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Gauge Invariance of Fractionally Charged Quasiparticles and Hidden Topological $Z_n$ Symmetry

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Using the braid-group formalism we study the consequences of gauge invariance for fractionally charged anyonic quasiparticles in a two-dimensional multiply connected system. It is shown that gauge invariance requires multicomponent wave functions, and leads to the emergence of a hidden topological $Z_n$ symmetry with associated quantum number and unavoidable occurrence of level crossings for many-body eigenstates. In certain situations, it relates the fractional charge to anyon statistics. The implications for the fractional quantum Hall effect are also discussed.

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Gauge invariance is known to play a fundamental role in macroscopic quantum phenomena, such as flux quantization\(^1\) in a superconducting ring and the integral quantum Hall effect.\(^2\) However, there is little understanding of its role in the fractional quantum Hall effect (FQHE),\(^3\) though some suggestions have been made in the literature.\(^4-6\) Recently, direct experimental evidence\(^7\) has been reported for a quasiparticle of fractional charge $e^* = e/3$ in the FQHE with the Landau-level filling factor $\nu = 1/3$, as predicted by Laughlin’s theory.\(^8,9\) This makes urgent the need for a better theoretical understanding of gauge invariance for fractionally charged quasiparticles.

Consider a cylindrical (or toroidal) system with a magnetic flux $\Phi$ through the hole. Gauge invariance\(^1,10\) implies that all physical properties of the system are periodic functions of $\Phi$ with the period (flux quantum) $\Phi_0 = hc/e$, where $e$ is the constituent (electron) charge. The problem is to determine what conditions can a system of quasiparticle excitations with fractional charge $e^*$ have, as required by gauge invariance, at $q$ times, if $e^* = e/q$ smaller period $\Phi_0$ than the naturally expected $\Phi_0^* = hc/e^*$.

To study this problem, we use the braid-group formalism\(^11\) on a cylinder\(^12\) (or on a torus\(^13\)), appropriate for anyonic\(^14,15\) quasiparticles in FQHE.\(^16-18\) We will show that if the anyon system is described by a one-component wave function, there is no period smaller than $\Phi_0^*$. However, with the use of a multicomponent wave function, we can derive the condition for the existence of a smaller period, which lends itself, in the cylinder case, the emergence of a hidden topological $Z_n$ symmetry with associated quantum number, the $n$-ality, in the spectrum for the many-body eigenstates. Here $n$ is the smallest integer satisfying $n(e^*/e) = \text{integer}$. On a torus, gauge invariance implies a relation between $e^*$ and the statistics $\theta$ for an irreducible braid-group representation (BGR), and in the thermodynamic limit there are two noncommuting topological symmetries which lead to the groundstate degeneracy. The revelation of topological $Z_n$ symmetry lends strong support to a “broken-symmetry” scenario for the FQHE, proposed by Tao and Wu\(^4\) some time ago and refined recently by Thouless.\(^6\)

According to the path-integral formalism,\(^11\) it is the braid group which plays the same basic role for anyons as the permutation group for usual bosons and fermions. The braid group is nothing but the first homotopy group of the $N$-anyon configuration space; and the anyon wave function forms a representation of it. On a cylinder, the braid-group generators consist of not only the usual $\gamma_i$ ($i = 1, \ldots, N-1$), which interchanges the $i$th and $(i+1)$th particles, but also of additional $\rho_j$ ($j = 1, \ldots, N$) which represents moving a particle along a simple loop around the hole once with $j-1$ particles to its left (see Fig. 1). These generators satisfy

$$
\gamma_i \gamma_j = \gamma_j \gamma_i \quad (i \neq j \pm 1), \quad \gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1}, \quad (1)
$$
$$
\rho_i \rho_j = \rho_j \rho_i, \quad \rho_i \gamma_j = \gamma_j \rho_i \gamma_{i-j} \quad (i \neq j, j-1), \quad (2)
$$
$$
\rho_{j+1} = \gamma_j \rho_j \gamma_j \quad (3)
$$

1D unitary BGR's are characterized\(^12\) by two parameters $\theta$ and $\Phi$:

$$
\sigma_j = \exp(i\theta), \quad (4)
$$
$$
\rho_j = \exp[i(2\theta(j-1)+2\pi e^* \Phi/hc)].
$$

It is easy to see that the minimal change in $\Phi$ that leads to the same BGR is $\Phi_0$. This is compatible with gauge invariance (the existence of another period $\Phi_0$), if and

![FIG. 1. Braid-group generators on a cylinder. $\Phi$ is the central flux through the hole.](image-url)
only if $e^\ast$ is an integral multiple of the constituent charge $e$. Conversely, when the anyon quasiparticles carry fractional charge $e^\ast$, for the system to possess a period smaller than $\Theta_0$, the wave function must have more than one component. So we assume the $M$-dimensional representation for $\sigma_i$ and $\rho_i$. We assume the anyons obey scalar statistics: $\sigma_i = e^{i\rho} I_M$ with $I_M$ being the $M \times M$ unit matrix. The representation of $\rho_j$ has a factorized $\Phi$ dependence:

$$\rho_j(\Phi) = \exp[i2\pi e^\ast \Phi/h\epsilon] T_j,$$

where $T_j$ are $\Phi$-independent $M \times M$ matrices satisfying $T_{j+1} = T_j e^{2i\theta}$. If $e^\ast/e = m/n$ with $m,n$ mutually prime, a smaller period $\Theta_0$ implies the minimal period $\Theta_0^\ast/n$ and the unitary equivalence of the two BGR’s,

$$\rho_j(\Phi + \Theta_0^\ast/n) = \rho_j(\Phi) \ (j = 1, \ldots, N).$$

It is sufficient to have this only for $j = 1$. Multiplying the eigenvalues of $T_1$ by exp$(i2\pi/n)$ should just shuffle them, so $M$ must be divisible by $n$. If $M = n$, the eigenvalues of $T_1$ must consist of $n$ phases $e^{i\lambda_0}$ cyclic under the multiplication of exp$(i2\pi/n)$: $\lambda_{k+1} = \lambda_k + 2\pi k/n$ ($k = 0, 1, \ldots, n - 1$) with $\lambda_0$ a constant determined by the underlying microscopic physics. While reducible as a BGR, this situation is irreducible under the large gauge transformation which shifts $\Phi$ by $\Theta_0^\ast/n$. The form of $\rho_j(\Phi)$ is thus determined as

$$\rho_j(\Phi) = \exp[i\lambda_0 + i2\pi(j-1) + i2\pi e^\ast \Phi/h\epsilon] W,$$

with $W$ a diagonal $n \times n$ matrix given by

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 0 \end{bmatrix}.$$

or in an appropriate basis,

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 0 \end{bmatrix}.$$

In such a basis, each base state changes into another if one moves one anyon around the hole once, and returns to itself after $n$ rounds (up to a possible phase). For a reducible case with $M = nM'$, there are $M'$ copies like this.

Similarly, we can discuss the torus case, in which anyons require a multicompontent wave function. If the latter forms an irreducible BGR, gauge invariance relates the anyon charge to its (scalar) statistics $\Theta$. In fact, for anyons on a torus with $\sigma_i = e^{i\rho} I_M$, $(\Theta/\pi) \Phi_0^\ast$ is always a period for $\Phi$. Note that on a torus, besides $\sigma_i$, there are generators $\tau_i, \rho_i$ ($i = 1, \ldots, N$) corresponding to moving the $i$th particle along one of the fundamental noncontractible loops, as shown in Fig. 2. They satisfy $\tau_{i+1} = \tau_i e^{-2i\theta}$, $\rho_{i+1} = \rho_i e^{2i\theta}$,

$$\tau_i \rho_i = \rho_i \tau_i$$

and $\tau_i$ ($\rho_i$) have a factorized $\Phi_i (\Phi_i)$ dependence as in

(5). Changing $\Phi_i$ by $(\Theta/\pi) \Phi_0^\ast$ just gives rise to a phase factor $\exp(2i\theta)$, which shifts $\tau_i$ to $\tau_{i-1}$. This only turns the BGR into an equivalent one and therefore does not alter any physical properties. For an irreducible BGR, $M = q$ for $\Theta = \pi(p/q)$ with $p$ and $q$ mutually prime, there are no other periods and the minimal period is $\Phi_0^\ast/q$. So the period $\Phi_0$ required by gauge invariance must be an integral multiple of the latter, and therefore $e^\ast/e = m/q$ with integer $m$. If the anyon charge does not satisfy this condition, one has to use a collection of copies of the same $q$-dimensional irreducible BGR. For the case irreducible under gauge transformations, the number of components of the wave function has to be $M = nq$, where $n$ is the “denominator” of $e^\ast/e$, and the minimal period of $\Phi$ is $\Phi_0^\ast/L$, where $L$ is the least common multiple of $n$ and $q$.

The above conditions have very profound physical implications. First, let us consider the cylinder case. For fractionally charged quasiparticles to exist, their wave function has to have $n$ components with $n > 1$. To specify their many-body states, besides the positions one needs an extra index, the index of components (or “sheets”) $s = 1, \ldots, n$. We emphasize that this index generally is not associated with individual quasiparticles. Equation (7) shows that the operation of moving one anyon (with others held fixed) around the hole is given by, up to some phase, the winding operator $W$ which acts on the sheet indices. Note that $W^n = 1$, so its eigenvalues are $n$ cyclic like $Z_n$. Normally the Hamiltonian $H$, no matter how complicated it may be, with various interactions, impurities or defects, or external field all included, involves only the coordinates (or other degrees of freedom like spin) of individual particles, but not the sheet index $s$ of the many-body wave function. So $H$ always commutes with $W$, even if the tunneling process involving the creation and annihilation of a virtual pair of quasiparticle and quasihole propagating across the cylinder is included. The eigenvalues of $W$, $e^\pm (i2\pi k/n)$ or simply $k$ (mod $2n$), give us a good quantum number, the so-called $n$-ality, for the many-body energy eigenstates.

Because the $(k, \Phi)$ dependence of the eigenvalues of $\rho_i$ is through the combination $\Phi + k \Phi_0^\ast/n$, the energy or any
physical property satisfies
\[ E(k, \Phi, \{a\}) = f(\Phi + k\Phi_0^\pi/n, \{a\}) , \]
where \( \{a\} \) is a set of usual quantum numbers. Thus the
energy spectrum is actually a collection of \( n \) sectors, each
corresponding to a one-component system with a central
flux differing from each other by \( \Phi_0^\pi/n \), and therefore
admits a smaller period \( \Phi_0^\pi/n \).

Moreover, (11) implies the existence of level crossings
or spectral flow, since the energy \( E(k, \Phi, \{a\}) \) with \( \{a\} \)
fixed is not necessarily a function of \( \Phi + k\Phi_0^\pi/n \) with
period \( \Phi_0^\pi/n \). Because the topological \( n \)-ality is a good
quantum number that can never be violated by impurities,
defects, or whatever, a gap can never be open at the
level-crossing points unless the two levels involved have
the same \( n \)-ality. This unavoidableness of level crossing
violates the Von Neumann–Wigner theorem,\textsuperscript{21} which
asserts that generically levels do not cross when two pa-
rameters in a complex Hamiltonian are varied. Indeed
this theorem is not applicable here, because we have a
topological quantum number which does not exist gener-
cally.

A numerical result showing the pattern of level cross-
ings is given by Fig. 3. (The details will be presented
elsewhere.)\textsuperscript{20} Note in particular that the three
lowest levels flow into each other with a period \( \frac{1}{3} \), but for each
fixed level the period is 3 times larger. Though this kind
of pattern is not typical for an anyon system, there are
good reasons to believe that the ground states of a cylin-
drical FQH system have such a pattern of level cross-
ings. First, we argue that the ground state of a cylin-
drical FQH system must also carry a topological \( n \)-ality \( k \),
since it exists for each excited state containing fraction-
ally charged quasiparticles but is not associated with
individual quasiparticles. Furthermore, at least when
\( v = 1/q \) (\( q \) odd), the energy of a given ground state
should be periodic in the flux with period \( \Phi_0^\pi = qhc/e \), as
Thouless recently has shown\textsuperscript{6} for the Laughlin state on a
sphere with two small holes at the poles. Then the ex-
istence of a period \( \Phi_0^\pi = \Phi_0^\pi/q \) implies that each time \( \Phi \)
changes by \( \Phi_0 \), there is a ground state equivalent to the
original one. Hence there must be a group of \( q \)
equivalent ground states whose energies depend on flux
\( \Phi \) and flow into each other, in a manner like the lowest
two levels in Fig. 3. Each of them has a different \( q \)-
ality. Since the topological \( Z_q \) symmetry prevents tran-
sitions between the equivalent ground states, energy-level
repulsion can never happen and level crossing will persist
even in the presence of impurities, defects, and so on.
Such a scenario is essentially what was proposed by Tao
and Wu\textsuperscript{4} several years ago and recently refined by Thou-
less.\textsuperscript{6} This has been shown by Niu, Thouless, and Wu\textsuperscript{22}
to be sufficient to give the fractional quantization of the
Hall conductance in the topological approach. We have
a broken symmetry in the sense that originally one would
expect that the winding operator \( W \) is proportional to the
unit matrix, but actually it has unequal eigenvalues. The
previous failure to uncover this quantum number is due
to the fact that it does not exist on a disk or sphere.

A consequence of our results is that the \( v = 1/q \) FQH
edge states on a cylinder must, like the bulk states, carry
a \( Z_q \)-like quantum number, as pointed out by Wen\textsuperscript{23} in
a different approach.

When put on a torus,\textsuperscript{24} the Laughlin states for \( v = 1/q \)
with \( q \) odd correspond to an irreducible BGR.\textsuperscript{13} So from
gauge invariance alone we can infer that the fractional
charge of quasiparticles must be an integral multiple of
\( e/q \). In the torus case, we have two noncommuting wind-
ing operators \( W_1 \) and \( W_2 \), similar to (7), respectively,
corresponding to moving an anyon along different funda-
mental loops.\textsuperscript{25} \( (W_1 W_2)^\pi = W_2 W_1 e^{2\pi i} \), since \( \tau_1 \) and \( \rho_1 \)
satisfy a similar relation.\textsuperscript{13} They are not symmetries of
a finite system: Because of tunneling of virtual quasipar-
icle and quasi-hole pairs across the torus, \( H \) should con-
tain terms proportional to \( W_1 \) and \( W_2 \). But such terms are
negligible in the thermodynamic limit,\textsuperscript{26} so we have
two noncommuting symmetries, and the spectrum has
\( q \)-fold exact degeneracy for each level of anyons. As
argued before, we expect the same is true for the electronic
ground states that can support fractionally charged
anyonic excitations. Note that our topological sym-
metries giving rise to the ground-state degeneracy are
not the magnetic translation operators proposed in the
literature.\textsuperscript{26,27} The latter can be broken by impurities,
but ours cannot.

In summary, we have shown that gauge invariance im-
poses significant constraints on the structure of the Hil-
bert space and the spectrum of a system if it supports
fractionally charged (anyonic) excitations. The fascinat-
ning consequences include the necessity of multicom-
ponent wave functions, the emergence of hidden topolog-
ical \( Z_q \) symmetry with associated quantum number (the
\( n \)-ality) for many-body eigenstates, and the unavoidable

![Figure 3: Spectral flow of a three-anyon system on a 3x3 cylindrical lattice. The statistics is \( \pi/5 \) and the charge is \( e/3 \).](image-url)
occurrence of level crossings. In certain situations, gauge invariance relates the fractional charge to anyon
statistics. Our topological discussion is quite general and model independent, but does not tell what underlying
dynamical mechanism will give rise to the spectrum required by gauge invariance. Finally, it would be
interesting to speculate on the possible relevance to quarks, which are also fractionally charged and carry a triality.
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7J. A. Simmons, H. P. Wei, L. W. Engel, D. C. Tsui, and M. Shayegan, Phys. Rev. Lett. 63, 1731 (1989); see also R. G.
8R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983); F. D. M.
9Generally the elementary quasiparticle charge is $e^* = e/q$
when $v$ is near $p/q$ with $q$ odd.
12Y. Hatsugai, M. Kohmoto, and Y. S. Wu, Institute for
Solid State Physics, University of Tokyo, Report Ser. A, No.
2306, 1990 (to be published).
14F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982); 49, 957
(1982).
19X. G. Wen, E. Dagotto, and E. Fradkin, Institute for
Theoretical Physics Report No. ITP-90-05, Institute for
Advanced Study Report No. IASSNS-HEP-90/38 (to be
published).
20Y. Hatsugai, M. Kohmoto, and Y. S. Wu (to be published).
21J. Von Neumann and E. P. Wigner, Z. Phys. 30, 467
(1929).
24F. D. M. Haldane and E. H. Rezayi, Phys. Rev. B 31, 2529
(1985).
25A similar situation occurs in U(1) Chern-Simons theory;