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THE SERRE DUALITY THEOREM
FOR A NON-COMPACT WEIGHTED CR MANIFOLD

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Abstract. It is proved that the Hodge decomposition and Serre duality hold on a non-compact weighted CR manifold with negligible boundary. A complete CR manifold has negligible boundary. Some examples of complete CR manifolds are presented.

1. Introduction

Let $M$ be a strongly pseudo-convex CR manifold, an s.p.c. CR manifold for short, without boundary. A weighted CR manifold is an s.p.c. CR manifold endowed with a measure $\mu$, which has a smooth positive density $\eta$ with respect to the volume form of the CR structure. Then the space $(M, \mu)$ has a natural weighted Kohn Laplacian $\square_{\mu}$, which we call the Witten-Kohn Laplacian.

In this article, we are interested in Serre duality and Hodge decomposition on a non-compact weighted CR manifold. The Serre duality of a compact s.p.c. CR manifold was proved by Tanaka [15] for the case of a trivial line bundle, and recently, the first and the third named authors generalized it to any holomorphic vector bundle $E$ [8]. On the other hand, Kohn’s Hodge decomposition for a compact s.p.c. CR manifold was extended to a general s.p.c. CR manifold with negligible boundary (Definition 2.4) when $E$ is a trivial line bundle by the second author [13].

The aim of the present article is to extend these results to an arbitrary holomorphic vector bundle $E$ over a general weighted CR manifold with negligible boundary, and to relate them to each other. Namely, by denoting $H^{p,q}(E)$ the space of $E$-valued $L^2$-harmonic forms of $(p,q)$-type, we will show

Main Theorem. Let $M$ be a $(2n-1)$-dimensional weighted CR manifold with negligible boundary, and let $E$ be a holomorphic vector bundle over $M$. Then the $L^2$-Hodge decomposition

\[ L^2(\Omega^{\bullet, q}(E)) = \mathbb{H}^{\bullet, q} \oplus \text{range } (\overline{\partial}^{q-1})^{L^2} \oplus \text{range } (\delta_{\mu}^{q+1})^{L^2} \]

holds for $0 < q < n - 1$, and the Serre duality

\[ \#_\mu : H^{p,q}(E) \cong H^{n-p,n-(q+1)}(E^*) \]

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holds for every $0 \leq p, q \leq n - 1$, where $E^*$ is the dual bundle of $E$ and $\ast_\mu$ is the complex-conjugate weighted Hodge star operator. In addition, it follows that

$$\ker \left( \partial^* \right) \bigg/ \text{range} \left( \partial^{-1} \right)^{L^2} \cong \mathbb{H}^{p,q}(E) \cong \mathbb{H}^{n-p,n-\left(q+1\right)}(E^*), \text{ for } 0 < q < n - 1.$$  

We say that $M$ is complete and $M$ is Riemannian complete if it is complete with respect to the Carnot-Carathéodory distance $d_{CC}$ and Riemannian distance $d_g$ associated to the CR structure, respectively (see Section 2). Then we have

**Theorem (Proposition 4.1).**

(i) If $M$ is Riemannian complete, then it is complete

(ii) If $M$ is complete, then $M$ has negligible boundary.

Therefore, the Main Theorem is applicable, for example, to the Heisenberg group, Sasakian space forms, spherical orbits, and unbranched covering over a compact s.p.c. CR manifold with any weight. These are very important s.p.c. CR manifolds, but they have been excluded from the literature because of their non-compactness (see Section 3). Two main points in the proof are: identification of the weak solution of the Laplace equation $\Box_\mu \alpha = 0$ with the harmonic form (Corollary 2.7); explicit expressions for $\partial$ and $\delta_\mu$ in terms of $\overline{\partial}_\mu$ and $\ast_\mu$ via $\ast_\mu$ (Lemma 3.3).

We organize the article in the following manner: In Section 2, we recall some necessary notions which we will use in the article. Some new results are given, including the self-adjointness of the $E$-valued Witten-Kohn Laplacian. In Section 3, we will prove the Serre duality, and finally, in Section 4, we present the examples.

2. **Strongly pseudo-convex CR manifolds**

This section contains preliminary results. We recall some definitions related to a strongly pseudo-convex CR manifold $M$, focusing on the construction of the space $\Omega^{p,q}(E)$ of $E$-valued $(p,q)$-forms, its holomorphic structure $\overline{\partial}^*$, and the Witten-Kohn Laplacian $\Box_\mu$. For a thorough discussion on a geometrical analysis of an s.p.c. CR manifold, we refer the reader to [5] and [15]. We also establish the essential self-adjointness of $\Box_\mu$ and characterize the harmonic forms as the solutions of the Laplace equation with respect to $\Box_\mu$. These results are important steps when we extend our knowledge of a compact manifold to a non-compact one.

A $(2n-1)$-dimensional **strongly pseudo-convex CR manifold $M$**, we call it simply an **s.p.c. CR manifold**, is an oriented smooth manifold which carries a structure $(P,J,\theta)$, that is: $P = \ker(\theta) \subset TM$ is an $(n-1)$-dimensional real subbundle with an almost complex structure $J : P \rightarrow P$ satisfying:

$[X,Y] - [JX,JY] - J([JX,Y] - J[X,JY]) \in \Gamma(TM/P)$, for $X,Y \in \Gamma(P)$,

and a contact form $\theta \in \Gamma((TM/P)^*)$ whose **Levi-form** $L(X,Y) = -d\theta(JX,Y)$, for $X,Y \in P$, is positive definite.

Consider the complexification of $J$ and its eigenspace $S = \{X - \sqrt{-1}JX : X \in P\} \subset \mathbb{C}TM$. Then $S \cap \overline{S} = \{0\}$ and $[\Gamma(S),\Gamma(S)] \subset \Gamma(S)$, where $\overline{S}$ is the complex conjugation of $S$. With the assumption of the strong convexity of $M$, there exist the following implications:

- a Riemannian metric $g = -d\theta + \theta \otimes \theta$;
- a volume form $dv = (n-1)! \theta \wedge (d\theta)^{n-1}$;
a distance $d_{CC}$ on $M$.

Indeed, since the Levi form $L$ is positive definite, it follows for $0 \neq X \in P_x$ that

$$0 \neq 2L(X, X) = -(JX(\theta(X))) - X(\theta(JX)) - \theta([JX, X]) = \theta([JX, X]).$$

This shows that $[JX, X] \notin P_x$, and hence, $P$ satisfies the Hörmander condition [7]. Due to the Chow theorem [4], $P$ implies a non-degenerate distance $d_{CC}$ on $M$ defined as

$$d_{CC}(x, y) := \sup\{u(x) - u(y) : u \in C^\infty(M), \|\pi\nabla u\|_{L^\infty} \leq 1\},$$

where $\pi : TM \to P$ is the projection with respect to $g$. We say $M$ is complete if the distance space $(M, d_{CC})$ is complete.

**Definition 2.1.** A complex vector bundle $E$ over $M$ is called holomorphic if it admits a linear differential operator $\overline{\partial}_E : \Gamma(E) \to \Gamma(E \otimes \overline{\mathcal{S}})$ satisfying:

1. $\overline{\partial}_u f u = f \overline{\partial}_u u + (\overline{f}) u$;
2. $\overline{\partial}_u (\overline{\partial}_u u) - \overline{\partial}_u (\overline{\partial}_u u) - \overline{\partial}_u (\overline{\partial}_u u) = 0$;

where $f \in C^\infty(M), u \in \Gamma(E)$, and $X, Y \in \Gamma(S)$, where $\overline{\partial}_u := \overline{\partial}_E u (X)$.

**Example 2.2 (E.g. [15]).** Let $M$ be a boundary of a strongly pseudoconvex complex manifold and $E$ be a holomorphic vector bundle on the neighbourhood of $M$. Then $E|_M$ is holomorphic in the above sense.

Hereafter, $E$ stands for a holomorphic vector bundle over $M$. Consider the vector bundle $\mathcal{T}M = CTM/\mathcal{S}$, which is holomorphic (e.g. p. 15 in [15]) together with the operator $\overline{\partial} = \overline{\partial}_M$:

$$\overline{\partial}_u = \omega((\overline{X}, Z)),$$

for $u \in \Gamma(\mathcal{T}M)$ with $Z \in \Gamma(CTM)$ such that $\omega(Z) = u$ and $X \in \Gamma(S)$. Here $\omega : \mathcal{T}M \to \mathcal{T}M$ is the canonical projection. The distinguished vector bundle $\mathcal{E} \otimes \wedge^p(\mathcal{T}M)^*$ with $0 \leq p \leq n - 1$ carries a holomorphic structure:

$$\overline{\partial}_E \otimes id_{\wedge^p} + id_E \otimes \overline{\partial}_{\wedge^p},$$

where $id$ is the identity operator on the indicated space, and $\wedge^p = \wedge^p \mathcal{T}M$. Hereafter we assume additionally that $(E, \overline{\partial}_E)$ is furnished with a smooth Hermitian fiber metric $(\cdot, \cdot)_E$. The bundle which we will study is

$$\Omega^{p,q}(E) = \Omega^{p,q}(M; E) = \Gamma(M; E \otimes \wedge^p \mathcal{T}M^* \otimes \wedge^q \overline{\mathcal{S}}),$$

with the holomorphic structure $\overline{\partial} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ defined as

$$(\overline{\partial} \alpha)(\overline{X}_1, \cdots, \overline{X}_{q+1}) := \sum (-1)^r \overline{\partial}_u \left( \alpha \left( \overline{X}_1, \cdots, \overline{X}_i, \cdots, \overline{X}_{q+1} \right) \right) + \sum_{i<j} (-1)^{i+j} \alpha \left( \overline{X}_i, \overline{X}_j, \overline{X}_1, \cdots, \overline{X}_i, \cdots, \overline{X}_j, \cdots, \overline{X}_{q+1} \right),$$

where $\overline{\partial}$ is the holomorphic structure of $E \otimes \wedge^p \mathcal{T}M^*$, $\alpha \in \Omega^{p,q}(E)$, and $X_1, \cdots, X_{q+1}$ belong to $\Gamma(S)$. If $E$ is the trivial line bundle, we simply denote $\Omega^{p,q}(M) = \Omega^{p,q}(M; \mathbb{C})$. Set

$$\Omega^{p,q}(E) = \bigoplus_q \Omega^{p,q}(E), \quad \Omega(E) := \bigoplus_q \Omega^{p,q}(E);$$

$$\Omega^{pq}_0(E) = \{ \alpha \in \Omega^{p,q}(E) \mid \alpha \text{ has compact support} \}.$$
Let $\eta$ be the weight, which is a positive smooth function on $M$, and consider the measure \( d\mu = \eta dv \). The associated inner product \((\alpha, \beta)\) of $\alpha, \beta \in \Omega_0^{p,q}(E)$ is
\[
(\alpha, \beta) = \int_M (\alpha, \beta)(x) \, d\mu(x),
\]
where \((\alpha, \beta)(x)\) is the pointwise inner product of $\alpha$ and $\beta$ at $x \in M$. Denote by \( \|\alpha\|_2 \) the norm $\sqrt{(\alpha, \alpha)}$, and by \( L^2(\Omega^{p,q}(E)) = L^2(\Omega^{p,q}(E), \mu) \) the set of square integrable $E$-valued measurable $(p,q)$-forms, which coincides with the completion of $\Omega_0^{p,q}(E)$ with respect to \( \| \cdot \|_2 \).

Let $\delta^\mu_q : \Omega^{*, q+1}(E) \to \Omega^{*, q}(E)$ be the formal adjoint of $\overrightarrow{\partial}_\mu$ in $L^2(\Omega(E))$. The Witten-Kohn Laplacian $\square^\mu_p : \Omega^{*, q}(E) \to \Omega^{*, q}(E)$ is defined by
\[
\square^\mu_p := \overrightarrow{\partial}^\mu \delta^\mu_q - \delta^\mu_q \overrightarrow{\partial}^\mu.
\]
In abbreviation, we remove the super index $q$ when the operator is acting on the space of mixed degree forms. The operator $\square^\mu_p$ is called subelliptic (e.g. [5], [15]) if there are positive numbers $\sigma$ and $C_\sigma$ such that
\[
\|\alpha\|_{(\sigma)}^2 \leq C_\sigma \left( \|\square^\mu_p \alpha\|_2^2 + \|\alpha\|_2^2 \right), \quad \text{for every } \alpha \in \Omega_0^{*, q}(E),
\]
where \( \cdot \) is the Sobolev norm of order $\sigma$.

**Proposition 2.3.** The Witten-Kohn Laplacian $\square^\mu_p$ is subelliptic for $0 < q < n - 1$.

**Proof.** By Lemma 3.3 the Kohn Laplacian $\square = \square_{dv}$, which is the Witten-Kohn Laplacian with $\eta = 1$, has the form:
\[
(2.3) \quad \square = z^* \overrightarrow{\partial}_* z + z^* \delta_* z,
\]
where $z = z_{dv}$ (see Definition 3.1), $\overrightarrow{\partial}_*$ is the holomorphic structure of $\Omega(E^*)$ and $\delta_*$ is its adjoint operator in $L^2(\Omega(E^*), dv)$. Again by Lemma 3.3 and (2.3),
\[
\square^\mu p \alpha = z^* \overrightarrow{\partial}_* z^* \alpha + \eta \overrightarrow{\partial}_* (\overrightarrow{\partial}_* \eta \wedge z^* \delta_* z) + z^* \delta_* (\eta \overrightarrow{\partial}_*(\overrightarrow{\partial}_* \eta \wedge z \alpha)) + z^* \delta_* (\eta^{-1} \overrightarrow{\partial}_*(\overrightarrow{\partial}_* \eta \wedge z \alpha)),
\]
for $\alpha \in \Omega(E)$. Therefore, $\square^\mu$ and $\square$ have the same principal symbols. Since $\square$ is subelliptic [10], we can draw this conclusion.

We consider the following domains:
\[
D(\overrightarrow{\partial}) = \{ \alpha \in \Omega^{*, q} : \alpha \text{ and } \overrightarrow{\partial} \alpha \text{ are square integrable} \};
\]
\[
D(\delta^\mu_q) = \{ \alpha \in \Omega^{*, q+1} : \alpha \text{ and } \delta^\mu_q \alpha \text{ are square integrable} \};
\]
\[
D(\square^\mu_p) = \{ \alpha \in D(\overrightarrow{\partial}) \cap D(\delta^\mu_q) : \overrightarrow{\partial} \alpha \in D(\delta^\mu_q) \text{ and } \delta^\mu_q \alpha \in D(\square^\mu_p) \}.
\]

We need the following assumption so that $\square^\mu_p$ is symmetric:

**Definition 2.4.** We say $M$ has negligible boundary if
\[
(\overrightarrow{\partial} \alpha, \beta) = (\alpha, \delta^\mu \beta), \quad \text{for every } \alpha \in D(\overrightarrow{\partial}) \text{ and } \beta \in D(\delta^\mu).\]

We say $\square^\mu_p$ is essentially self-adjoint if its $L^2$-closure is self-adjoint, and $\square^\mu_p$ is hypoelliptic if, whenever the distribution $\square^\mu_p \alpha$ is smooth, then $\alpha$ is smooth.

It is proved in [11] that

**Lemma 2.5.** A subelliptic operator is hypoelliptic.
The assumption such that $M$ has negligible boundary implies a stronger property to $\Box_\mu$:

**Proposition 2.6** (e.g. [13]). If $M$ has negligible boundary, then $\Box^q_\mu$ is essentially self-adjoint in $L^2(\Omega^{\bullet,q}(E))$ with $0 < q < n - 1$.

**Outline of the proof.** Set $\alpha_\epsilon = e^{-\Box^2_\mu}\alpha$ for $\alpha \in D(\delta^2_\mu)$. By Proposition 2.3 and Lemma 2.5, $\alpha_\epsilon$ is smooth for every $\epsilon > 0$ (here we need the assumption: $0 < q < n - 1$). Therefore, since $\delta_\mu^2 \alpha_\epsilon = \Box_\mu \alpha_\epsilon \to \Box_\mu \alpha$ as $\epsilon \to 0$, we deduce that $\delta_\mu^2 \subset \Box_\mu$. Since $M$ has negligible boundary, $\delta^2_\mu = \overline{\partial}^*$, and by von Neumann’s theorem (e.g. [14]), $\overline{\partial}^* \delta_\mu$ is self-adjoint. Moreover, it follows that

$$\Box^2_\mu = \overline{\partial}^* \delta_\mu + \delta_\mu \overline{\partial}^*,$$

where the right-hand side is self-adjoint. \hfill $\Box$

We say $\alpha$ is harmonic if $\overline{\partial} \alpha = 0$ and $\delta_\mu \alpha = 0$ in the weak sense. A harmonic form always solves the Laplace equation $\Box_\mu \alpha = 0$, but in general, the converse does not need to be true. However, it follows that

**Corollary 2.7.** If $M$ has negligible boundary, then the following conditions are equivalent:

(i) $\overline{\partial} \alpha = 0$ and $\delta_\mu \alpha = 0$ pointwise;

(ii) $\alpha$ is harmonic;

(iii) $\alpha$ solves the Laplace equation;

here, $\alpha \in L^2(\mathcal{C}^{\bullet,q}(E))$ and $0 < q < n - 1$.

**Proof.** (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii) If $\alpha$ is harmonic, then $(\alpha, \overline{\partial} \beta) = 0$ and $(\alpha, \Box_\mu \gamma) = 0$ for every $\beta \in D(\delta_\mu)$ and $\gamma \in D(\overline{\partial})$. This implies $(\alpha, \Box_\mu \beta) = 0$ for every $\beta \in D(\Box_\mu)$; that is, $\alpha$ is the solution of the Laplace equation.

(iii) $\Rightarrow$ (i) Let $\alpha$ be a solution of the Laplace equation. By Proposition 2.6 there exists a sequence $\alpha_l \in D(\Box_\mu)$ such that

$$\alpha_l \to \alpha \quad \text{and} \quad \Box_\mu \alpha_l \to 0, \quad \text{as} \quad l \to \infty.$$

Due to the fact that $M$ has negligible boundary,

$$\|\overline{\partial} \alpha_l\|_2^2 + \|\delta_\mu \alpha_l\|_2^2 = (\Box_\mu \alpha_l, \alpha_l) \to 0, \quad \text{as} \quad l \to \infty.$$

This shows that $\alpha \in D(\overline{\partial}) \cap D(\delta_\mu)$, and $\overline{\partial} \alpha = \delta_\mu \alpha = 0 \mu$-a.e. Due to the hypoellipticity of $\Box_\mu$, $\alpha$ is smooth, and hence, $\overline{\partial} \alpha = \delta_\mu \alpha = 0$ pointwise. \hfill $\Box$

A consequence of the celebrated Kohn’s harmonic theory [10] is the Hodge decomposition of a vector bundle over a compact s.p.c. CR manifold. The corresponding result on a non-compact manifold, which is a consequence of Corollary 2.7, is the $L^2$-Hodge decomposition in the Main Theorem. Since the proof is similar to the case where $\eta \equiv 1$ and $E$ is trivial (e.g. [13]), we will omit the proof here.

3. **Serre duality**

In this section, we study Serre duality and complete the proof of the Main Theorem. Our method is to relate the operators on $E$ to those on $E^*$ via the weighted complex-conjugate Hodge star operator $\sharp_\mu$ (see e.g. [8], [6]). Together with results from the previous section, we obtain the Main Theorem.
We start from the construction of $\sharp_\mu$. Let $\ast : \bigwedge^k T^*M \to \bigwedge^{2n-1-k} T^*M$ be the Hodge star operator of $M$ with respect to $g$, which is uniquely determined by $g(\ast \alpha, \beta) dv = (n-1)! \alpha \wedge \beta$, for $\alpha \in \bigwedge^k T^*M$ and $\beta \in \bigwedge^{2n-1-k} T^*M$. $\ast$ is isometric and involutive, i.e. $g(\ast \alpha, \ast \beta) = g(\alpha, \beta)$ and $\ast^2 = id$, because $M$ is odd-dimensional. As the complexification of $\ast$ exchanges the set of holomorphic forms and the set of anti-holomorphic forms, the linear map $\sharp = \sharp_M := - \circ \ast$ satisfies (e.g. Lemma 7.1 [E5]):

$$(3.1) \quad \sharp (\Omega^{p,q}(M)) = \Omega^{n-p,n-(q+1)}(M).$$

We extend (3.1) to

**Definition 3.1.** Define

$$\sharp_\mu : \Omega^{p,q}(E) \to \Omega^{n-p,n-(q+1)}(E^*)$$

by

$$\sharp_\mu \alpha := \sum_{1 \leq i,j \leq r} \eta_{aj} (\sharp_\alpha^i) \otimes s^j, \text{ for } \alpha \in \Omega^{p,q}(E),$$

where $\alpha = \sum \alpha^i \otimes s_i$, $\{s_i\}_{1 \leq i \leq r}$ is a local frame of $E$, $\{s^i\}$ is its dual frame of $E^*$, and $a_{ij} = \langle s_i, s_j \rangle_E$. Moreover, define $\sharp^*_\mu : \Omega^{n-p,n-(q+1)}(E^*) \to \Omega^{p,q}(E)$ by

$$\sharp^*_\mu \phi := \sum \eta^{-1} \alpha^{ij} (\sharp_\phi^i) \otimes s_i,$$

where $\phi = \sum \phi_j \otimes s^j$ and $\alpha^{ij} = \langle s^i, s^j \rangle_{E^*}$, which is the entry of the inverse-matrix of $(a_{ij})$. Here $E^*$ is furnished with the Hermitian fiber metric induced from $E$:

$$(\phi, \psi)_{E^*} = (\phi, \psi)_{E^*,d\mu^-} = \frac{1}{(n-1)!} \int \sum \phi_i \psi_j d\mu^-, \quad \text{where } \phi = \sum \phi_j \otimes s^j, \psi = \sum \psi_i \otimes s^i, \text{ and } d\mu^- = \eta^{-1} dv.$$

For $\alpha \in \Omega^{p,q}(E)$ and $\phi \in \Omega^{p,q}(E^*)$, the product $\alpha \wedge \phi$ is defined by

$$(\alpha, \phi)_{E} = \frac{1}{(n-1)!} \int \alpha \wedge \sharp_\mu \phi, \quad \text{for } \alpha, \beta \in \Omega(E),$$

and

$$(\phi, \psi)_{E^*} = \frac{1}{(n-1)!} \int \phi \wedge \sharp^*_\mu \psi, \quad \text{for } \phi, \psi \in \Omega(E^*).$$

The operators $\sharp_\mu$ and $\sharp^*_\mu$ satisfy the following properties.

**Proposition 3.2.** It follows that

(i) $\sharp_\mu^* \sharp_\mu = id_{\Omega(E)}$ and $\sharp^*_\mu \sharp^*_\mu = id_{\Omega(E^*)}$;

(ii) $(\alpha, \sharp^*_\mu \phi)_{E} = (\sharp^*_\mu \alpha, \phi)_{E^*}$, for every $\alpha \in L^2(\Omega(E))$ and $\phi \in L^2(\Omega(E^*))$.

**Proof.**

(i) $\sharp^*_\mu \sharp_\mu \alpha = \sharp^*_\mu \left( \sum \eta_{aj} \ast \alpha^i \otimes s^j \right) = \sum \eta_{aj} \ast (a_{ji} \ast \alpha^i) \otimes s_k = \alpha$.

(ii) $\alpha, \sharp^*_\mu \phi)_{E} = (\alpha, \sharp^*_\mu \phi)_{E, dv} = (\sharp^*_\mu \alpha, \phi)_{E^*, dv} = (\sharp^*_\mu \alpha, \phi)_{E^*}. \quad \square$

We denote by $\delta_\mu$ and $\delta^*_\mu$ the holomorphic structure of $L^2(\Omega(E^*))$ and its formal adjoint, respectively.
Lemma 3.3. It follows that
\[(3.4) \quad (\alpha, \ast \beta) = (\partial \alpha, ( -1)^{p+q+1} \beta)_E,\]
for \(\alpha \in \Omega^{p,q}(E)\) and \(\beta \in \Omega_0^{p,q+1}(E)\), and
\[(3.3) \quad ( -1)^{p+q+1} (\psi, \delta \mu, \phi)_E = (\psi, \ast \beta, \partial \phi)_E.
\]
for \(\phi \in \Omega^{p,q}(E^*)\) and \(\psi \in \Omega_0^{p,q-1}(E^*)\).

Proof. Recall that the holomorphic structure \(\overline{\partial}_{\lambda_c}\) coincides with the tangential Cauchy-Riemann operator \((-1)^n d''\) (e.g. Proposition 1.1 in [15]) defined as:
\[d''f := df|_{\Sigma}, \text{ for } f \in C(M).\]
Therefore, since \(\bar{\ast} \beta \in \Omega^{n-p,n-(q+2)}(E^*)\) and \(\alpha \ast \bar{\ast} \beta \in \Omega_0^{n,n-2}(M),\)
\[0 = (-1)^n \int d(\alpha \ast \bar{\ast} \beta) = (-1)^n \int d''(\alpha \ast \bar{\ast} \beta)
= \int \partial \overline{\partial}_{\lambda_c} (\alpha \ast \bar{\ast} \beta)
= \int \overline{\partial} \alpha \ast \bar{\ast} \beta + (-1)^{p+q} \alpha \ast \overline{\partial}_\mu \bar{\ast} \beta
= (\overline{\partial} \alpha, \beta)_E + (\alpha, (-1)^{p+q} \overline{\partial}_\mu \bar{\ast} \beta)_E.
\]
We used (3.3) and Proposition 3.2 for the last step. We have (3.4).

Next, by taking into account that \(\int d(\phi \ast \bar{\ast} \psi) = 0\), we can prove (3.4) in a similar way. \(\square\)

We are now in a position to show:

Theorem 3.4 (Serre duality). It follows for every \(0 \leq p, q \leq n - 1\) that
\[\bar{\ast} : \mathbb{H}^{p,q}(E) \cong \mathbb{H}^{n-p,n-(q+1)}(E^*).\]

Proof. Take \(\alpha \in \mathbb{H}^{p,q}(E)\). Then \(\alpha \in D(\overline{\partial}) \cap D(\delta), \overline{\partial} \alpha = 0\) and \(\delta \mu \alpha = 0\). By Proposition 3.2 and (3.4), for \(\beta \in \Omega_0(E)\) we have that
\[(\delta \mu \ast \bar{\ast} \alpha, \bar{\ast} \beta)_E = (\bar{\ast} \mu \bar{\ast} \beta, \partial \phi)_E
= (\partial \alpha, ( -1)^{p+q} \beta)_E = 0.
\]
Since \(\{\bar{\ast} \beta : \beta \in \Omega_0(E)\}\) is dense in \(L^2(E^*)\) by Proposition 3.2 we have \(\delta \mu \ast \bar{\ast} \alpha = 0\).

On the other hand, for \(\phi \in \Omega_0(E^*)\), we have by Proposition 3.2 and (3.4) that
\[(-1)^{p+q+1} (\overline{\partial} \ast \bar{\ast} \alpha, \phi)_E = (-1)^{p+q+1} (\bar{\ast} \mu \alpha, \delta_n \phi)_E
= (\bar{\ast} \mu \alpha, \overline{\partial}_\mu \bar{\ast} \phi)_E
= (\alpha, \overline{\partial}_\mu \bar{\ast} \phi)_E
= (\delta_n \phi, \bar{\ast} \mu \alpha)_E = 0.
\]
Therefore, \(\overline{\partial} \ast \bar{\ast} \alpha = 0\), and we deduce that \(\bar{\ast} \mu \alpha \in \mathbb{H}^{n-p,n-(q+1)}(E^*).\)

The reverse implication can be shown by running the argumentation above from the bottom to the top. Now we obtain
\[\bar{\ast} : \mathbb{H}^{p,q}(E) \cong \mathbb{H}^{n-p,n-(q+1)}(E^*),\]
where \(\bar{\ast} \mu \) is a complex conjugate linear isomorphism. \(\square\)
4. Examples

In this section, we present examples of non-compact s.p.c. CR manifolds with negligible boundary. Our argumentation relies on the following proposition.

**Proposition 4.1.**

(i) If $M$ is Riemannian complete, then it is complete with respect to $d_{CC}$.

(ii) If $M$ is complete, then $M$ has negligible boundary.

**Proof.** (i) Assume that $M$ is Riemannian complete, and let ${\{x_n\}}_{n>0} \subseteq M$ be a Cauchy sequence with respect to $d_{CC}$. Since $d_{\gamma}$ has the alternative expression \[ d_{\gamma}(x, y) = \sup\{u(x) - u(y) : u \in C^\infty(M), \|\nabla u\|_{L^\infty} \leq 1\}, \] it follows that $d_{\gamma} \leq d_{CC}$ by Equation (2.1). Hence, ${\{x_n\}}_{n>0}$ is a Cauchy sequence with respect to $d_{\gamma}$, and thus, the limit belongs to $M$, by the assumption. Since the topologies generated by $d_{\gamma}$ and $d_{CC}$ are the same, we have the assertion.

(ii) Due to the fact that $M$ is complete, there exists a sequence $\{\chi_l\}_{l>0}$ of smooth functions with compact support such that $0 \leq \chi_l \leq 1$, $\chi_l \rightarrow 1$, and $\partial_l \chi_l \rightarrow 0$ as $l \to \infty$ (\[\text{(1)}, \ [\text{12}]\). For $\alpha \in D(\partial_l)$, set $\alpha_l = \chi_l \alpha$. Since $\alpha_l$ has compact support for every $l > 0$, \[(\partial_l \alpha, \beta) = (\alpha_l, \delta_{\mu} \beta) \rightarrow (\alpha, \delta_{\mu} \beta), \text{ as } l \to \infty \text{ for every } \beta \in D(\delta_{\mu}), \] where the left-hand side tends to $(\partial \alpha, \beta)$ as $l \to \infty$. \[\square\]

The most fundamental example is

**Example 4.2 (Heisenberg group).** The Heisenberg group $H(n)$ is

$H(n) = \{(w, z) \in \mathbb{C}^n \times \mathbb{C} : \text{Im} z = ||w||^2\}$

with the group structure

$(w, z) \circ (w', z') = (w + w', z + z' + 2\sqrt{-1}w \cdot w')$.

It is a quadratic CR submanifold (see e.g. \[\text{[2]}\]), whose defining function is

$f(w, z) = \text{Im} z - ||w||^2$.

Consider the following CR manifold which is CR-equivalent to $H(n) : \mathbb{C}^n \times \mathbb{R}$ with the contact form

$\theta = dt + 2 \sum (x_i dy_i - y_i dx_i)$.

Then since the orthonormal frame $\{X_i, Y_i\}$ of $P$ and the characteristic direction $\xi$ are given by

$X_i = \frac{1}{2} \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial t}$, \[ Y_i = \frac{1}{2} \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial t}, \text{ and } \xi = \frac{\partial}{\partial t}, \]

no geodesics with respect to $g = -d\theta + \theta \otimes \theta$ reach $\infty$ in finite time. Due to the Hopf-Rinow theorem, $H(n)$ is Riemannian complete, and we conclude by Proposition 5.1 that $H(n)$ is complete. Moreover, it was proved in \[\text{[13]}\] that $\mathbb{H}^p(E) = 0$ for $0 < q < n - 1$ when $E$ is the trivial bundle over $H(n)$.

**Remark 4.3 (\[\text{[13]}\]).** If $M$ has negligible boundary and additionally either

(1) the Ricci operator is positive on $\Omega^{p,q}$ or

(2) the Ricci operator is non-negative on $\Omega^{p,q}$ and $M$ has infinite volume,

then $\mathbb{H}^p(E) = 0$ for $0 < q < n - 1$ when $E$ is the trivial bundle over $M$. 

Example 4.4 (Sasakian space forms). There exist exactly three types of Riemannian complete simply connected Sasakian space forms: $S^{2n+1}$, $\mathbb{R}^{2n+1}$, and $D^n \times \mathbb{R}$, where $D \subset \mathbb{C}^n$ is a simply connected bounded domain with Kähler form $d\omega$ (e.g. [3]). The latter two space forms $\mathbb{R}^{2n+1}$ and $D^n \times \mathbb{R}$ are non-compact and they have the contact form $dt - \sum y_idx_i$ and $\omega + dt$, respectively. They have negligible boundary.

Example 4.5 (Spherical orbits). Let $O$ be the orbit of an $n^2$-dimensional automorphism in an $n$-dimensional non-homogeneous hyperbolic manifold. If $O$ is spherical, i.e. each point of $O$ has a neighbourhood which is CR-equivalent to an open set of $S^{2n-1}$, then $O$ is CR-equivalent to one of the following hypersurfaces (e.g. [4]):

1. A lens space $S^{2n-1}/\mathbb{Z}_m$;
2. $\sigma = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Re} z = \|w\|^2\}$;
3. $\sigma' = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : |z| = \exp \|w\|^2\}$;
4. $\omega = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|w\|^2 + \exp(\text{Re} z) = 1\}$;
5. $\omega_\alpha = \{(w, z) \in \mathbb{C}^{n-1} \times \mathbb{C} : \|w\|^2 + |z|^\alpha = 1, z \neq 0\}$, for some $\alpha > 0$.

We are interested in $\sigma$ and $\sigma'$ because they are non-compact. First we show that $\sigma$ is complete. Consider the map $\Phi : \mathcal{H}(n) \to \sigma$ defined as

$$\Phi(w, z) := (w, -\sqrt{-1}z).$$

Clearly, $\Phi$ preserves the holomorphic structure. Moreover, since

$$f_\sigma \Phi(w, z) = f_H(w, z),$$

where $f_\sigma = \text{Re} z - \|w\|^2$ is the defining function of $\sigma$, the contact structure is also preserved. Since $\mathcal{H}(n)$ is complete, so is $\sigma$.

Next we proceed to show that $\sigma'$ is complete. The differential $df_{\sigma'}$ of the $\sigma'$'s defining function $f_{\sigma'} = |z|^2 - \exp \|w\|^2$ is

$$df_{\sigma'} = \frac{1}{2} \left( \frac{\bar{z}}{|z|} dz + \frac{z}{|z|} d\bar{z} \right) - \exp \|w\|^2 \left( \sum (\bar{w}_i dw_i + w_i d\bar{w}_i) \right),$$

and its pull-back $\Psi^* df_{\sigma'}$ by the covering map $\Psi : \sigma \to \sigma'$, defined as $\Psi(w, z) = (w, \exp z)$, is

$$\frac{1}{2} \exp(R\text{e} z)^{-1} (\exp \Re \Psi^* dz + \exp z \Psi^* d\bar{z}) - \exp \|w\|^2 \left( \sum (\bar{w}_i dw_i + w_i d\bar{w}_i) \right)$$

$$= \frac{1}{2} \exp(R\text{e} z)^{-1} \exp(2\text{Re} z) (dz + d\bar{z}) - \exp \|w\|^2 \left( \sum (\bar{w}_i dw_i + w_i d\bar{w}_i) \right)$$

$$= \exp \|w\|^2 df_{\sigma'},$$

where we have used the fact that $\text{Re} z = \|w\|^2$ for the last step. This shows that $\ker(\theta_{\sigma'}) = \ker(\theta_{\sigma'})$ via $\Psi$. Moreover, since

$$\Psi^* d\theta_{\sigma'} = d\exp \|w\|^2 \wedge \theta_{\sigma'} + \exp \|w\|^2 d\theta_{\sigma'},$$

where the first term on the right-hand side vanishes on $P$ and $\exp \|w\|^2 \geq 1$, it follows that the distance associated to $\Psi^* d\theta_{\sigma'}$ is not less than the one associated to $d\theta_{\sigma'}$.

Thus, since these two distances generate the same topology and $\sigma$ is complete with respect to $d\theta_{\sigma'}$, we may conclude that the distance associated to $\Psi^* d\theta_{\sigma'}$ is complete by the same reason as in the proof of Proposition [4].

We can also show that $\omega$ is complete, where the proof will appear in a forthcoming paper.
Example 4.6. If $M$ is a compact s.p.c. CR manifold and $M' \to M$ is an unbranched covering, then $M'$ has negligible boundary.

Remark 4.7. Since the distance structure of $M$ is independent of the choice of the weight, all of those examples have negligible boundary with an arbitrary weight.

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