

REGULARITY OF ULTRAFILTERS AND FIXED POINTS OF ELEMENTARY EMBEDDINGS

By

Yoshihiro ABE and Masahiro SHIOYA*

1. Introduction

Let κ be a compact cardinal, λ a cardinal $\geq \kappa$ and U a fine ultrafilter on $\mathcal{P}_\kappa \lambda$. As is well known, the canonical embedding $j_U : V \rightarrow \text{Ult}(V, U)$ is the identity below κ and $j_U(\kappa) \geq 2^{\lambda^{<\kappa}}$ holds. What about the action of j_U on higher cardinals? Barbanel [2] determined the class of fixed cardinals when U is normal. Abe [1] solved the general case:

THEOREM 1. *Let κ , λ and U be as above. Then j_U moves a cardinal $\mu > 2^{\lambda^{<\kappa}}$ exactly when $\kappa \leq \text{cf}(\mu) \leq \lambda^{<\kappa}$ or $\mu = \nu^+$ with $\kappa \leq \text{cf}(\nu) \leq \lambda^{<\kappa}$.*

The main advantage of [1] over [2] lies in the assertion that the embedding moves such cardinals. In this paper we pursue this subject in somewhat greater generality, or in terms of regularity instead of fineness. In particular we give two alternative proofs of the main claim of Theorem 1, which are both much simpler than the original one. On the other hand Abe [1] noted that some of his arguments work even for a uniform ultrafilter. We also show that such extension may fail in other cases.

2. Preliminaries

In this paper we follow the terminology of Kanamori [5] closely. We let κ denote a regular cardinal $> \omega$ and λ a cardinal $\geq \kappa$. We use μ and ν to denote a cardinal $\geq \omega$. By inaccessibility and compactness of a cardinal we always mean the stronger notions.

*The authors were partially supported by Grant-in-Aid for Scientific Research (No. 09640299), Ministry of Education, Science, Sports and Culture.

Received April 10, 1996

Revised June 9, 1997

As shown by Solovay [10], cardinal arithmetic above a compact cardinal is very simple:

THEOREM 2. *Let κ be compact and $v \geq \max\{\kappa, 2^\mu\}$. Then $v^\mu = v^+$ when $\text{cf}(v) \leq \mu$, or else $v^\mu = v$.*

We sketch a proof of the following observation essentially due to Barbanel [2].

LEMMA 1. *Let U be a κ -complete ultrafilter on I . Assume $v^{|I|} = v^+$ for any $\mu > 2^{|I|}$ with $\kappa \leq \text{cf}(v) \leq |I|$. Then j_U fixes $\mu > 2^{|I|}$ unless $\kappa \leq \text{cf}(\mu) \leq |I|$ or $\mu = v^+$ with $\kappa \leq \text{cf}(v) \leq |I|$.*

PROOF. We proceed by induction on μ . First observe $j_U(\mu) = \sup j_U''\mu = \mu$ for $\mu = (2^{|I|})^+$ by the cofinality argument and the cardinality calculation. Similarly j_U fixes v^{++} when $v > 2^{|I|}$ and $\kappa \leq \text{cf}(v) \leq |I|$. Next note that the successor of a fixed point is also fixed. Finally we have the equalities $j_U(\mu) = \sup j_U''\mu = \mu$ for a limit μ by the cofinality arguments and the inductive hypothesis. \square

3. Regular ultrafilters

The main result of this section is as follows:

THEOREM 3. *For a κ -complete (κ, λ) -regular ultrafilter U we have*

- (1) $j_U(\kappa) \geq 2^{\lambda^{<\kappa}}$ and
- (2) $j_U(v) \geq v^+$ when $\kappa \leq \text{cf}(v) \leq \lambda^{<\kappa}$.

We remind the reader that a filter F is (κ, λ) -regular if there is a family $\{X_\alpha : \alpha < \lambda\} \subset F$ such that $\bigcap_{\alpha \in S} X_\alpha = \emptyset$ for any $S \in [\lambda]^\kappa$. Note that a fine filter on $\mathcal{P}_\kappa \lambda$ is (κ, λ) -regular, as witnessed by $\{\{x : \alpha \in x\} : \alpha < \lambda\}$. Hence together with Theorem 2 and Lemma 1, Theorem 3(2) establishes Theorem 1.

We prove Theorem 3 via two lemmas on a regular filter. For the rest of this section we fix a κ -complete (κ, λ) -regular filter F on I witnessed by $\{X_\alpha : \alpha < \lambda\}$ and write κ^* for $\sup_{\alpha < \kappa} (2^\alpha)^+$ for brevity. Our first lemma is a modification of a classical result of Frayne, Morel and Scott [4]:

LEMMA 2. *There is a family $H \subset I^{(<\kappa^* v)}$ of size $v^{\lambda^{<\kappa}}$ such that $\{i : f(i) \neq g(i)\} \in F$ for any distinct $f, g \in H$.*

PROOF. First fix a bijection $\pi_i : |\{x : i \in \bigcap_{\alpha \in x} X_\alpha\}| \rightarrow \{x : i \in \bigcap_{\alpha \in x} X_\alpha\}$ for $i \in I$. Note that $\text{dom } \pi_i < \kappa^*$. For $s : \mathcal{P}_\kappa \lambda \rightarrow \nu$ and $i \in I$ define $h_s(i) : \text{dom } \pi_i \rightarrow \nu$ by $(h_s(i))(\xi) = s(\pi_i(\xi))$. We show that $H = \{h_s : s : \mathcal{P}_\kappa \lambda \rightarrow \nu\}$ is as desired. Fix distinct $s, t : \mathcal{P}_\kappa \lambda \rightarrow \nu$. Take $x \in \mathcal{P}_\kappa \lambda$ with $s(x) \neq t(x)$. Then $\bigcap_{\alpha \in x} X_\alpha \subset \{i : h_s(i) \neq h_t(i)\}$. \square

We remark here that the combination of Theorem 2, Lemmas 1 and 2 provides the third proof of Theorem 1 via Theorem 3(2) for a compact κ .

LEMMA 3. *Let $\kappa^* \leq \text{cf}(\nu) \leq \lambda^{<\kappa}$. Then there is a family $H \subset {}^I \nu$ of size ν^+ such that $\{i : f(i) \neq g(i)\} \in F$ for any distinct $f, g \in H$.*

PROOF. First we construct a family $A \subset [\nu]^\nu$ of size ν^+ such that $a \cap b$ is bounded in ν for any distinct $a, b \in A$. We may assume that ν is singular. Let $\{v_\xi : \xi < \text{cf}(\nu)\} \subset \nu$ be a club set of cardinals with $v_0 = 0$. By induction on $\alpha < \nu^+$ define $f_\alpha : \nu \rightarrow \nu$ with $f_\alpha'' v_\xi \subset v_\xi$ for any $\xi < \text{cf}(\nu)$ as follows: Suppose that $\{f_\beta : \beta < \alpha\} = \{g_\beta^\alpha : \beta < \nu\}$ is defined. Take $f_\alpha(\gamma) \in v_{\xi+1} - \{g_\beta^\alpha(\gamma) : \beta < v_\xi\}$ for $\gamma \in v_{\xi+1} - v_\xi$. By construction $A = \{\pi'' f_\alpha : \alpha < \nu^+\}$ is as desired, where $\pi : \nu^2 \rightarrow \nu$ is the canonical bijection.

Now fix an unbounded $\{\delta_x : x \in \mathcal{P}_\kappa \lambda\} \subset \nu$. For $a \in A$ define $h_a : I \rightarrow \nu$ by $h_a(i) = \min(a - \sup\{\delta_x : i \in \bigcap_{\alpha \in x} X_\alpha\})$. We show that $H = \{h_a : a \in A\}$ is as desired. Fix distinct $a, b \in A$. Take $x \in \mathcal{P}_\kappa \lambda$ with $a \cap b \subset \delta_x$. Then $\bigcap_{\alpha \in x} X_\alpha \subset \{i : h_a(i) \neq h_b(i)\}$. \square

PROOF OF THEOREM 3. First observe $\kappa^* = \kappa$, as U is not κ^+ -complete by (κ, λ) -regularity and hence $\text{crit}(j_U) = \kappa$. Then items (1) and (2) follow from Lemma 2 with $\nu = \kappa$ and Lemma 3 respectively. \square

4. Uniform ultrafilters

In this section we construct a model in which Theorem 3(1) fails for a uniform ultrafilter on $\mathcal{P}_\kappa \kappa^{+3}$:

THEOREM 4. *Let κ be supercompact and $\lambda > \kappa$ huge. Then some poset forces “ κ is supercompact and $\mathcal{P}_\kappa \kappa^{+3}$ carries a uniform κ -complete ultrafilter U with $j_U(\kappa) < \kappa^{+4}$.”*

First we fix our notation. Let μ be regular and $\lambda > \mu$ inaccessible. By $C(\mu, \lambda)$ we denote the Levy collapse forcing “ $\lambda = \mu^+$,” i.e. the set of functions

$p : d \times \delta \rightarrow \lambda$ with $\forall \xi, \zeta p(\xi, \zeta) \in \xi \cup \{0\}$ for some $d \in \mathcal{P}_\mu \lambda$ and $\delta < \mu$ ordered by reverse inclusion. Also we set $A(\mu) = (\overset{< \mu^+}{2}, \supset)$, which forces “ $2^\mu = \mu^+$.” Let P and Q be posets. We write a P -name as x , \dot{x} or \dot{x}^P depending on the context. For example $\dot{C}(\mu, \lambda)^P$ denotes a P -name for the Levy collapse. We let “ $P \leq Q$ ” mean that P is a complete suborder of Q . Then $\dot{C}(\mu, \lambda)^P$ is naturally a Q -name whose interpretation $(\dot{C}(\mu, \lambda)^P)_G$ under a V -generic $G \subset Q$ would be $C(\mu, \lambda)^{V[G \cap P]}$. Also the iteration $Q * \dot{R}^P$ for a P -name \dot{R}^P for a poset makes sense, which we redefine equivalently as the set of canonical representatives from $Q \times \{\dot{p}^P : \Vdash_P \text{“}\dot{p}^P \in \dot{R}^P\text{”}\}$.

For the rest of this section we assume familiarity with the methods of Silver’s reverse Easton forcing (see Baumgartner’s exposition [3]) and of Kunen’s universal collapse [6]. We incorporate Laver’s chain condition argument [8] to generalize Magidor’s universal collapse [9] as follows:

LEMMA 4. *Let $\lambda > \kappa$ be inaccessible. Then there is a κ -directed closed poset $P \subset V_\lambda$ satisfying λ -c.c. and $\Vdash_P \text{“}\lambda = \kappa^+\text{”}$ such that $Q * \dot{C}(\mu, \lambda)^Q \leq P$ for any regular $\mu \in \lambda - \kappa$ and $Q \leq P$ of size $< \mu$.*

PROOF. Inductively we build a $< \kappa$ -support iteration $(P_\gamma : \gamma < \lambda)$ of posets and for $\alpha < \lambda$ a list $\{(\mu_{\alpha\beta}, Q_{\alpha\beta}) : \beta < \lambda\}$ of all pairs (μ, Q) of a regular $\mu \in \lambda - \kappa$ and $Q \leq P_\alpha$ of size $< \mu$, so that $P_{\gamma+1} \simeq P_\gamma * \dot{C}(\mu_{\alpha\beta}, \lambda)^{Q_{\alpha\beta}}$ when $\pi(\alpha, \beta) = \gamma$, where $\pi : \lambda^2 \rightarrow \lambda$ is the canonical bijection. We claim that $P = \bigcup_\gamma P_\gamma$ is as desired.

To see κ -directed closure, fix a directed $D \in \mathcal{P}_\kappa P$. Inductively construct a common extension $p \in P$ of D with support $s = \bigcup_{q \in D} \text{supp}(q)$ as follows: Suppose that $p|_\gamma$ with $\gamma = \pi(\alpha, \beta) \in s$ is defined and $p|_\gamma \leq q|_\gamma$ for any $q \in D$. Then a reduction $r \in Q_{\alpha\beta}$ of $p|_\gamma$ forces in $Q_{\alpha\beta}$ “ $\{q(\gamma) : q \in D\} \subset \dot{C}(\mu_{\alpha\beta}, \lambda)^{Q_{\alpha\beta}}$ is directed,” as $p|_\gamma$ forces the same formula in P_γ . Take a canonical $Q_{\alpha\beta}$ -name $p(\gamma)$ with $r \Vdash_{Q_{\alpha\beta}} \text{“}p(\gamma) \leq q(\gamma)\text{”}$ for any $q \in D$. Then $p|_{(\gamma+1)} \leq q|_{(\gamma+1)}$ for any $q \in D$, as desired.

Next we have $P_\gamma \subset V_\lambda$ inductively, as canonical representatives from $\{\dot{p}^Q : \Vdash_Q \text{“}\dot{p}^Q \in \dot{C}(\mu, \lambda)^Q\text{”}\}$ belong to V_λ for such a pair (μ, Q) as above.

Also we get the universality property by $Q * \dot{C}(\mu, \lambda)^Q \leq P_{\pi(\alpha, \beta)+1}$ for a pair $(\mu, Q) = (\mu_{\alpha\beta}, Q_{\alpha\beta})$.

To see λ -c.c., fix $X \in [P]^\lambda$. Take $Y \in [X]^\lambda$ and $s \in \mathcal{P}_\kappa \lambda$ with $\gamma = \sup s + 1$ such that $\text{supp}(p) \cap \gamma = s$ for any $p \in Y$ and $\{\text{supp}(p) - \gamma : p \in Y\}$ is pairwise disjoint. Set $\mu = \sup\{\mu_{\alpha\beta} : \pi(\alpha, \beta) \in s\} < \lambda$. For $p \in Y$ we have $d_p \in \mathcal{P}_{\mu^+} \lambda$ with $\Vdash_{Q_{\alpha\beta}} \text{“}\text{dom } p(\pi(\alpha, \beta)) \subset d_p \times \mu\text{”}$ for any $\pi(\alpha, \beta) \in s$. Take $d_{p\alpha\beta} \in \mathcal{P}_{\mu_{\alpha\beta}} \lambda$ with $\Vdash_{Q_{\alpha\beta}} \text{“}\text{dom } p(\pi(\alpha, \beta)) \subset d_{p\alpha\beta} \times \mu_{\alpha\beta}\text{”}$ for any $\pi(\alpha, \beta) \in s$ and set $d_p = \bigcup \{d_{p\alpha\beta} : \pi(\alpha, \beta) \in s\}$. Take $Z \in [Y]^\lambda$ such that $\{d_p : p \in Z\}$ forms a Δ -system

with root d . The standard argument yields distinct $p, q \in Z$ such that $\Vdash_{\mathcal{Q}_{\alpha\beta}} "p(\pi(\alpha, \beta)) \mid (d \times \mu) = q(\pi(\alpha, \beta)) \mid (d \times \mu)"$ for any $\pi(\alpha, \beta) \in s$. Now p and q are the desired compatible elements in X by the compatibility of $p \mid \gamma$ and $q \mid \gamma$ in P_γ .

Finally $\Vdash_P "\lambda = \kappa^+"$ follows from $C(\kappa, \lambda) \leq P$ and λ -c.c. of P . \square

PROOF OF THEOREM 4. We may assume that κ remains supercompact in any κ -directed closed extension of the universe V , for Laver's poset [7] forcing this property is small enough to preserve hugeness of λ . We may also assume $2^\lambda = \lambda^+$ in V , as we show next.

Let $j: V \rightarrow M$ witness hugeness of λ . Construct a forcing iteration $(P_\alpha: \alpha \leq j(\lambda) + 1)$ taking direct limits at inaccessible steps and inverse limits otherwise so that $P_{\alpha+1} \simeq P_\alpha * \dot{A}(\alpha)^{P_\alpha}$ when $\alpha \geq \kappa$ is inaccessible, and $P_{\alpha+1} \simeq P_\alpha$ otherwise.

Fix a V -generic $G_\lambda \subset P_\lambda$ and $V[G_\lambda]$ -generic $G_{\lambda+1}^\lambda \subset A(\lambda)^{V[G_\lambda]}$. In $V[G_{\lambda+1}] = V[G_\lambda * G_{\lambda+1}^\lambda]$, set $P_{j(\lambda)}^{\lambda+1} = P_{j(\lambda)}/G_{\lambda+1}$. Fix a $V[G_{\lambda+1}]$ -generic $G_{j(\lambda)}^{\lambda+1} \subset P_{j(\lambda)}^{\lambda+1}$ and work in $V[G_{j(\lambda)}] = V[G_{\lambda+1} * G_{j(\lambda)}^{\lambda+1}]$. By abuse of notation $j: V \rightarrow M$ lifts to $j: V[G_\lambda] \rightarrow M[G_{j(\lambda)}]$ by $j''G_\lambda = G_\lambda$ and $j(P_\lambda) = P_{j(\lambda)}$, and $j^{(\lambda)}M[G_{j(\lambda)}] \subset M[G_{j(\lambda)}]$ by $j^{(\lambda)}M \subset M$ and $j(\lambda)$ -c.c. of $P_{j(\lambda)}$ in V . Hence $p = \bigcup j''G_{\lambda+1}^\lambda \in j(A(\lambda)^{V[G_\lambda]})$, as $j''G_{\lambda+1}^\lambda \subset j(A(\lambda)^{V[G_\lambda]})$ is directed and of size $\leq \lambda^+$. Fix a $V[G_{j(\lambda)}]$ -generic $p \in G_{j(\lambda)+1}^{j(\lambda)} \subset j(A(\lambda)^{V[G_\lambda]})$. We claim that $V[G_{j(\lambda)+1}] = V[G_{j(\lambda)} * G_{j(\lambda)+1}^{j(\lambda)}]$ is the desired model.

First observe $2^\lambda = \lambda^+$, as $2^\lambda = \lambda^+$ in $V[G_{j(\lambda)}]$ by $2^\lambda = \lambda^+$ and λ^+ -closure of $P_{j(\lambda)}^{\lambda+1}$ in $V[G_{\lambda+1}]$. Next we construct a normal ultrafilter U on $[j(\lambda)]^\lambda$.

Again by abuse of notation $j: V[G_\lambda] \rightarrow M[G_{j(\lambda)}]$ lifts to $j: V[G_{\lambda+1}] \rightarrow M[G_{j(\lambda)+1}]$ by $j''G_{\lambda+1}^\lambda \subset G_{j(\lambda)+1}^{j(\lambda)}$, and $j^{(\lambda)}M[G_{j(\lambda)+1}] \subset M[G_{j(\lambda)+1}]$ by $j^{(\lambda)}M[G_{j(\lambda)}] \subset M[G_{j(\lambda)}]$ and $j(\lambda)^+$ -closure of $j(A(\lambda)^{V[G_\lambda]})$ in $V[G_{j(\lambda)}]$. Hence we have a common extension $q \in j(P_{j(\lambda)}^{\lambda+1})$ of a directed $j''G_{j(\lambda)}^{\lambda+1} \subset j(P_{j(\lambda)}^{\lambda+1})$ of size $\leq j(\lambda)$ by $j(\lambda)^+$ -directed closure of $j(P_{j(\lambda)}^{\lambda+1})$. We claim that $q \Vdash$ "there is a $V[G_{j(\lambda)}]$ -normal $V[G_{j(\lambda)}]$ -ultrafilter \mathcal{U} on $([j(\lambda)]^\lambda)^{V[G_{j(\lambda)}]}$ ".

Fix a $V[G_{j(\lambda)+1}]$ -generic $q \in G_{j(\lambda)+1}^{j(\lambda)+1} \subset j(P_{j(\lambda)}^{\lambda+1})$. Then in $V[G_{j^2(\lambda)}] = V[G_{j(\lambda)+1} * G_{j^2(\lambda)}^{j(\lambda)+1}]$, $j: V[G_{\lambda+1}] \rightarrow M[G_{j(\lambda)+1}]$ lifts to $j: V[G_{j(\lambda)}] \rightarrow M[G_{j^2(\lambda)}]$ (once again by abuse of notation) by $j''G_{j(\lambda)}^{\lambda+1} \subset G_{j^2(\lambda)}^{j(\lambda)+1}$, and $\mathcal{U} = \{X \subset [j(\lambda)]^\lambda : j''j(\lambda) \in j(X)\}$ is as desired.

Now in $V[G_{j(\lambda)+1}]$, we have $2^{j(\lambda)} = j(\lambda)^+$, as $2^{<j(\lambda)} = j(\lambda)$ and $2^{j(\lambda)} = j(\lambda)^+$ in $M[G_{j(\lambda)+1}]$ by $2^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$ in $V[G_{\lambda+1}]$, and $j^{(\lambda)}M[G_{j(\lambda)+1}] \subset M[G_{j(\lambda)+1}]$. Let $\{X_\alpha: \alpha < j(\lambda)^+\}$ list the set $\mathcal{P}([j(\lambda)]^\lambda) = \mathcal{P}([j(\lambda)]^\lambda)^{V[G_{j(\lambda)}]}$ and a $j(P_{j(\lambda)}^{\lambda+1})$ -name $\dot{\mathcal{U}}$ witness the above claim. By $j(\lambda)^+$ -closure of $j(P_{j(\lambda)}^{\lambda+1})$ build a descending $\{q_\alpha: \alpha < j(\lambda)^+\} \subset j(P_{j(\lambda)}^{\lambda+1})$ so that $q_\alpha \leq q$ decides " $X_\alpha \in \dot{\mathcal{U}}$ " for any $\alpha < j(\lambda)^+$. Then $U = \{X_\alpha: q_\alpha \Vdash "X_\alpha \in \dot{\mathcal{U}}"\}$ is as desired.

Now we restart from the universe V as above. Let $j : V \rightarrow M$ witness hugeness of λ and P be as in Lemma 4. We show that $P * \dot{C}(\lambda^+, j(\lambda))^P$ is the desired poset. First note in V , $P * \dot{C}(\lambda^+, j(\lambda))^P \leq j(P)$, as $j(P)$ has the universality property by $j^{(\lambda)}M \subset M$, and $P \leq j(P)$ by λ -c.c. of P and $P \subset V_\lambda$.

Fix a V -generic $G_P \subset P$ and a $V[G_P]$ -generic $G_C \subset C(\lambda^+, j(\lambda))^{V[G_P]}$. We show that in $V[G_P * G_C]$, $R = j(P)/G_P * G_C$ forces “there is a uniform $V[G_P * G_C]$ - λ -complete $V[G_P * G_C]$ -ultrafilter \mathcal{U} on $j(\lambda)$.”

Fix a $V[G_P * G_C]$ -generic $G_R \subset R$ and work in $V[G_{j(P)}] = V[G_P * G_C * G_R]$. By abuse of notation $j : V \rightarrow M$ lifts to $j : V[G_P] \rightarrow M[G_{j(P)}]$ by $j''G_P = G_P$, and $j^{(\lambda)}M[G_{j(P)}] \subset M[G_{j(P)}]$ by $j^{(\lambda)}M \subset M$ and $j(\lambda)$ -c.c. of $j(P)$ in V . Hence $r = \bigcup j''G_C \in j(C(\lambda^+, j(\lambda))^{V[G_P]})$, as $j''G_C \subset j(C(\lambda^+, j(\lambda))^{V[G_P]})$ is directed and of size $\leq j(\lambda)$. We claim that $r \Vdash$ “there is a uniform $V[G_P * G_C]$ - λ -complete $V[G_P * G_C]$ -ultrafilter \mathcal{V} on $j(\lambda)$.”

Fix a $V[G_{j(P)}]$ -generic $r \in G_{j(C)} \subset j(C(\lambda^+, j(\lambda))^{V[G_P]})$. Then in $V[G_{j(P)} * G_{j(C)}]$, $j : V[G_P] \rightarrow M[G_{j(P)}]$ lifts to $j : V[G_P * G_C] \rightarrow M[G_{j(P)} * G_{j(C)}]$ (again by abuse of notation) by $j''G_C \subset G_{j(C)}$, and $\mathcal{V} = \{X \subset j(\lambda) : \sup j''j(\lambda) \in j(X)\}$ is as desired.

Work again in $V[G_{j(P)}]$. We have $2^{j(\lambda)} = j(\lambda)^+$, as $2^{j(\lambda)} = j(\lambda)^+$ in $M[G_{j(P)}]$ by $2^\lambda = \lambda^+$ in $V[G_P]$, and $j^{(\lambda)}M[G_{j(P)}] \subset M[G_{j(P)}]$. Let $\{X_\alpha : \alpha < j(\lambda)^+\}$ list the set $\mathcal{P}(j(\lambda))^{V[G_P * G_C]}$ and a $j(C(\lambda^+, j(\lambda))^{V[G_P]})$ -name $\dot{\mathcal{V}}$ witness the above claim. By $j(\lambda)^+$ -closure of $j(C(\lambda^+, j(\lambda))^{V[G_P]})$ build a descending $\{r_\alpha : \alpha < j(\lambda)^+\} \subset j(C(\lambda^+, j(\lambda))^{V[G_P]})$ so that $r_\alpha \leq r$ decides “ $X_\alpha \in \dot{\mathcal{V}}$ ” for any $\alpha < j(\lambda)^+$. Then $\mathcal{U} = \{X_\alpha : r_\alpha \Vdash “X_\alpha \in \dot{\mathcal{V}}”\}$ is as desired.

Now work in $V[G_P * G_C]$. Let an R -name $\dot{\mathcal{U}}$ witness the claim. Then the filter $F = \{X \subset j(\lambda) : \Vdash “X \in \dot{\mathcal{U}}”\}$ is λ -complete and contains all final segments. Also $|j^{(\lambda)}\kappa/F| \leq |\text{ro}(R)|^\kappa \leq j(\lambda)$, as the map $[f] \mapsto (\|f^{-1}\{\alpha\} \in \dot{\mathcal{U}}\| : \alpha < \kappa)$ is injective and $|\text{ro}(R)| \leq j(\lambda)^{<j(\lambda)} = j(\lambda)$. By compactness of κ we have a uniform κ -complete ultrafilter $U \supset F$ on $j(\lambda) = \kappa^{+3}$ with $j_U(\kappa) < \kappa^{+4}$, and hence on $\mathcal{P}_\kappa \kappa^{+3}$ through a bijection. \square

In conclusion we ask whether the integers 3 and 4 in Theorem 4 may be replaced by smaller ones.

References

- [1] Y. Abe, Strongly compact cardinals, elementary embeddings and fixed points, *J. Symbolic Logic* 49 (1984), 808–812.
- [2] J. Barbanel, Supercompact cardinals, elementary embeddings and fixed points, *J. Symbolic Logic* 47 (1982), 84–88.
- [3] J. Baumgartner, Iterated forcing, in: A. Mathias (ed.), *Surveys in Set Theory*, London Math. Soc. Lecture Note Ser. 87, Cambridge Univ. Press, Cambridge, 1983, pp. 1–59.

- [4] T. Frayne, A. Morel and D. Scott, Reduced direct products, *Fund. Math.* **51** (1962), 195–228.
- [5] A. Kanamori, *The Higher Infinite*, Springer, Berlin, 1994.
- [6] K. Kunen, Saturated ideals, *J. Symbolic Logic* **43** (1978), 65–76.
- [7] R. Laver, Making the supercompactness of κ indestructible under κ -directed closed forcing, *Israel J. Math.* **29** (1978), 385–388.
- [8] R. Laver, An $(\aleph_2, \aleph_2, \aleph_0)$ -saturated ideal on ω_1 , in: D. van Dalen et al. (eds.) *Logic Colloquium '80*, North-Holland, Amsterdam, 1982, pp. 173–180.
- [9] M. Magidor, On the existence of nonregular ultrafilters and the cardinality of ultrapowers, *Trans. Amer. Math. Soc.* **249** (1979), 97–111.
- [10] R. Solovay, Strongly compact cardinals and the GCH, in: L. Henkin et al. (eds.), *Proceedings of the Tarski Symposium*, Proc. Symp. Pure Math. 25, Amer. Math. Soc., Providence, 1974, pp. 365–372.

Yoshihiro Abe
Department of Mathematics
Kanagawa University
Yokohama, 221
Japan

Masahiro Shioya
Institute of Mathematics
University of Tsukuba
Tsukuba, 305
Japan