ON CURVATURE PROPERTIES OF CERTAIN GENERALIZED ROBERTSON-WALKER SPACETIMES

Dedicated to the memory of Professor Dr. Georgii Ionovich Kruchkovich

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1. Introduction

The warped product $\tilde{M} \times_F N$, of a 1-dimensional manifold $(\tilde{M}, \tilde{g}), \tilde{g}_{11} = -1$, with a warping function $F$ and a 3-dimensional Riemannian manifold $(N, \tilde{g})$ is said to be a generalized Robertson-Walker spacetime (cf. [2], [32]). In particular, when the manifold $(N, \tilde{g})$ is a Riemannian space of constant curvature, the warped product $\tilde{M} \times_F N$ is called a Robertson-Walker spacetime. In [11] it was shown that at every point of a generalized Robertson-Walker spacetime $\tilde{M} \times_F N$ the following condition is satisfied:

\[ (*) \quad \text{the tensors } R \cdot R - Q(S, R) \text{ and } Q(g, C) \text{ are linearly dependent.} \]

This condition is equivalent to the relation

\[ R \cdot R - Q(S, R) = L_1 Q(g, C) \quad (1) \]

on the set $\mathcal{U}_C$ consisting of all points of the manifold $\tilde{M} \times_F N$ at which its Weyl tensor $C$ is non-zero, where $L_1$ is a certain function on $\mathcal{U}_C$. For precise definitions of the symbols used, we refer to the Sections 2 and 3. $(*)_1$ is a curvature condition of pseudosymmetry type. In this paper we will investigate generalized Robertson-Walker spacetimes realizing a condition of pseudosymmetry type introduced in [25]. Namely, semi-Riemannian manifolds $(M, g), n \geq 4$, fulfilling at every point of $M$ the following condition

\[ (*) \quad \text{the tensors } R \cdot C \text{ and } Q(S, C) \text{ are linearly dependent.} \]

were considered in [25]. This condition is equivalent to the relation

\[ R \cdot C = L Q(S, C) \quad (2) \]
on the set \( \mathcal{U} = \{ x \in M | Q(S, C) \neq 0 \text{ at } x \} \), where \( L \) is a certain function on \( \mathcal{U} \). We note that every semisymmetric manifold \((R \cdot R = 0)\) as well as every Weyl-semisymmetric manifold \((R \cdot C = 0)\) realizes \((*)\) trivially (see [25]). There exist also non-semisymmetric and non Weyl-semisymmetric manifolds realizing \((*)\) ([25]). We mention that warped products realizing curvature conditions of pseudosymmetry type were studied in: [7], [8], [9], [11], [13], [14], [15], [16], [17], [19], [20], [21], [24], [26], [28] and [29].

In Section 2 we present a review of the family of curvature conditions of pseudosymmetry type. In the next section we give results on warped products which we apply in the last two sections. In Section 4 we find necessary and sufficient conditions for a warped product to be a manifold satisfying (2). Finally, in Section 5 we present our main results.

Let \((M, g)\) be a semi-Riemannian manifold satisfying \((*)\). We denote by \(\mathcal{U}_L\) the set of all points of the set \(\mathcal{U} \subset M\) at which the function \(L\) is non-zero. It is clear that the tensors \(R \cdot C\) and \(Q(S, C)\) are non-zero at every point of the set \(\mathcal{U}_L\). Moreover, let \((M, g)\) be a 4-dimensional warped product \(\bar{M} \times_F N\), \(\dim \bar{M} = 1\). We denote by \(\mathcal{U}_F\) the subset of \(\mathcal{U}_L\) consisting of all non-critical points of \(F\). Our main result states (see Theorem 5.1) that if the 4-dimensional warped product \(\bar{M} \times_F N\), \(\dim \bar{M} = 1\), satisfies \((*)\) and the set \(\mathcal{U}_F\) is a dense subset of \(\mathcal{U}_L\) then the open submanifold \(U_L\) of the manifold \(\bar{M} \times_F N\) is a pseudosymmetric warped product of the 1-dimensional manifold, with the function \(F\), defined by \(F(x^1) = a \exp(bx^1), \quad a = \text{const.} > 0, \quad b = \text{const.} \neq 0\), and a 3-dimensional semi-Riemannian manifold such that its Ricci tensor is of rank one and its scalar curvature vanishes identically. From this statement it follows immediately (see Corollary 5.1) that if a generalized Robertson-Walker spacetime \(\bar{M} \times_F N\) realizes above assumptions then at every point of \(\bar{M} \times_F N\) at least one of the tensors \(R \cdot C\) or \(Q(S, C)\) must vanish. Finally, using this fact we prove (see Theorem 5.2) that every Robertson-Walker spacetime satisfying \((*)\) is a pseudosymmetric manifold.

2. Curvature Conditions of Pseudosymmetry Type

Let \((M, g)\) be a connected \(n\)-dimensional, semi-Riemannian manifold of class \(C^\infty\) and let \(\nabla\) be its Levi-Civita connection. We define on \(M\) the endomorphisms \(X \wedge Y, \mathcal{R}(X, Y)\) and \(\mathcal{C}(X, Y)\) by

\[
(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y, \quad \mathcal{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,
\]

\[
\mathcal{C}(X, Y) = \mathcal{R}(X, Y) - \frac{1}{n-2} \left( X \wedge \mathcal{H}Y + \mathcal{H}X \wedge Y - \frac{\kappa}{n-1} X \wedge Y \right),
\]
On curvature properties of certain
respectively, where \( X, Y, Z \in \mathfrak{X}(M), \mathfrak{X}(M) \) being the Lie algebra of vector fields of \( M \). The Ricci operator \( \mathcal{R} \) is defined by \( S(X, Y) = g(X, \mathcal{R}Y) \), where \( S \) is the Ricci tensor and \( \kappa \) the scalar curvature of \((M, g)\), respectively. Next, we define the tensors \( U, G \), the Riemann-Christoffel curvature tensor \( R \) and the Weyl conformal tensor \( C \) of \((M, g)\), by
\[
U(X_1, X_2, X_3, X_4) = g(X_1, X_4)S(X_2, X_3) + g(X_2, X_3)S(X_1, X_4) \\
- g(X_1, X_3)S(X_2, X_4) - g(X_2, X_4)S(X_1, X_3),
\]
\[
G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4),
\]
\[
R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4),
\]
\[
C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4),
\]
respectively. Now we can present the Weyl tensor \( C \) in the following form
\[
C = R - \frac{1}{n-2} U + \frac{\kappa}{(n-2)(n-1)} G. \tag{3}
\]
For a \((0,k)\)-tensor field \( T, k \geq 1 \), we define the \((0,k+2)\)-tensors \( R \cdot T \) and \( Q(g, T) \) by
\[
(R \cdot T)(X_1, \ldots, X_k; X, Y) = (\mathcal{R}(X, Y) \cdot T)(X_1, \ldots, X_k) \\
- T(\mathcal{R}(X, Y)X_1, X_2, \ldots, X_k) \\
- \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X, Y)X_k),
\]
\[
Q(g, T)(X_1, \ldots, X_k; X, Y) = ((X \wedge Y) \cdot T)(X_1, \ldots, X_k) \\
= - T((X \wedge Y)X_1, X_2, \ldots, X_k) \\
- \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge Y)X_k).
\]
Putting in the last formulas \( T = R, T = S \) or \( T = C \), we obtain the tensors \( R \cdot R, R \cdot S, R \cdot C, Q(g, R), Q(g, S) \) and \( Q(g, C) \), respectively. The tensor \( C \cdot C \) we define in the same way as the tensor \( R \cdot R \).
Let \((M, g)\) be a Riemannian manifold covered by a system of charts \( \{ U; x' \} \). We denote by \( g_{rs}, \Gamma^r_{st}, R_{rstu}, S_{st}, G_{rstu} = g_{ru}g_{st} - g_{rt}g_{su} \) and
\[
C_{rstu} = R_{rstu} - \frac{1}{n-2} (g_{ru}S_{st} - g_{rt}S_{su} + g_{st}S_{ru} - g_{su}S_{rt}) + \frac{\kappa}{(n-2)(n-1)} G_{rstu},
\]
the local components of the metric $g$, the Levi-Civita connection $\nabla$, the Riemann-Christoffel curvature tensor $\mathbf{R}$, the Ricci tensor $\mathbf{S}$, the tensor $\mathbf{G}$, and the Weyl conformal curvature tensor $\mathbf{C}$ of $(M, g)$, respectively, where $p, g, r, s, t, u, v, w \in \{1, 2, \ldots, n\}$. The local components of the tensors $\mathbf{R} \cdot \mathbf{R}$ and $Q(g, R)$ are given by the following formulas

$$(R \cdot R)_{rstuvw} = \nabla_w \nabla_v R_{rstu} - \nabla_v \nabla_w R_{rstu}$$

$$= g^{pq} (R_{pstu} R_{qrvw} - R_{prt} R_{qsvu} - R_{pqrst} R_{quvw}),$$

$$Q(g, R)_{rstuvw} = g_{rq} R_{rstw} + g_{sv} R_{rstw} + g_{tv} R_{rstw} + g_{uw} R_{rstw} - g_{rw} R_{rstw} - g_{sw} R_{rstw} - g_{uw} R_{rstw}.$$

A semi-Riemannian manifold $(M, g)$, $n \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{\kappa}{n} g$$

holds on $M$. According to [4] (p. 432), (4) is called the Einstein metric condition. Einstein manifolds form a natural subclass of various classes of semi-Riemannian manifolds determined by a curvature condition imposed on their Ricci tensor ([4], Table, pp. 432–433). For instance, every Einstein manifold belongs to the class of semi-Riemannian manifolds $(M, g)$ realizing the following relation

$$\nabla \left( S - \frac{\kappa}{2(n-1)} g \right)(X, Y; Z) = \nabla \left( S - \frac{\kappa}{2(n-1)} g \right)(X, Z; Y),$$

which means that $S - (\kappa/(2(n-1)))g$ is a Codazzi tensor on $M$. Manifolds of dimension $\geq 4$ fulfilling (5) are called manifolds with harmonic Weyl tensor ([4], p. 440). It is known that every warped product $S^1 \times_F M$ of the sphere $S^1$, with a positive smooth function $F$, and an Einstein manifold $(M, g)$, $\dim M \geq 2$, realizes (5) ([4], p. 433). Such warped product is a non-Einstein manifold, in general. We say that (5) is a generalized Einstein metric condition ([4], chapter XVI). On the other hand, such warped product realizes a condition of pseudosymmetry type too. Namely, the warped product $S^1 \times_F M$ of the sphere $S^1$, with a positive smooth function $F$, and an Einstein manifold $(M, g)$, $\dim M \geq 2$, is a Ricci-pseudosymmetric manifold ([24], Corollary 3.2). Thus, in particular, the warped product $S^1 \times_F \mathbb{C}P^n$ of $S^1$, with a positive smooth function $\mathcal{F}$, and the complex projective space $\mathbb{C}P^n$ (considered with its standard Riemannian locally symmetric metric) is a Ricci-pseudosymmetric manifold.
A semi-Riemannian manifold \((M, g), n \geq 3\), is said to be Ricci-pseudosymmetric ([14], [24]) if at every point of \(M\) the following condition is satisfied:

\[ (*)_2 \quad \text{the tensors } R \cdot S \quad \text{and} \quad Q(g, S) \quad \text{are linearly dependent.} \]

Evidently, any Einstein manifold is Ricci-pseudosymmetric. Thus we see that \((*)_2\) is a generalized Einstein metric condition. The manifold \((M, g)\) is Ricci-pseudosymmetric if and only if

\[ R \cdot S = L_S Q(g, S) \tag{6} \]

holds on the set \(U_S = \{ x \in M \mid S - (\kappa/n)g \neq 0 \text{ at } x \}\), where \(L_S\) is some function on \(U_S\). Warped products realizing \((*)_2\) were considered in [14], [17], [24] and [26]. Certain examples of compact and non-Einstein Ricci-pseudosymmetric manifolds were found in [26] and [30]. For instance, in [30] (Theorem 1) it was shown that the Cartan hypersurfaces \(M\) in the spheres \(S^7, S^{17}\) or \(S^{25}\) are non-pseudosymmetric, Ricci-pseudosymmetric manifolds with non-pseudosymmetric Weyl tensor. The Cartan hypersurfaces \(M\) in \(S^4\) are non-semisymmetric, pseudosymmetric manifolds. Ricci-pseudosymmetric hypersurfaces immersed isometrically in a semi-Riemannian manifolds of constant curvature were investigated in [10].

A very important subclass of the class of Ricci-pseudosymmetric manifolds form pseudosymmetric manifolds. The semi-Riemannian manifold \((M, g), n \geq 3\), is said to be pseudosymmetric ([21]) if at every point of \(M\) the following condition is satisfied:

\[ (*)_3 \quad \text{the tensors } R \cdot R \quad \text{and} \quad Q(g, R) \quad \text{are linearly dependent.} \]

The manifold \((M, g)\) is pseudosymmetric if and only if

\[ R \cdot R = L_R Q(g, R) \tag{7} \]

holds on the set \(U_R = \{ x \in M \mid R - (\kappa/(n(n - 1)))G \neq 0 \text{ at } x \}\), where \(L_R\) is some function on \(U_R\). It is clear that any semisymmetric manifold \((R \cdot R = 0, [36])\) is pseudosymmetric. Very recently the theory of Riemannian semisymmetric manifolds has been presented in [6]. The condition \((*)_3\) arose during the study of totally umbilical submanifolds of semisymmetric manifolds ([1]) as well as when we consider geodesic mappings of semisymmetric manifolds ([18], [37]). There exist many examples of pseudosymmetric manifolds which are not semisymmetric ([13], [19], [20], [21], [28]). Among these examples we can distinguish also compact pseudosymmetric manifolds (for instance, see [19], Example 3.1 and Theorem 4.1). Another example of a compact pseudosymmetric manifold is the warped product \(S^1 \times_F S^{n-1}\), with a positive smooth function \(F\), as well as \(n\)-dimensional
tori $T^n$ with a certain metric (see [19], Examples 4.1 and 4.2). It is clear that if at a point $x$ of a manifold $(M, g)$ $(*)_3$ is satisfied then also $(*)_2$ holds at $x$. The converse statement is not true. E.g. every warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 \geq 3$, of a manifold $(M_1, \tilde{g})$ and a non-pseudosymmetric, Einstein manifold $(M_2, \tilde{g})$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold (cf. [24], Remark 3.4 and [21], Theorem 4.1).

It is easy to see that if $(*)_3$ holds on a semi-Riemannian manifold $(M, g)$, $n \geq 4$, then at every point of $M$ the following condition is satisfied:

$$(*)_4 \quad \text{the tensors } R \cdot C \text{ and } Q(g, C) \text{ are linearly dependent.}$$

Manifolds fulfilling $(*)_4$ are called Weyl-pseudosymmetric. Weyl-pseudosymmetric manifolds has been studied in [15], [17] and [22]. The manifold $(M, g)$ is a Weyl-pseudosymmetric manifold if and only if the relation $R \cdot C = L_2 Q(g, C)$ holds on the set $\mathcal{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where $L_2$ is some function on $\mathcal{U}_C$.

A semi-Riemannian manifold $(M, g)$, $n \geq 4$, is said to be a manifold with pseudosymmetric Weyl tensor ([29]) if at every point of $M$ the following condition is satisfied:

$$(*)_5 \quad \text{the tensors } C \cdot C \text{ and } Q(g, C) \text{ are linearly dependent.}$$

Thus $(M, g)$ is a manifold with pseudosymmetric Weyl tensor if and only if the relation $C \cdot C = L_3 Q(g, C)$ holds on the set $\mathcal{U}_C$, where $L_3$ is a certain function on $\mathcal{U}_C$. The condition $(*)_5$ arose during the study of 4-dimensional warped products ([17]). Namely, in [17] (Theorem 2) it was shown that at every point of a warped product $M_1 \times_F M_2$, with $\dim M_1 = \dim M_2 = 2$, $(*)_5$ is fulfilled. Many examples of manifolds satisfying $(*)_5$ are presented in [9]. For instance, the Cartesian product of two manifolds of constant curvature is a manifold realizing $(*)_5$. Warped products satisfying $(*)_5$ were considered in [29]. In [9] it was shown that the classes of manifolds realizing $(*)_3$ and $(*)_5$ do not coincide. However, there exist pseudosymmetric manifolds fulfilling $(*)_5$, e.g. Einsteinian pseudosymmetric manifolds ([9], Theorem 3.1). Curvature properties of pseudosymmetric manifolds with pseudosymmetric Weyl tensor were obtained in [31].

For a $(0, k)$-tensor field $T, k \geq 1$, and a symmetric $(0, 2)$-tensor field $A$, we define the $(0, k + 2)$-tensor $Q(A, T)$ by

$$Q(A, T)(X_1, \ldots, X_k; X, Y) = ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k)$$

$$= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k)$$

$$- \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k),$$
where $X \wedge_A Y$ is the endomorphism defined by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y.$$ 

In particular, we have $X \wedge g Y = X \wedge Y$. Putting in the above formula $A = S$ and $T = R, T = C$ or $T = G$, we obtain the tensors $Q(S, R), Q(S, C)$ and $Q(S, G)$, respectively.

A semi-Riemannian manifold $(M, g)$ is said to be Ricci-generalized pseudosymmetric ([7]) if at every point of $M$ the following condition is satisfied:

$$(*)_6 \quad \text{the tensors } R \cdot R \text{ and } Q(S, R) \text{ are linearly dependent.}$$

A very important subclass of Ricci-generalized pseudosymmetric manifolds form manifolds fulfilling the following relation $R \cdot R = Q(S, R)$ ([7], [8], [23]). Every 3-manifold $(M, g)$ as well as every hypersurface $M$ immersed isometrically in an $(n + 1)$-dimensional semi-Euclidean space $E_{s}^{n+1}$, of index $s, n \geq 3$, fulfils the last equality, see [16] (Theorem 3.1) and [27] (Corollary 3.1), respectively.

As it was shown in [27], every hypersurface $M$ in a semi-Riemannian space of constant curvature $M^{n+1}(c), n \geq 4$, fulfils (1). More precisely, we have the following

**Remark 2.1** ([27], Proposition 3.1). *Every hypersurface $M$ immersed isometrically in a semi-Riemannian space of constant curvature $M^{n+1}(c), n \geq 4$, satisfies the equality $R \cdot R - Q(S, R) = -(((n - 2)\tilde{k})/(n(n + 1)))Q(g, C)$, where $\tilde{k}$ is the scalar curvature of $M^{n+1}(c)$ and $R, S$ and $C$ are the curvature tensor, the Ricci tensor and the Weyl tensor of $M$, respectively.*

Using Theorem 3.1 of [16], which was mentioned above, and the fact that the Weyl tensor of every 3-dimensional semi-Riemannian manifold vanishes identically, we conclude that $(*)_1$ is trivially satisfied on any 3-dimensional semi-Riemannian manifold. Recently, warped products realizing $(*)_1$ were considered in [11].

The relations $(*)_1, (**)_1 - (**)_6$ are called conditions of pseudosymmetry type. We refer to [12], [18] and [37] as the review papers on semi-Riemannian manifolds satisfying such conditions. A hypersurface fulfilling a curvature condition of pseudosymmetry type is said to be a hypersurface of pseudosymmetry type ([12]).

We finish this section with the following

**Lemma 2.1.** *Let $(M, g), n = \text{dim} M \geq 3$, be a semi-Riemannian manifold.*

(i) ([13], Lemma 1.2; [23], Lemma 2) *If the Weyl tensor $C$ of $(M, g)$ vanishes at a point $x \in M$ then at $x$ any of the following three identities is equivalent to each*
\[ R \cdot R = aQ(g, R), \quad R \cdot S = aQ(g, S), \]
\[ \left( \frac{\kappa}{n-1} + (n-2) \alpha \right) \left( S - \frac{\kappa}{n} \bar{g} \right) = S^2 - \frac{1}{n} \text{tr}(S^2) \bar{g}, \]

where \( \alpha \in \mathbb{R} \).

(ii) ([3], Lemma 3.1) The following identity is fulfilled on \( M : Q(S, G) = -Q(g, U) \).

(iii) ([16], Theorem 3.1) If \( \text{dim } M = 3 \) then \( R \cdot R = Q(S, R) \) holds on \( M \).

(iv) If the following conditions are fulfilled at a point \( x \in M : C = 0, \quad \text{rank} (S) = 1 \quad \text{and} \quad \kappa = 0, \) then \( R \cdot R = 0 \) holds at \( x \).

**Proof.** (iv) The condition \( \text{rank}(S) = 1 \) we can present in the following form

\[ S_{ij} = \beta u_i u_j, \quad u \in T_x^*(M), \quad \beta \in \mathbb{R}, \]

(8)

where \( u_i \) are the local components of \( u \). From (8), by \( \kappa = 0 \), it follows that \( \beta g^{ij} u_i u_j = 0 \). Transvecting now (8) with \( u^i = g^{ij} u_j \) we get \( u^i S_{ij} = 0 \). Next, transvecting (8) with \( S'_k \) and using the last relation we get \( S'_{ij} = 0 \) which, in view of (i), completes the proof.

3. Warped Products

Let now \((\bar{M}, \bar{g}) \) and \((N, \bar{g}) \), \( \text{dim } \bar{M} = p, \text{dim } N = n - p, 1 \leq p < n \), be semi-Riemannian manifolds covered by systems of charts \( \{U; x^a\} \) and \( \{V; y^\beta\} \), respectively. Let \( F \) be a positive smooth function on \( \bar{M} \). The warped product \( \bar{M} \times_F N \) of \((\bar{M}, \bar{g}) \) and \((N, \bar{g}) \) ([5], [33]) is the product manifold \( \bar{M} \times N \) with the metric \( g = \bar{g} \times_F \bar{g} \) defined by

\[ \bar{g} \times_F \bar{g} = \pi_1^* \bar{g} + (F \circ \pi_1) \pi_2^* \bar{g}, \]

where \( \pi_1 : \bar{M} \times N \to \bar{M} \) and \( \pi_2 : \bar{M} \times N \to N \) are the natural projections on \( \bar{M} \) and \( N \), respectively. Let \( \{U \times V; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\} \) be a product chart for \( \bar{M} \times N \). The local components of the metric \( g = \bar{g} \times_F \bar{g} \) with respect to this chart are the following \( g_{rs} = \bar{g}_{ab} \) if \( r = a \) and \( s = b, g_{rs} = F \bar{g}_{ab} \) if \( r = a \) and \( s = \beta \), and \( g_{rs} = 0 \) otherwise, where \( a, b, c, \ldots \in \{1, \ldots, p\}, \alpha, \beta, \gamma, \ldots \in \{p + 1, \ldots, n\} \) and \( r, s, t, \ldots \in \{1, 2, \ldots, n\} \). We will denote by bars (resp., by tildes) tensors formed from \( \bar{g} \) (resp., \( \bar{g} \)). The local components \( \Gamma^r_{st} \) of the Levi-Civita connection \( \nabla \) of \( \bar{M} \times_F N \) are the following ([34]):
On curvature properties of certain

\[ \Gamma^a_{bc} = \tilde{\Gamma}^a_{bc}, \quad \Gamma^a_{\beta\gamma} = \tilde{\Gamma}^a_{\beta\gamma}, \quad \Gamma^a_{\alpha\beta} = -\frac{1}{2} \tilde{g}^{ab} F_b \tilde{g}_{a\beta}, \quad \Gamma^a_{\alpha\beta} = \frac{1}{2F} F_d \delta^a_{\beta}, \]

\[ \Gamma^a_{\alpha\beta} = \Gamma^a_{\alpha\beta} = 0, \quad F_a = \partial_a F = \frac{\partial F}{\partial x^a}, \quad \partial_a = \frac{\partial}{\partial x^a}. \]

The local components

\[ R_{rstu} = g_{rw} R^w_{stu} = g_{rw} (\partial_u \Gamma^w_{st} - \partial_t \Gamma^w_{su} + \Gamma^v_{su} \Gamma^w_{tv} - \Gamma^v_{st} \Gamma^w_{uv}), \quad \partial_u = \frac{\partial}{\partial x^u}, \]

of the Riemann-Christoffel curvature tensor \( R \) and the local components \( S_{rs} \) of the Ricci tensor \( S \) of the warped product \( \overline{M} \times_F N \) which may not vanish identically are the following:

\[ R_{abcd} = \tilde{R}_{abcd}, \quad R_{a\beta\gamma} = -\frac{1}{2} T_{ab} \tilde{g}_{a\beta}, \quad R_{a\beta\gamma} = F \tilde{R}_{a\beta\gamma} - \frac{1}{4} \Delta_1 F \tilde{G}_{a\beta\gamma}, \quad (9) \]

\[ S_{ab} = \tilde{S}_{ab} - \frac{n-p}{2} \frac{T_{ab}}{F}, \quad S_{a\beta} = \tilde{S}_{a\beta} - \frac{1}{2} \left( tr(T) + \frac{n-p-1}{2F} \Delta_1 F \right) \tilde{g}_{a\beta}, \quad (10) \]

where

\[ T_{ab} = \tilde{V}_b F_a - \frac{1}{2F} F_a F_b, \quad tr(T) = \tilde{g}^{ab} T_{ab}, \quad \Delta_1 F = \Delta_1 F = \tilde{g}^{ab} F_a F_b, \quad (11) \]

and \( T \) is the \((0,2)\)-tensor with the local components \( T_{ab} \). The scalar curvature \( \kappa \) of \( \overline{M} \times_F N \) satisfies the following relation

\[ \kappa = \tilde{\kappa} + \frac{1}{F} \tilde{\kappa} - \frac{n-p}{F} \left( tr(T) + \frac{n-p-1}{4F} \Delta_1 F \right). \quad (12) \]

From now we assume that \( \dim \overline{M} \times_F N = 4 \) and \( \dim \overline{M} = 1 \). Then (9), (10) and (12) turn into

\[ R_{x11\beta} = -\frac{1}{2} T_{11} \tilde{g}_{a\beta}, \quad R_{a\beta\gamma\delta} = F \tilde{R}_{a\beta\gamma\delta} - \frac{1}{4} \Delta_1 F \tilde{G}_{a\beta\gamma\delta}, \quad (13) \]

\[ S_{11} = -\frac{3}{2F} T_{11}, \quad S_{a\beta} = \tilde{S}_{a\beta} - \frac{1}{2} \left( tr(T) + \frac{\Delta_1 F}{F} \right) \tilde{g}_{a\beta}, \quad (14) \]

\[ \kappa = \frac{1}{F} \tilde{\kappa} - \frac{3}{F} \left( tr(T) + \frac{1}{2} \frac{\Delta_1 F}{F} \right). \quad (15) \]

respectively. Further, by making use of (13), (14) and (15), we obtain the following relations (see [17], Lemma 6):
\[ C_{a11\beta} = -\frac{1}{2} \bar{g}_{11} \left( \tilde{S}_{2\beta} - \frac{1}{3} \tilde{k} \tilde{g}_{2\beta} \right), \]
\[ C_{a\beta\gamma\delta} = \frac{1}{2} F (\tilde{g}_{a\beta} \tilde{S}_{\beta\gamma} - \tilde{g}_{a\beta} \tilde{S}_{\beta\gamma} - \tilde{g}_{\gamma\delta} \tilde{S}_{a\beta} - \tilde{g}_{\gamma\delta} \tilde{S}_{a\beta} - \frac{1}{3} F \tilde{k} \tilde{G}_{a\beta\gamma\delta}. \quad (16) \]

On the other hand, from (3) it follows that
\[ \tilde{C}_{a\beta\gamma\delta} = \tilde{R}_{a\beta\gamma\delta} - \tilde{U}_{a\beta\gamma\delta} + \frac{1}{2} \tilde{k} \tilde{G}_{a\beta\gamma\delta}. \quad (17) \]

Since \( \tilde{C}_{a\beta\gamma\delta} = 0 \), the last identity reduces to
\[ \tilde{U}_{a\beta\gamma\delta} = \tilde{R}_{a\beta\gamma\delta} + \frac{1}{2} \tilde{k} \tilde{G}_{a\beta\gamma\delta}. \quad (18) \]

Now (16) turns into
\[ C_{a11\beta} = -\frac{1}{2} \bar{g}_{11} (\tilde{S}_{a\beta} - \frac{1}{3} \tilde{k} \tilde{g}_{a\beta}), \quad C_{a\beta\gamma\delta} = \frac{1}{2} F \tilde{R}_{a\beta\gamma\delta} - \frac{1}{12} F \tilde{k} \tilde{G}_{a\beta\gamma\delta}. \quad (19) \]

4. Preliminary Results

Let \( \tilde{M} \times_F N \) be a 4-dimensional warped product with 1-dimensional base manifold \((\tilde{M}, \bar{g})\). Using (13), (14), (18) and (19), we can verify that the local components of the tensors \( R \cdot C \) and \( Q(S, C) \) of the manifold \( \tilde{M} \times_F N \), which may not vanish identically are the following:

\[ (R \cdot C)_{\alpha\beta\gamma\delta\mu} = \frac{1}{2} F (\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\mu} - \frac{1}{8} \Delta_1 F Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\mu}, \quad (20) \]

\[ Q(S, C)_{\alpha\beta\gamma\delta\mu} = -\frac{1}{4} F \left( tr(T) + \frac{\Delta_1 F}{F} \right) Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\mu} \]
\[ + \frac{1}{2} F Q(\tilde{S}, \tilde{R})_{\alpha\beta\gamma\delta\mu} - \frac{1}{12} F \tilde{k} Q(\tilde{S}, \tilde{G})_{\alpha\beta\gamma\delta\mu}, \quad (21) \]

\[ (R \cdot C)_{a\beta\gamma\delta\mu} = -\frac{1}{2} F T_{11} C_{a\beta\gamma\delta\mu} + \frac{1}{12} \tilde{k} T_{11} \tilde{G}_{a\beta\gamma\delta\mu} - \frac{1}{4} T_{11} (\tilde{g}_{\gamma\delta} \tilde{S}_{a\beta} - \tilde{g}_{\beta\mu} \tilde{S}_{a\gamma}), \quad (22) \]

\[ Q(S, C)_{a\beta\gamma\delta\mu} = -\frac{3}{2} \frac{1}{2} F T_{11} C_{a\beta\gamma\delta} + \frac{1}{2} \bar{g}_{11} \left( \tilde{S}_{\gamma\delta} \tilde{S}_{a\beta} - \tilde{S}_{\delta\mu} \tilde{S}_{a\gamma} - \frac{1}{3} \tilde{k} (\tilde{g}_{a\beta} \tilde{S}_{\gamma\delta} - \tilde{g}_{a\gamma} \tilde{S}_{\delta\beta}) \right) \]
\[ + \frac{1}{12} \tilde{k} \left( tr(T) + \frac{\Delta_1 F}{F} \right) \bar{g}_{11} \tilde{G}_{a\beta\gamma\delta} \]
\[ - \frac{1}{4} \bar{g}_{11} \left( tr(T) + \frac{\Delta_1 F}{F} \right) (\tilde{g}_{\gamma\delta} \tilde{S}_{a\beta} - \tilde{g}_{\beta\mu} \tilde{S}_{a\gamma}), \quad (23) \]
On curvature properties of certain

\[ (R \cdot C)_{a11 \beta \gamma} = -\frac{1}{2} \frac{1}{F} \bar{g}_{11} \left( F(R \cdot \bar{S})_{a \beta \gamma} - \frac{1}{4} \Delta_1 F Q(\bar{g}, \bar{S})_{a \beta \gamma} \right), \quad (24) \]

\[ Q(S, C)_{a11 \beta \gamma} = -\frac{1}{2} \bar{g}_{11} \left( \frac{1}{6} \tilde{k} - \frac{1}{2} \left( t r(T) + \frac{\Delta_1 F}{F} \right) \right) Q(\bar{g}, \bar{S})_{a \beta \gamma}, \quad (25) \]

From Lemma 2.1(ii), it follows that \( Q(\bar{S}, \bar{G})_{a \beta \gamma} = -Q(\bar{g}, \bar{G})_{a \beta \gamma}, \) which by making use of (18), turns into \( Q(\bar{S}, \bar{G})_{a \beta \gamma} \). Now (21) takes the form

\[ Q(S, C)_{a \beta \gamma} = \frac{1}{2} F \left( Q(\bar{S}, \bar{G})_{a \beta \gamma} + \left( \frac{1}{6} \tilde{k} - \frac{1}{2} \left( t r(T) + \frac{\Delta_1 F}{F} \right) \right) Q(\bar{g}, \bar{G})_{a \beta \gamma} \right). \quad (26) \]

In view of Lemma 2.1(iii), we have also \( (\bar{R} \cdot \bar{R})_{a \beta \gamma} = Q(\bar{S}, \bar{G})_{a \beta \gamma}. \) Substituting this in (26) we obtain

\[ Q(S, C)_{a \beta \gamma} = \frac{1}{2} F (\bar{R} \cdot \bar{G})_{a \beta \gamma} + \left( \frac{1}{6} \tilde{k} - \frac{1}{2} \left( t r(T) + \frac{\Delta_1 F}{F} \right) \right) Q(\bar{g}, \bar{G})_{a \beta \gamma}, \]

whence

\[ Q(S, C)_{a \beta \gamma} = \frac{1}{2} F (\bar{R} \cdot \bar{G})_{a \beta \gamma} - \frac{1}{2} F \tau Q(\bar{g}, \bar{G})_{a \beta \gamma}, \quad (27) \]

\[ \tau_1 = \frac{1}{2} \left( -\frac{1}{3} \tilde{k} + t r(T) + \frac{\Delta_1 F}{F} \right). \quad (28) \]

Now, the equality \( (R \cdot C)_{a \beta \gamma} = L Q(S, C)_{a \beta \gamma}, \) in virtue of (20) and (27), gives

\[ (1 - L)(\bar{R} \cdot \bar{G})_{a \beta \gamma} = \left( \frac{1}{4} \frac{\Delta_1 F}{F} - \tau_1 L \right) Q(\bar{g}, \bar{G})_{a \beta \gamma}. \quad (29) \]

By (22) and (23) the relation \( (R \cdot C)_{1a \beta \gamma} = L Q(S, C)_{1a \beta \gamma} \) is equivalent to

\[ -\frac{1}{2} T_{11} C_{a \beta \gamma} + \frac{1}{12} \tilde{k} T_{11} G_{a \beta \gamma} - \frac{1}{4} T_{11} (\tilde{g}_{\gamma \delta} \tilde{S}_{a \beta} - \tilde{g}_{\beta \delta} \tilde{S}_{a \gamma}) \]

\[ = -\frac{3}{2} \frac{1}{F} L T_{11} C_{a \beta \gamma} + \frac{1}{2} L \tilde{g}_{11} (\tilde{S}_{\gamma \delta} \tilde{S}_{a \beta} - \tilde{S}_{\beta \delta} \tilde{S}_{a \gamma}) + \frac{1}{12} \tilde{k} \left( t r(T) + \frac{\Delta_1 F}{F} \right) L \tilde{g}_{11} \tilde{G}_{a \beta \gamma} \]

\[ + \frac{1}{2} L \tilde{g}_{11} \left( -\frac{1}{2} \left( t r(T) + \frac{\Delta_1 F}{F} \right) (\tilde{g}_{\beta \delta} \tilde{S}_{a \gamma} - \tilde{g}_{\gamma \delta} \tilde{S}_{a \beta}) - \frac{1}{3} \tilde{k} (\tilde{g}_{\alpha \beta} \tilde{S}_{a \gamma} - \tilde{g}_{\alpha \gamma} \tilde{S}_{a \beta}) \right). \quad (30) \]

Further, we can check that the relation \( (R \cdot C)_{a11 \beta \gamma} = L Q(S, C)_{a11 \beta \gamma} \) turns into

\[ (\bar{R} \cdot \bar{S})_{a \beta \gamma} = \left( \frac{1}{4} \frac{\Delta_1 F}{F} - \tau_1 L \right) Q(\bar{g}, \bar{S})_{a \beta \gamma}. \quad (31) \]
This, in view of Lemma 2.1(iii), is equivalent to

\[ (\tilde{R} \cdot \tilde{R})_{\beta \gamma \delta \mu} = \left( \frac{1}{4} \frac{\Delta_1 F}{F} - \tau_1 \lambda \right) Q(\tilde{g}, \tilde{R})_{\beta \gamma \delta \mu}. \]

(32)

Thus we have the following

**Proposition 4.1.** A 4-dimensional warped product \( \overline{M} \times F \overline{N} \), \( \dim \overline{M} = 1 \), satisfies the condition \( R \cdot C = LQ(S, C) \) if and only if (29), (30) and (32) hold on \( \mathcal{U} \).

5. Main Results

**Example 5.1.** (i) We present an example of a 4-dimensional warped product, with 1-dimensional base manifold, realizing (*) and (**). Let \((N, \tilde{g}), \dim N = 3\), be a semi-Riemannian manifold such that its Ricci tensor \( \tilde{S} \) is of rank one and its scalar curvature \( \tilde{k} \) vanishes identically on \( N \). Then, in view of Lemma 2.1(iv), \((N, \tilde{g})\) is a semisymmetric manifold. Furthermore, let \( F \), defined by \( F(x^1) = a \exp(bx^1), \) \( a = \text{const.} > 0, b = \text{const.} \neq 0 \), be a function on a 1-dimensional manifold \((\overline{M}, g_1)\). It is easy to check, that \( \overline{M} \times F \overline{N} \) realizes (29), (30) and (32), with \( L = 1/3 \). Thus, in view of Proposition 4.1, \( \overline{M} \times F \overline{N} \) fulfills \( R \cdot C = (1/3)Q(S, C) \). From Corollary 4.2 of [21] it follows that the manifold \( \overline{M} \times F \overline{N} \) is pseudosymmetric too. Next, using (3.12) of [21] and (15), we get \( R \cdot R = (1/12)\kappa Q(g, R) \), where \( \kappa \) is the scalar curvature of \( \overline{M} \times F \overline{N} \).

(ii) We present an example of a 3-dimensional semisymmetric warped product such that the rank of its Ricci tensor is one and its scalar curvature vanishes identically. Let \( M_2 = \{(x^2, x^3) : x^2, x^3 \in \mathbb{R}\} \) be a connected, non-empty, open subset of \( \mathbb{R}^2 \), equipped with the metric tensor \( g_2 \) defined by \( g_{2,22} = g_{2,33} = 0 \), \( g_{2,23} = g_{2,32} = 1 \), and let \( H = H(x^2) \) be a smooth function on \( M_2 \). Moreover, let \((M_3, g_3)\) be a 1-dimensional manifold. In [35] (see p. 177) it was shown that the rank of the Ricci tensor \( \tilde{S} \) of the warped product \( M_2 \times_H M_3 \) is equal to one and that the scalar curvature of this manifold vanishes identically. Moreover, we have (cf. [35], p. 177)

\[ \tilde{S}_{22} = -\frac{1}{H} \left( \frac{\partial H}{\partial x^2} \frac{1}{2H} H_2 H_2 \right), \quad H_2 = \frac{\partial H}{\partial x^2}, \quad \tilde{S}_{33} = 0, \quad \tilde{S}_{44} = 0. \]

Furthermore, from Lemma 2.1(iv) it follows that \( M_2 \times_H M_3 \) is a semisymmetric manifold. (iii) We consider the warped product \( \overline{M} \times F \overline{N} \), where \( \dim \overline{M} \)
On curvature properties of certain manifolds, the warping function $F$ is defined by $F(x^1) = a \exp(b x^1), a = \text{const.} > 0, b = \text{const.} \neq 0,$ and $(N, \tilde{g})$ is a semisymmetric manifold defined in (ii). We can verify that the tensor $\tilde{S} - (\kappa/4)g$ is of rank one, i.e. the warped product $\tilde{M} \times_F N$ is a quasi-Einstein manifold.

In this section we prove, that under certain assumptions every 4-dimensional warped product $\tilde{M} \times_F N, \dim \tilde{M} = 1$, realizing $(*)$ is the manifold described in Example 5.1(i).

Symmetrizing (30) in $\alpha, \delta$ we obtain

$$\left(\frac{1}{2} T_{11} - \tau L \tilde{g}_{11}\right) Q(\tilde{g}, \tilde{S})_{\alpha \beta \gamma \delta} = 0,$$

(33)

where

$$\tau = \frac{1}{2} \left( -\frac{2}{3} \kappa + tr(T) + \frac{\Delta_1 F}{F} \right).$$

(34)

From (19) it follows that the Weyl tensor $C$ of every 4-dimensional warped product $\tilde{M} \times_F N, \dim \tilde{M} = 1$, vanishes at a point $x \in M_1 \times_F N$ if and only

$$\tilde{S}_{\alpha \beta} = \frac{1}{3} \kappa \tilde{g}_{\alpha \beta}.$$  

(35)

holds at $x$. We note also that $Q(\tilde{g}, \tilde{S})$ vanishes at $x$ if and only if (35) is satisfied at $x$. So, if the tensor $C$ is non-zero at the point $x \in M_1 \times_F N$ then from (33) it follows that

$$\frac{1}{2} T_{11} = \tau L \tilde{g}_{11}.$$ 

(36)

holds at $x$. Applying (36) in (30) we obtain

$$-\frac{\tau}{F} C_{\delta \sigma \delta \gamma} + \frac{\tau}{6} \kappa \tilde{g}_{\delta \sigma \delta \gamma} - \frac{\tau}{2} \left( \tilde{g}_{\delta \sigma} \tilde{S}_{\gamma \delta \gamma} - \tilde{g}_{\gamma \delta} \tilde{S}_{\delta \gamma} \right)$$

$$= -3 \frac{\tau}{F} C_{\delta \sigma \delta \gamma} + \frac{1}{2} \left( \tilde{S}_{\delta \gamma} \tilde{S}_{\delta \gamma} - \tilde{S}_{\delta \gamma} \tilde{S}_{\delta \gamma} \right) + \frac{1}{12} \kappa \left( tr(T) + \frac{\Delta_1 F}{F} \right) \tilde{g}_{\delta \sigma \delta \gamma}$$

$$- \frac{1}{4} \left( tr(T) + \frac{\Delta_1 F}{F} \right) (\tilde{g}_{\delta \gamma} \tilde{S}_{\delta \gamma} - \tilde{g}_{\delta \gamma} \tilde{S}_{\delta \gamma}) - \frac{1}{6} \kappa (\tilde{g}_{\delta \gamma} \tilde{S}_{\delta \gamma} - \tilde{g}_{\delta \gamma} \tilde{S}_{\delta \gamma}).$$ 

(37)

If $x \in U_L$ then the last equality reduces to
\[
\frac{1}{F} \tau(3L - 1) C_{\alpha \beta \gamma} + \frac{1}{6} \kappa \left( \tau - \frac{1}{2} \left( tr(T) + \frac{\Delta F}{F} \right) \right) \tilde{G}_{\alpha \beta \gamma} \\
= \frac{1}{2} (\tilde{S}_{\gamma \delta} \tilde{S}_{\alpha \beta} - \tilde{S}_{\alpha \beta} \tilde{S}_{\delta \gamma} - \frac{1}{6} \kappa (\tilde{g}_{\alpha \beta} \tilde{S}_{\delta \gamma} - \tilde{g}_{\gamma \delta} \tilde{S}_{\alpha \beta} - \tilde{g}_{\beta \gamma} \tilde{S}_{\alpha \delta} - \tilde{g}_{\delta \alpha} \tilde{S}_{\gamma \beta})),
\]

which, by (34), turns into
\[
\frac{2}{F} \tau(3L - 1) C_{\alpha \beta \gamma} - \frac{1}{9} \kappa^2 \tilde{G}_{\alpha \beta \gamma} = \tilde{S}_{\gamma \delta} \tilde{S}_{\alpha \beta} - \tilde{S}_{\alpha \beta} \tilde{S}_{\delta \gamma} - \frac{1}{3} \kappa \tilde{U}_{\alpha \beta \gamma}.
\]

On the other hand (18) and (19) give
\[
C_{\alpha \beta \gamma} = \frac{1}{2} F \left( \tilde{U}_{\alpha \beta \gamma} - \frac{2}{3} \kappa \tilde{G}_{\alpha \beta \gamma} \right).
\]

Applying this in (38) we obtain
\[
\tau(3L - 1) \tilde{U}_{\alpha \beta \gamma} - \frac{2}{3} \tau(3L - 1) \kappa \tilde{G}_{\alpha \beta \gamma} - \frac{1}{9} \kappa^2 \tilde{G}_{\alpha \beta \gamma} = \tilde{S}_{\gamma \delta} \tilde{S}_{\alpha \beta} - \tilde{S}_{\alpha \beta} \tilde{S}_{\delta \gamma} - \frac{1}{3} \kappa \tilde{U}_{\alpha \beta \gamma},
\]
whence
\[
\tilde{S}_{\gamma \delta} \tilde{S}_{\alpha \beta} - \tilde{S}_{\beta \gamma} \tilde{S}_{\alpha \delta} = \rho \tilde{U}_{\alpha \beta \gamma} + \mu \tilde{G}_{\alpha \beta \gamma},
\]

where
\[
\rho = \tau(3L - 1) + \frac{1}{3} \kappa, \quad \mu = -\frac{1}{3} \kappa \left( 2 \tau(3L - 1) + \frac{1}{3} \kappa \right).
\]

We put \( \tilde{A}_{\alpha \beta} = \tilde{S}_{\alpha \beta} - \rho \tilde{g}_{\alpha \beta} \). Thus, by (39), we have
\[
\tilde{A}_{\gamma \delta} \tilde{A}_{\alpha \beta} - \tilde{A}_{\beta \delta} \tilde{A}_{\gamma \alpha} = \tilde{S}_{\gamma \delta} \tilde{S}_{\alpha \beta} - \tilde{S}_{\alpha \beta} \tilde{S}_{\delta \gamma} + \rho^2 \tilde{G}_{\alpha \beta \gamma} \\
- \rho (\tilde{g}_{\gamma \delta} \tilde{S}_{\alpha \beta} + \tilde{g}_{\alpha \beta} \tilde{S}_{\gamma \delta} - \tilde{g}_{\beta \delta} \tilde{S}_{\gamma \alpha} - \tilde{g}_{\delta \gamma} \tilde{S}_{\alpha \beta}) \\
= (\rho^2 + \mu) \tilde{G}_{\alpha \beta \gamma},
\]

which leads to
\[
Q(\tilde{A}, A) = (\rho^2 + \mu) Q(\tilde{A}, \tilde{G}),
\]

where the (0,4)-tensor \( A \) is defined by
\[
A_{\alpha \beta \gamma \delta} = \tilde{A}_{\alpha \beta} \tilde{A}_{\gamma \delta} - \tilde{A}_{\alpha \gamma} \tilde{A}_{\beta \delta}.
\]

Since the tensor \( Q(\tilde{A}, A) \) vanishes identically, we have \( (\rho^2 + \mu) Q(\tilde{A}, \tilde{G}) = 0 \), whence we get easily \( (\rho^2 + \mu)(\tilde{A} - (1/3) tr(\tilde{A}) \tilde{g}) = 0 \). Since \( \tilde{S} \neq (1/3) \kappa \tilde{g} \) holds at
On curvature properties of certain $x$, the last relation yields

$$\rho^2 + \mu = 0. \quad (42)$$

Further, using (29) and (32) we deduce that

$$(a) \quad (\bar{R} \cdot \bar{R})_{\alpha \beta \gamma \delta \mu} = 0, \quad (b) \quad \frac{1}{4} \frac{\Delta F}{F} = \tau L, \quad (43)$$

hold at $x \in \mathcal{H}_L$. Further, contracting (39) with $\bar{g}^{\alpha \beta}$ we obtain

$$\bar{S}^2_{\alpha \beta} = (\bar{\kappa} - \rho) \bar{S}_{\alpha \beta} + (2\mu + \rho \bar{\kappa}) \bar{g}_{\alpha \beta},$$

which, by making use of (42), can be rewritten in the following form

$$\bar{S}^2_{\alpha \beta} - \frac{1}{3} tr(\bar{S}^2) \bar{g}_{\alpha \beta} = \left( \frac{\bar{\kappa}}{2} + \alpha \right) \left( \bar{S}_{\alpha \beta} - \frac{\bar{\kappa}}{3} \bar{g}_{\alpha \beta} \right), \quad \alpha = (\bar{\kappa} - \rho) - \frac{\bar{\kappa}}{2} = \frac{\bar{\kappa} - \rho}{2}. \quad (44)$$

From (44), in view of Lemma 2.1(i), it follows that $(\bar{R} \cdot \bar{S})_{\alpha \beta \gamma \delta} = \alpha Q(\bar{g}, \bar{S})_{\alpha \beta \gamma \delta}$ holds at $x \in \mathcal{H}_L$ and in a consequence, $(\bar{R} \cdot \bar{R})_{\alpha \beta \gamma \delta \mu} = \alpha Q(\bar{g}, \bar{R})_{\alpha \beta \gamma \delta \mu}$. The last relation, by (43)(a), implies $\alpha = 0$, i.e. $\rho = \bar{\kappa}/2$. Applying the last equality and (42) in (40) we find

$$(3L - 1)\tau = \frac{5}{6} \bar{\kappa}, \quad \frac{2}{3} \bar{\kappa}(3L - 1)\tau = -\frac{13}{36} \bar{\kappa}^2, \quad (45)$$

which gives $(5/6)\bar{\kappa}^2 = -(13/36)\bar{\kappa}^2$, whence $\bar{\kappa} = 0$. Now (45) reduces to $(3L - 1)\tau = 0$ and in a consequence, from (40) we get $\rho = \mu = 0$. So, (39) reduces to $\text{rank}(\bar{S}) = 1$. Since $\bar{\kappa} = 0$, (28) and (34) leads to $\tau_1 = \tau = (1/2)\left(tr(T) + (\Delta F/F)\right)$. Further, we denote by $\mathcal{H}_F$ the set consisting of all points of $\mathcal{H}_L$ at which $F' \neq 0$. We suppose that $\tau$ vanishes at $x \in \mathcal{H}_F$. Then (43)(b) implies $F' = 0$, a contradiction. Thus $L = 1/3$ holds on $\mathcal{H}_F$. We note that if $L = 1/3$ then only the functions $F$, defined by $F(x^1) = a \exp(bx^1), a = \text{const.} > 0, b = \text{const.} \neq 0$, are non-constant solutions of (36) and (43)(b). Thus we have the following.

**Theorem 5.1.** Let the set $\mathcal{H}_F$ be a dense subset of the set $\mathcal{H}_L$ of a 4-dimensional warped product $\bar{M} \times_F N$, $\dim \bar{M} = 1$. Then the warped product $\bar{M} \times_F N$ satisfies the condition $R \cdot C = LQ(S, C)$ on the set $\mathcal{H}_L \subset \mathcal{H} \subset \bar{M} \times N$ if and only if $L = 1/3, F(x^1) = a \exp(bx^1), a = \text{const.} > 0, b = \text{const.} \neq 0$, and $(N, \bar{g})$ is a 3-dimensional semi-Riemannian manifold fulfilling $\text{rank}(\bar{S}) = 1$ and $\bar{\kappa} = 0$.

**Remark 5.1.** Let $(N, \bar{g}), \dim N = 3$, be a semisymmetric manifold with vanishing identically on $N$ scalar curvature $\bar{\kappa}$. Suppose that $\bar{g}$ is a Riemannian
metric. Using this fact we can easily deduce that the condition rank $\tilde{S} \leq 1$ implies $\tilde{S} = 0$. Therefore, if the assumption rank $\tilde{S} = 1$ is fulfilled on $(N, \tilde{g})$ then the metric $\tilde{g}$ must be necessary indefinite, more precisely, $\tilde{g}$ is a Lorentzian metric.

Now from Theorem 5.1, in view of the above remark, follows the following

**Corollary 5.1.** If a generalized Robertson-Walker spacetime satisfies (*) then at every point of this spacetime at least one of the tensors $R \cdot C$ or $Q(S, C)$ must vanish.

Let $x$ be a point of a 4-dimensional warped product $\overline{M} \times_F N, \dim \overline{M} = 1$. If at $x$ the conditions: $C \neq 0$ and $R \cdot C = 0$ are satisfied then $R \cdot R = 0$ holds at $x$ ([17], Theorem 3). If at $x$ the conditions: $C \neq 0, S \neq 0$ and $Q(S, C) = 0$ are satisfied then $R \cdot R = (\kappa/3)Q(g, R)$ holds at $x$ ([25], Theorem 3.1). If at $x$ the condition $S = 0$ is satisfied then $C = 0$ holds at $x$. This statement is an immediate consequence of (14) and (32). Finally, if at $x$ the condition $C = 0$ is satisfied then $R \cdot R = \alpha Q(g, R), \alpha \in R$, holds at $x$ ([13], Lemma 3.1). These facts, together with Corollary 5.1, leads to the following

**Theorem 5.2.** Every generalized Robertson-Walker spacetime satisfying (*) is a pseudosymmetric manifold.

**Remark 5.2.** (i) Theorem 2 of [29] implies that the warped product $\overline{M} \times_F N$, of a 1-dimensional base manifold $(\overline{M}, \overline{g})$, a warping function $F$ and a 3-dimensional manifold $(N, \tilde{g})$ with the Ricci tensor $\tilde{S}$ of rank one realizes (*)$_S$, i.e. $\overline{M} \times_F N$ is a manifold with pseudosymmetric Weyl tensor.

(ii) We can also check that the Weyl tensor of the warped product defined above is not of harmonic curvature.

**References**


On curvature properties of certain


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