Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate linearly normal smooth surface of degree \( \geq \Delta - 1 \) with \( g_H = (1/2)\Delta \geq 1 \) (\( \Delta \) is even). Then \( S \) is a scroll over a curve of genus \( g_H. \)

**Proof.** Since \( \deg(\mathcal{O}_H(1)) \geq 2g_H - 1, \) \( h₁(\mathcal{O}_H(1)) = 0. \) So \( h₁(\mathcal{O}_S) \geq \Delta - g_H \) by (2.2, (b)). Since \( g_H \geq h₁(\mathcal{O}_S), \) we have \( h₁(\mathcal{O}_S) = g_H. \) Therefore \( S \) is a scroll over a curve of genus \( g_H \) by ([20], (1.52)).

§ 3. Projective normality of projective surfaces

**Lemma 3.1.** Let \( S \subseteq \mathbb{P}^n \) be a nondegenerate surface, and let \( H \in |\mathcal{O}_S(1)| \) be a smooth hyperplane section of \( S. \) If \( H \subseteq \mathbb{P}^{n-1} \) is projectively normal, then \( S \subseteq \mathbb{P}^n \) is projectively normal as well.
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**Proof.** We consider the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & H^0(P^n, \mathcal{O}(k-1)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^0(S, \mathcal{O}_S(k-1))
\end{array}
\]

It is easy to prove the lemma by induction on \( k \).

**Theorem 3.2.** Let \( S \subset P^n \) be a nondegenerate linearly normal smooth surface of \( \Delta \)-genus \( \Delta \). Let \( H \in |\mathcal{O}_S(1)| \) be a smooth hyperplane section of \( S \), and let \( g_H \) be its genus.

If \( \deg(S) = 2\Delta \) and \( g_H \geq \Delta \geq 4 \), then \( S \) is projective normal.

**Proof.** We can prove the theorem by combining the proofs of (2.3), (2.9) and (3.1).

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