TSUKUBA J. MATH.
Vol. 17 No. 2 (1993), 299—310

ON THE SUM OF A PRIME AND A SQUARE

By

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1. Introduction.

In 1923 G. H. Hardy and J. E. Littlewood [3] conjectured that every large integer, not being a square, may be expressed as the sum of a prime and a square. Let $\nu(n)$ be the number of representations of an integer n in this manner. They further stated the hypothetical asymptotic formula; As $n(\neq k^2)\to\infty$,

$$\nu(n)\sim\mathfrak{S}(n)\frac{\sqrt{n}}{\log n}$$

with

$$\mathfrak{S}(n)=\prod_{p>2}\left(1-\frac{(n/p)}{p-1}\right)$$

where $(-)$ is the Legendre symbol.

Define $\mathfrak{S}(k^2)=0$. In 1968 R. J. Miech [5] proved that

$$(1)\qquad \sum_{n\leq x}\left|\nu(n)-\mathfrak{S}(n)\frac{\sqrt{n}}{\log n}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)\right|^2\ll x^2(\log x)^{-A}$$

for any $A>0$, from which it follows that

$$(2)\qquad E(x)\ll x(\log x)^{-A}$$

where $E(x)$ denotes the number of integers $n\leqq x$ with $\nu(n)=0$. It seems difficult to sharpen the right hand side of (1). However (2) may be improved, see [1, 9, 12].

A. I. Vinogradov [12; p. 35] remarked that, for any $\varepsilon>0$,

$$E(x)\ll x^{2/3+\varepsilon}$$

under the extended Riemann hypothesis. First of all we shall show

PROPOSITION. *Assume the extended Riemann hypothesis. Then*

$$(3)\qquad E(x)\ll x^{1/2}(\log x)^5.$$

Received June 16, 1992.

It is the main aim of this paper to prove the following unconditional results.

THEOREM 1. *Let $1/2<\Theta\leqq 1$ and $A>0$ be given. We have*

$$\sum_{x-x^{\Theta}<n\leqq x}\left|\nu(n)-\mathfrak{S}(n)\frac{\sqrt{n}}{\log n}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)\right|^2\ll x^{\Theta+1}(\log x)^{-A}$$

where the O-constant is absolute and the $\ll$-constant depends on Θ and A only.

THEOREM 2. *Let $7/24<\theta\leqq 1$ and $A>0$ be given. We have*

$$E(x+x^{\theta})-E(x)\ll x^{\theta}(\log x)^{-A}$$

where the implied constant depends on θ and A only.

Our assertion may be regarded as a refinement of Miech's work (1)(2), and must be compared with a conditional bound (3). Within the frame of Circle method, we appeal to the large sieve [7, 8] and R.C. Vaughan's method [11; Chap. 4] on Weyl sums.

I would like to thank Professor Uchiyama and Dr. Kawada for suggestion and encouragement.

2. Singular series.

In this section we collect the facts of $\mathfrak{S}(n)$. For the proof, see [1, 5, 9, 12].

For integers q and n, let $\rho(q, n)$ be the number of solutions of the congruence $x^2\equiv n \pmod q$, and ρ_1 be the convolution inverse of ρ with respect to q. Define, for $Q\geqq 3$,

$$\mathfrak{S}(n, Q)=\sum_{q\leqq Q}\frac{\mu(q)}{\varphi(q)}\rho_1(q, n). \tag{2.1}$$

Then, uniformly for n,

$$\mathfrak{S}(n, Q)\ll\log Q. \tag{2.2}$$

Let $\mathcal{D}$ be the set of fundamental discriminants. An integer n may be uniquely written as $n=n_1n_2^2$ with a square-free n_1. Put

$$\delta(n)=\begin{cases} n_1 & \text{if } n_1\equiv 1 \pmod 4\\ 4n_1 & \text{otherwise}.\end{cases}$$

Thus, if $n\neq k^2$ then $\delta(n)\in\mathcal{D}$. For $d\in\mathcal{D}$, the Kronecker symbol $(d/\cdot)$ is a primitive character to modulus d. Let $\mathcal{L}=\mathcal{L}(T)$, $T\geqq 3$, denote the set of $d\in\mathcal{D}$ for which $L(s, (d/\cdot))$ the Dirichlet L-function has no zero in the region:

$\mathrm{Re}(s) \geqq 29/30$ and $|\mathrm{Im}(s)| \leqq T$. Suppose $x \asymp T$. If $\delta(n) \in \mathcal{L}$ then there exists a constant $\eta > 0$ such that

$$\mathfrak{S}(n, Q) = \mathfrak{S}(n) + O(Q^{-\eta} \exp(\sqrt{\log x})) \tag{2.3}$$

uniformly for $n \leqq x$. Moreover,

$$\#\{d : d \in \mathcal{D} \backslash \mathcal{L}, d \leqq 4x\} \ll x^{1/4} (\log x)^{14}. \tag{2.4}$$

Finally, for $n \neq k^2$,

$$\frac{\varphi(n)}{n} \ll \mathfrak{S}(n) L\left(1, \left(\frac{\delta(n)}{\cdot}\right)\right) \ll \frac{n}{\varphi(n)}. \tag{2.5}$$

3. A conditional estimate.

In this section we illustrate our device with the proof of Proposition. We employ the Circle method [11].

Let x be a large parameter. We divide the unit interval by the Farey dissections of order

$$Q = x^{1/2} (\log x)^2 .$$

For $(a, q) = 1$, write

$$I_{q,a} = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].$$

Put

$$\boldsymbol{M} = \bigcup_{q \leqq P} \bigcup_{\substack{0 < a \leqq q \\ (a,q)=1}} I_{q,a}, \qquad P = x/100Q,$$

$$\boldsymbol{m} = [Q^{-1}, 1 + Q^{-1}] \backslash \boldsymbol{M}.$$

We define the exponential sum

$$W(\alpha) = \sum_{m^2 \leqq x} e(\alpha m^2)$$

where $e(t) = e^{2\pi i t}$. By Weyl's inequality, we see that

$$|W(\alpha)|^2 \ll \left(\frac{x}{q} + x^{1/2} + q\right) \log qx, \tag{3.1}$$

for $|\alpha - (a/q)| \leqq q^{-2}$ with $(a, q) = 1$. When $\alpha \in I_{q,a}$, $W(\alpha)$ is approximated by

$$V(\alpha) = q^{-1} g(a, q) v\left(\alpha - \frac{a}{q}\right)$$

where

$$g(a, q) = \sum_{m(q)} e\left(\frac{a}{q} m^2\right) \quad \text{and} \quad v(\beta) = \sum_{m \leqq x} \frac{e(\beta m)}{2\sqrt{m}}.$$

Actually it follows from [11; Theorem 4.1] and [12; p. 38] that

(3.2) $$|W(\alpha)-V(\alpha)|^2 \ll q(\log q)^2$$

for $|\alpha-(a/q)| \leqq (4q\sqrt{x})^{-1}$. In addition, we note that

(3.3) $$|g(a, q)|^2 \ll q,$$

(3.4) $$|v(\beta)|^2 \ll \min(x, \|\beta\|^{-1})$$

where $\|t\|=\min_{n\in Z}|t-n|$.

Put

$$S(\alpha)=\sum_{n\leqq x}\Lambda(n)e(\alpha n)$$

where Λ is the von Mangoldt function. It is expected that, for $\alpha\in I_{q,a}$, $S(\alpha)$ is nearly equal to

$$T(\alpha)=\frac{\mu(q)}{\varphi(q)}t\left(\alpha-\frac{a}{q}\right)$$

where

(3.5) $$t(\beta)=\sum_{n\leqq x}e(\beta n)\ll\min(x, \|\beta\|^{-1}).$$

In order to show this, define

(3.6) $$J(q)=\sum_{a=1}^{q*}\int_{I_{q,a}}|S(\alpha)-T(\alpha)|^2 d\alpha$$

where $*$ in $\sum_a^*$ stands for $(a, q)=1$. If $\alpha\in I_{q,a}$,

$$S(\alpha)-T(\alpha)=\varphi(q)^{-1}\sum_{\chi(q)}\chi(a)\tau(\bar{\chi})\sum_{n\leqq x}^{\#}\chi(n)\Lambda(n)e\left(\left(\alpha-\frac{a}{q}\right)n\right)+O((\log x)^2).$$

Here $\#$ in $\sum_n^{\#}$ means that if χ is principal then $\chi(n)\Lambda(n)$ should be replaced by $\Lambda(n)-1$. When $q\leqq P$, by [2; Lemma 1], we have

$$J(q)\ll\varphi(q)^{-1}\sum_{\chi(q)}|\tau(\bar{\chi})|^2\int_{|\beta|\leqq 1/qQ}\left|\sum_{n\leqq x}^{\#}\chi(n)\Lambda(n)e(\beta n)\right|^2 d\beta+Q^{-1}(\log x)^4$$

$$\ll\varphi(q)^{-1}\sum_{\chi(q)}q(qQ)^{-2}\int_{-\infty}^{+\infty}\left|\sum_{\substack{n\leqq x\\ y<n\leqq y+qQ/2}}^{\#}\chi(n)\Lambda(n)\right|^2 dy+Q^{-1}(\log x)^4.$$

On noting $qQ\leqq PQ=x/100$, it is easy to show that the above integral is

$$\ll qQx(\log x)^4,$$

under the extended Riemann hypothesis. Therefore, uniformly for $q\leqq P$,

(3.7) $$J(q)\ll Q^{-1}x(\log x)^4.$$

Now, for $n\leqq x$,

(3.8) $$\sum_{l+m^2=n}\Lambda(n)=\int_{Q^{-1}}^{1+Q^{-1}}S(\alpha)W(\alpha)e(-n\alpha)d\alpha=\int_M+\int_m.$$

By Bessel's inequality and (3.1),

$$\sum_{n\leq x}\left|\int_{m}S(\alpha)W(\alpha)e(-n\alpha)d\alpha\right|^{2}\leq\int_{m}|S(\alpha)W(\alpha)|^{2}d\alpha \tag{3.9}$$

$$\leq\sup_{\alpha\in m}|W(\alpha)|^{2}\int_{m}|S(\gamma)|^{2}d\gamma$$

$$\ll Qx(\log x)^{2}.$$

On $\alpha\in\boldsymbol{M}$ we first exchange $S(\alpha)$ for $T(\alpha)$. Thus, by (3.1) and (3.7),

$$\sum_{n\leq x}\left|\int_{M}(S(\alpha)-T(\alpha))W(\alpha)e(-n\alpha)d\alpha\right|^{2} \tag{3.10}$$

$$\leq\sum_{q\leq P}\sum_{a=1}^{q*}\int_{I_{q,a}}|S(\alpha)-T(\alpha)|^{2}|W(\alpha)|^{2}d\alpha$$

$$\ll\sum_{q\leq P}\frac{x}{q}(\log x)J(q)$$

$$\ll x^{2}Q^{-1}(\log x)^{6}.$$

Next we replace $W(\alpha)$ by $V(\alpha)$. On using (3.2) and (3.5),

$$\sum_{n\leq x}\left|\int_{M}T(\alpha)(W(\alpha)-V(\alpha))e(-n\alpha)d\alpha\right|^{2} \tag{3.11}$$

$$\ll\sup_{\alpha\in M}|W(\alpha)-V(\alpha)|^{2}\int_{M}|T(\gamma)|^{2}d\gamma$$

$$\ll P(\log P)^{2}\sum_{q\leq P}\sum_{a=1}^{q*}\int_{|\beta|\leq1/qQ}\frac{\mu^{2}(q)}{\varphi^{2}(q)}|t(\beta)|^{2}d\beta$$

$$\ll Px(\log x)^{3}.$$

Finally we extend the Farey arc $I_{q,a}$ to $\boldsymbol{I}_{q,a}=[(a/q)-(1/2),(a/q)+(1/2)]$. The resulting remainder is then equal to

$$r_{n}=\sum_{q\leq P}\sum_{a=1}^{q*}\int_{\boldsymbol{I}_{q,a}\setminus I_{q,a}}T(\alpha)V(\alpha)e(-n\alpha)d\alpha \tag{3.12}$$

$$=\sum_{q\leq P}\sum_{a=1}^{q*}\int_{1/qQ<|\beta|\leq1/2}\frac{\mu(q)}{\varphi(q)}t(\beta)q^{-1}g(a,q)v(\beta)e\Big(-n\Big(\frac{a}{q}+\beta\Big)\Big)d\beta$$

$$=\int_{1/Q<|\beta|\leq1/2}t(\beta)v(\beta)e(-\beta n)\sum_{\substack{q\leq P\\|\beta|qQ>1}}\sum_{a=1}^{q*}\frac{\mu(q)}{q\varphi(q)}g(a,q)e\Big(-\frac{a}{q}n\Big)d\beta.$$

On using Cauchy's inequality and (3.4), we have

$$\sum_{n\leq x}|r_{n}|^{2}\leq\sum_{n\leq x}\int_{|\gamma|\leq1/2}|v(\gamma)|^{2}d\gamma\int_{1/Q<|\beta|\leq1/2}|t(\beta)|^{2}|\Sigma_{\beta}|^{2}d\beta$$

$$\ll(\log x)\int_{1/Q<|\beta|\leq1/2}|t(\beta)|^{2}\Big(\sum_{n\leq x}\Big|\sum_{\substack{q\leq P\\|\beta|qQ>1}}\sum_{a=1}^{q*}\frac{\mu(q)}{q\varphi(q)}g(a,q)e\Big(-\frac{a}{q}n\Big)\Big|^{2}\Big)d\beta.$$

The large sieve inequality [7, 8] yields that

$$\sum_{n\leq x}|r_n|^2 \ll (\log x)\int_{1/Q<|\beta|\leq 1/2}|t(\beta)|^2\Big(\sum_{\substack{q\leq P\\ |\beta|qQ>1}}\sum_{a=1}^{q*}(x+qP)\Big|\frac{\mu(q)}{q\varphi(q)}g(a,q)\Big|^2\Big)d\beta$$

$$\ll(\log x)\sum_{q\leq P}(x+qP)\frac{\mu^2(q)}{q\varphi(q)}\int_{1/qQ<|\beta|\leq 1/2}|t(\beta)|^2 d\beta$$

$$\ll(\log x)(x+P^2)Q\sum_{q\leq P}\frac{\mu^2(q)}{\varphi(q)}$$

$$\ll Qx(\log x)^2. \tag{3.13}$$

Here we used the bounds (3.3) and (3.5).

It remains to calculate

$$\sum_{q\leq P}\sum_{a=1}^{q*}\int_{I_{q,a}}T(\alpha)V(\alpha)e(-n\alpha)d\alpha \tag{3.14}$$

$$=\sum_{q\leq P}\sum_{a=1}^{q*}\frac{\mu(q)}{q\varphi(q)}g(a,q)e\Big(-\frac{a}{q}n\Big)\int_{-1/2}^{+1/2}\sum_{l,m\leq x}\sum\frac{e(\beta(l+m-n))}{2\sqrt{m}}d\beta.$$

The above sum is $\mathfrak{S}(n, P)$ with the definition (2.1). The integral is equal to

$$\sum_{l<n}\frac{1}{2\sqrt{n-l}}=\sqrt{n}+O(1).$$

Hence, by (2.2), (3.14) becomes

$$\mathfrak{S}(n, P)\sqrt{n}+O(\log x). \tag{3.15}$$

On summing up the above argument (3.9)-(3.15), we obtain

$$\sum_{n\leq x}\Big|\sum_{p+m^2=n}\log p-\mathfrak{S}(n,P)\sqrt{n}\Big|^2\ll Qx(\log x)^2+x^2Q^{-1}(\log x)^6+Px(\log x)^3$$

$$\ll x^{3/2}(\log x)^4. \tag{3.16}$$

Now, the extended Riemann hypothesis implies that $\mathcal{L}=\mathcal{D}$ the sets introduced in section 2, and that $\mathfrak{S}(n)\gg(\log\log n)^{-2}$ for $n\neq k^2$ and $n\gg 1$. By (2.3) we then have that, for $n\neq k^2$ $(\gg 1)$,

$$\mathfrak{S}(n, P)\gg(\log\log n)^{-2}.$$

Consequently (3.16) leads that

$$x^{3/2}(\log x)^4\gg\sum_{n\leq x}\Big|\sum_{p+m^2=n}\log p-\mathfrak{S}(n,P)\sqrt{n}\Big|^2$$

$$\geq\sum_{\substack{x/2<n\leq x\\ \nu(n)=0\\ n\neq k^2}}|\mathfrak{S}(n,P)|^2 n$$

$$\gg x(\log\log x)^{-4}\sum_{\substack{x/2<n\leq x\\ \nu(n)=0\\ n\neq k^2}}1$$

or

$$E(x)\leqq \sum_{\substack{n\leqq x\\ \nu(n)=0\\ n\neq k^2}} 1+\sum_{k^2\leqq x} 1 \ll \sum_{l}\left(\frac{x}{2^l}\right)^{1/2}(\log x\ \log\log x)^4+x^{1/2}$$
$$\ll x^{1/2}(\log x)^5,$$

as required.

4. Proof of Theorems.

In this section we derive Theorems from the known results mentioned in section 2 and our main lemma below. Lemma will be verified in the next section.

LEMMA. *Let x be a large parameter. For given $1/2<\Theta\leqq 3/4$ and $7/12<\Xi\leqq 1$, put $\Delta=y^{\Theta}$ and $y=x^{\Xi}$. Write*

$$\nu(n,\ y)=\#\{(p,\ m):\ x-y<p\leqq x,\ m^2\leqq y,\ p+m^2=n\},$$

and

$$K(n,\ y)=\int_1^{n-(x-y)-3}\frac{dt}{2\sqrt{t}\log(n-t)}.$$

Then, for any $A>0$, we have

$$\sum_{x-\Delta<n\leqq x}|\nu(n,\ y)-\mathfrak{S}(n,\ \sqrt{y})K(n,\ y)|^2\ll \Delta y(\log x)^{-A}$$

where the implied constant depends on Θ, Ξ and A only.

Proof of Theorem 1. Let $\mathcal{L}=\mathcal{L}(x)$ in section 2. Choose $\Xi=1$ in Lemma. Then, because of $\nu(n,\ x)=\nu(n)$,

$$\sum_{x-x^{\Theta}<n\leqq x}|\nu(n)-\mathfrak{S}(n,\ \sqrt{x})K(n,\ x)|^2\ll x^{\Theta+1}(\log x)^{-A}$$

for any $A>0$. We note that

$$K(n,\ x)=\frac{\sqrt{n}}{\log n}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)$$

with an absolute O-constant. Combining the above with (2.3) and (2.5) we have

$$S=\sum_{x-x^{\Theta}<n\leqq x}|\nu(n)-\mathfrak{S}(n)K(n,\ x)|^2$$
$$=\sum_{\delta(n)\in\mathcal{L}}+\sum_{\substack{\delta(n)\notin\mathcal{L}\\ n\neq k^2}}+\sum_{n=k^2}$$
$$\ll \sum_{\substack{x-x^{\Theta}<n\leqq x\\ \delta(n)\in\mathcal{L}}}|\nu(n)-\mathfrak{S}(n,\ \sqrt{x})K(n,\ x)|^2+\sum_{\substack{x-x^{\Theta}<n\leqq x\\ \delta(n)\in\mathcal{L}}}|\mathfrak{S}(n)-\mathfrak{S}(n,\ \sqrt{x})|^2\frac{n}{(\log n)^2}$$
$$+\sum_{\substack{x-x^{\Theta}<n\leqq x\\ \delta(n)\notin\mathcal{L}\\ n\neq k^2}}\left(\nu(n)^2+\left(\frac{\mathfrak{S}(n)}{\log n}\right)^2 n\right)+\sum_{x-x^{\Theta}<k^2\leqq x}\nu(k^2)^2$$

$$\ll \sum_{x-x^\Theta<n\leq x} |\nu(n)-\mathfrak{S}(n,\sqrt{x})K(n,x)|^2+x^{\Theta+1}\sup_{\substack{n\leq x\\ \delta(n)\in\mathcal{L}}} |\mathfrak{S}(n)-\mathfrak{S}(n,\sqrt{x})|^2$$

$$+x\Big(1+\sup_{\substack{n\leq x\\ \delta(n)\notin\mathcal{L}\\ n\neq k^2}}\Big(\frac{\mathfrak{S}(n)}{\log n}\Big)^2\Big)\sum_{\substack{d\leq 4x\\ d\in\mathcal{D}\setminus\mathcal{L}\\ \text{or } d=1}}\ \sum_{\substack{x-x^\Theta<n\leq x\\ \delta(n)=d}} 1$$

$$\ll x^{\Theta+1}(\log x)^{-A}+x\Big(1+\sup_{\substack{d\leq 4x\\ d\in\mathcal{D}}}\Big|L\Big(1,\Big(\frac{d}{\cdot}\Big)\Big)\Big|^{-2}\Big)\Big(1+\sum_{\substack{d\leq 4x\\ d\in\mathcal{D}\setminus\mathcal{L}}} 1\Big)x^{\Theta-1/2}.$$

By (2.4) and Siegel's theorem [10; Kap. IV, §8], S becomes

$$\ll x^{\Theta+1}(\log x)^{-A}+x^{\Theta+1/2}(x^{\varepsilon})(1+x^{1/4}(\log x)^{14})$$
$$\ll x^{\Theta+1}(\log x)^{-A}.$$

Hence we obtain Theorem 1 in case $1/2<\Theta\leqq 3/4$. If $3/4<\Theta\leqq 1$, Theorem 1 follows from the case of $\Theta=2/3$, by splitting up the interval $(x-x^\Theta, x]$ into the sum of smaller intervals of type $(u-u^{2/3}, u]$.

Proof of Theorem 2. Put $\theta=\Theta\Xi$ in Lemma. Then, $7/24<\theta\leqq 3/4$. It is sufficient to prove Theorem 2 for θ in the above range only. Since $\nu(n)=0$ implies $\nu(n, y)=0$, Lemma yields that

$$\sum_{\substack{x-x^\theta<n\leq x\\ \nu(n)=0}} |\mathfrak{S}(n,\sqrt{y})K(n,y)|^2\ll yx^\theta(\log x)^{-A-5}.$$

Here, $K(n, y)^2\asymp y(\log x)^{-2}$. Thus,

$$\sum_{\substack{x-x^\theta<n\leq x\\ \nu(n)=0\\ \delta(n)\in\mathcal{L}}} 1\ll x^\theta(\log x)^{-A-3}(\log x\ \log x)^2$$

by (2.3) and (2.5) with $\mathcal{L}=\mathcal{L}(x)$. Hence, by (2.4), we obtain

$$E(x)-E(x-x^\theta)\leqq \sum_{\substack{x-x^\theta<n\leq x\\ \nu(n)=0\\ \delta(n)\in\mathcal{L}}} 1+\sum_{\substack{x-x^\theta<n\leq x\\ \delta(n)\notin\mathcal{L}}} 1$$
$$\ll x^\theta(\log x)^{-A}+\sum_{\substack{d\leq 4x\\ d\in\mathcal{D}\setminus\mathcal{L}\text{ or } d=1}} (x^{\theta-1/2}+1)$$
$$\ll x^\theta(\log x)^{-A}+x^{1/4}(\log x)^{14}$$
$$\ll x^\theta(\log x)^{-A},$$

as required.

5. Proof of Lemma.

Put

$$S(\alpha)=\sum_{x-y<p\leq x} e(\alpha p).$$

And we define the exponential sums $W(\alpha)$ and $V(\alpha)$ by the similar way in section 3, except for changing the parameter x in section 3 by y. The Farey arcs are determined as follows:

$$Q=y^{1/2}(\log x),$$

$$\boldsymbol{M}=\bigcup_{q\le P}\bigcup_{\substack{0<a\le q\\(a,q)=1}} I_{q,a},\quad \boldsymbol{L}=\bigcup_{q\le P}\bigcup_{\substack{0<a\le q\\(a,q)=1}} \boldsymbol{I}_{q,a},\quad P=(\log x)^A$$

$$\boldsymbol{m}=\bigcup_{P<q\le R}\bigcup_{\substack{0<a<q\\(a,q)=1}} I_{q,a},\quad R=y/Q$$

$$\boldsymbol{n}=[Q^{-1},\ 1+Q^{-1}]\backslash(\boldsymbol{M}\cup\boldsymbol{m}).$$

Here $\boldsymbol{I}_{q,a}$ and $I_{q,a}$ are similar to that in section 3. We then have

$$\begin{aligned}\nu(n,\ y)&=\int_{Q^{-1}}^{1+Q^{-1}} S(\alpha)W(\alpha)e(-n\alpha)d\alpha\\&=\int_{\boldsymbol{L}} SV-\int_{\boldsymbol{L}\backslash\boldsymbol{M}} SV+\int_{\boldsymbol{m}} SV+\int_{\boldsymbol{M}\cup\boldsymbol{m}} S(W-V)+\int_{\boldsymbol{n}} SW\\&=J_1-J_2+J_3+J_4+J_5,\quad \text{say.}\end{aligned}$$

First we evaluate J_1. An elementary calculation leads that

$$\begin{aligned}J_1(n)&=\int_{\boldsymbol{L}} S(\alpha)V(\alpha)e(-n\alpha)d\alpha\\&=\sum_{q\le P}\sum_{a=1}^{q*}\int_{-1/2}^{+1/2} S\Big(\frac{a}{q}+\beta\Big)q^{-1}g(a,\ q)v(\beta)e\Big(-n\Big(\frac{a}{q}+\beta\Big)\Big)d\beta\\&=\sum_{q\le P}\sum_{a=1}^{q*} q^{-1}g(a,\ q)e\Big(-\frac{a}{q}n\Big)\sum_{x-y<p\le x}\sum_{m\le y}\frac{e((a/q)p)}{2\sqrt{m}}\int_{-1/2}^{+1/2} e(\beta(p+m-n))d\beta\\&=\sum_{q\le P}\sum_{a=1}^{q*} q^{-1}g(a,\ q)e\Big(-\frac{a}{q}n\Big)\sum_{x-y<p<n}\frac{e((a/q)p)}{2\sqrt{n-p}}\\&=\sum_{q\le P} q^{-1}\sum_{\substack{x-y<p<n\\m(q)}}\frac{c_q(p+m^2-n)}{2\sqrt{n-p}}\end{aligned}$$

$$=\sum_{q\le P} q^{-1}\sum_{m(q)}\sum_{\substack{d\mid q\\(d,\,m^2-n)=1}}\mu\Big(\frac{q}{d}\Big)d\sum_{\substack{x-y<p<n\\p\equiv n-m^2(d)}}\frac{1}{2\sqrt{n-p}}+O\Big(\sum_{q\le P}\tau(q)q\Big).\tag{5.2}$$

On using partial summation, the innermost sum is equal to

$$\frac{K(n,\ y)}{\varphi(d)}+O\Big(1+\sup_{(b,\,d)=1}\sup_{1\le t\le y} t^{-1/2}\Big|\sum_{\substack{n-t\le p<n\\p\equiv b(d)}}\log p-\frac{t}{\varphi(d)}\Big|\Big).\tag{5.3}$$

We now appeal to the well known result on primes in arithmetical progressions [10; Kap. IX. §3]. It follows from [10; Kap. VIII. Satz 6.2, Kap. IV. Satz 8.1] zero free region and [4, 6; Theorem 12.1] zero density estimates for the Dirichret L-functions that, for given positive constants ε, E and F,

$$\sum_{\substack{X<p\leq X+Y \\ p\equiv l(k)}} \log p=\frac{Y}{\varphi(k)}+O(Y(\log X)^{-E})$$

uniformly for $(l, k)=1$, $k\leqq(\log X)^F$ and $X^{7/12+\varepsilon}\leqq Y\leqq X$. Hence the O-term in (5.3) is at most

$$y^{1/2}(\log x)^{-3A-1},$$

and contributes to (5.2)

$$\begin{aligned}&\ll y^{1/2}(\log x)^{-3A-1}\sum_{q\leq P}\tau(q)q\\ &\ll y^{1/2}(\log x)^{-3A-1}P^2(\log P)\\ &\ll y^{1/2}P^{-1}.\end{aligned}$$

On combining this with (5.2) and (5.3) we have

$$(5.4)\qquad J_1(n)=K(n, y)\sum_{q\leq P} q^{-1}\sum_{m(q)}\sum_{\substack{d\mid q \\ (d, m^2-n)=1}}\mu\left(\frac{q}{d}\right)\frac{d}{\varphi(d)}+O(y^{1/2}P^{-1}).$$

Notice that the above sum is

$$(5.5)\qquad \sum_{q\leq P}\frac{\mu(q)}{q\varphi(q)}\sum_{m(q)}c_q(m^2-n)=\mathfrak{S}(n, P)=\sum_{q\leq P}\sum_{a=1}^{q*}\frac{\mu(q)}{q\varphi(q)}g(a, q)e\left(-\frac{a}{q}n\right).$$

We widen the range of q up to $\sqrt{y}$. Let $J_{11}(n)$ be the resulting cost. On employing the large sieve inequality [7, 8] and (3.3),

$$\begin{aligned}\sum_{x-\Delta<n\leq x}|J_{11}(n)|^2&\ll K(n, y)^2\sum_{x-\Delta<n\leq x}\left|\sum_{P<q\leq\sqrt{y}}\sum_{a=1}^{q*}\frac{\mu(q)}{q\varphi(q)}g(a, n)e\left(-\frac{a}{q}n\right)\right|^2\\ &\ll y(\log x)^{-?}\sum_{P<q\leq\sqrt{y}}\sum_{a=1}^{q*}(\Delta+q\sqrt{y})\left|\frac{\mu(q)}{q\varphi(q)}g(a, q)\right|^2\\ &\ll y(\log x)^{-2}\left(\frac{\Delta}{P}+\sqrt{y}\right)\sum_{q\leq\sqrt{y}}\frac{\mu^2(q)}{\varphi(q)}\end{aligned}$$

$$(5.6)\qquad \ll y(\Delta P^{-1}+\sqrt{y}).$$

In conjunction with (5.4), (5.5) and (5.6) we obtain

$$(5.7)\qquad \sum_{x-\Delta<n\leq x}|J_1(n)-\mathfrak{S}(n, \sqrt{y})K(n, y)|^2\ll\Delta yP^{-1}+y^{3/2}.$$

We proceed to J_2. On using Cauchy's inequality and (3.3),

$$\begin{aligned}J_2(n)&=\int_{L\setminus M}S(\alpha)V(\alpha)e(-n\alpha)d\alpha\\ &=\sum_{q\leq P}\sum_{a=1}^{q*}q^{-1}g(a, q)e\left(-\frac{a}{q}n\right)\int_{1/qQ<|\beta|\leq 1/2}S\left(\frac{a}{q}+\beta\right)v(\beta)e(-n\beta)d\beta\\ &\ll P\left(\sum_{q\leq P}q^{-1}\sum_{a=1}^{q*}\left|\int_{1/qQ<|\beta|\leq 1/2}S\left(\frac{a}{q}+\beta\right)v(\beta)e(-n\beta)d\beta\right|^2\right)^{1/2}.\end{aligned}$$

By Bessel's inequality and (3.4), we have

$$\sum_{x-\Delta<n\leq n}|J_2(n)|^2 \ll P^2\sum_{q\leq P} q^{-1}\sum_{a=1}^{q*}\int_{1/qQ<|\beta|\leq 1/2}\left|S\left(\frac{a}{q}+\beta\right)v(\beta)\right|^2 d\beta$$

$$\ll P^2 Q\sum_{q\leq P}\sum_{a=1}^{q*}\int_{|\beta|\leq 1/2}\left|S\left(\frac{a}{q}+\beta\right)\right|^2 d\beta$$

$$\ll P^4 Q y(\log x)^{-1}. \tag{5.8}$$

Next we consider J_3. Changing the order of summation and integration, we use Cauchy's inequality and (3.4). Thus,

$$J_3(n)=\int_{m} S(\alpha)V(\alpha)e(-n\alpha)d\alpha$$

$$=\sum_{P<q\leq R}\sum_{a=1}^{q*}\int_{|\beta|\leq 1/qQ} S\left(\frac{a}{q}+\beta\right)q^{-1}g(a,\ q)v(\beta)e\left(-n\left(\frac{a}{q}+\beta\right)\right)d\beta$$

$$=\int_{|\beta|PQ\leq 1} v(\beta)e(-n\beta)\sum_{\substack{P<q\leq R\\ |\beta|qQ\leq 1}}\sum_{a=1}^{q*} q^{-1}g(a,\ q)S\left(\frac{a}{q}+\beta\right)e\left(-\frac{a}{q}n\right)d\beta$$

or

$$\sum_{x-\Delta<n\leq x}|J_3(n)|^2$$

$$\ll(\log x)\int_{|\beta|PQ\leq 1}\sum_{x-\Delta<n\leq x}\left|\sum_{\substack{P<q\leq R\\ |\beta|qQ\leq 1}}\sum_{a=1}^{q*} q^{-1}g(a,\ q)S\left(\frac{a}{q}+\beta\right)e\left(-\frac{a}{q}n\right)\right|^2 d\beta.$$

The large sieve [7, 8] yields that

$$\sum_{x-\Delta<n\leq x}|J_3(n)|^2 \ll(\log x)\int_{|\beta|PQ\leq 1}\sum_{\substack{P<q\leq R\\ |\beta|qQ\leq 1}}\sum_{a=1}^{q*}(\Delta+qR)\left|q^{-1}g(a,\ q)S\left(\frac{a}{q}+\beta\right)\right|^2 d\beta$$

$$\ll(\log x)\sum_{P<q\leq R}\sum_{a=1}^{q*}\int_{|\beta|\leq 1/qQ}\left(\frac{\Delta}{q}+R\right)\left|S\left(\frac{a}{q}+\beta\right)\right|^2 d\beta$$

$$\ll(\log x)\left(\frac{\Delta}{P}+R\right)\int_{m}|S(\alpha)|^2 d\alpha$$

$$\ll(\Delta P^{-1}+R)y. \tag{5.9}$$

We turn to J_4.

$$\sum_{x-\Delta<n\leq x}|J_4(n)|^2=\sum_{x-\Delta<n\leq x}\left|\int_{M\cup m} S(\alpha)(W(\alpha)-V(\alpha))e(-n\alpha)d\alpha\right|^2$$

$$\leq\int_{M\cup m}|S(\alpha)|^2|W(\alpha)-V(\alpha)|^2 d\alpha$$

$$\ll R(\log x)^2\int_{M\cup m}|S(\alpha)|^2 d\alpha$$

$$\ll y^{3/2}, \tag{5.10}$$

by Bessel's inequality and (3.2). Similarly, by (3.1),

$$\sum_{x-\Delta<n\leq x}|J_5(n)|^2=\sum_{x-\Delta<n\leq x}\left|\int_{n} S(\alpha)W(\alpha)e(-n\alpha)d\alpha\right|^2$$

$$\leq\int_{n}|S(\alpha)W(\alpha)|^2 d\alpha$$

$$\ll\left(\frac{y}{R}+Q\right)(\log x)\int_{n}|S(\alpha)|^2 d\alpha$$

$$\ll y^{3/2}(\log x). \tag{5.11}$$

In conjunction with (5.7)-(5.11) and (5.1), we have that

$$\sum_{x-\Delta<n\leq x}|\nu(n, y)-\mathfrak{S}(n, \sqrt{y})K(n, y)|^2\ll\Delta yP^{-1}+P^4Qy(\log x)^{-1}+Ry+y^{3/2}(\log x)$$

$$\ll\Delta yP^{-1}+P^4y^{3/2}$$

$$\ll\Delta y(\log x)^{-A},$$

as required.

This completes our proof.

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