ON THE SUM OF A PRIME AND A SQUARE

By

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1. Introduction.

In 1923 G. H. Hardy and J. E. Littlewood [3] conjectured that every large integer, not being a square, may be expressed as the sum of a prime and a square. Let \( \nu(n) \) be the number of representations of an integer \( n \) in this manner. They further stated the hypothetical asymptotic formula; As \( n(\neq k^2) \to \infty \),

\[
\nu(n) \sim \mathcal{S}(n) \frac{n^{1/2}}{\log n}
\]

with

\[
\mathcal{S}(n) = \prod_{p|n} \left( 1 - \frac{n/p}{p-1} \right)
\]

where \((-\)) is the Legendre symbol.

Define \( \mathcal{S}(k^2) = 0 \). In 1968 R. J. Miech [5] proved that

\[
\sum_{n \leq x} \left| \nu(n) - \mathcal{S}(n) \frac{n^{1/2}}{\log n} \left( 1 + O\left( \frac{\log \log n}{\log n} \right) \right) \right|^2 \ll x^2 (\log x)^{-A}
\]

for any \( A > 0 \), from which it follows that

\[
E(x) \ll x (\log x)^{-A}
\]

where \( E(x) \) denotes the number of integers \( n \leq x \) with \( \nu(n) = 0 \). It seems difficult to sharpen the right hand side of (1). However (2) may be improved, see [1, 9, 12].

A. I. Vinogradov [12; p. 35] remarked that, for any \( \varepsilon > 0 \),

\[
E(x) \ll x^{2/3 + \varepsilon}
\]

under the extended Riemann hypothesis. First of all we shall show

**Proposition.** Assume the extended Riemann hypothesis. Then

\[
E(x) \ll x^{1+\varepsilon} (\log x)^{\delta}.
\]

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It is the main aim of this paper to prove the following unconditional results.

**Theorem 1.** Let $1/2 < \Theta \leq 1$ and $A > 0$ be given. We have

$$\sum_{x^{1/2} \leq \lambda \leq x} \nu(n) - \mathcal{S}(n) \frac{\sqrt{n}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) \ll x^{\Theta + \varepsilon} (\log x)^{-A}$$

where the $O$-constant is absolute and the $<-$constant depends on $\Theta$ and $A$ only.

**Theorem 2.** Let $7/24 < \theta \leq 1$ and $A > 0$ be given. We have

$$E(x + x^\theta) - E(x) \ll x^\theta (\log x)^{-A}$$

where the implied constant depends on $\theta$ and $A$ only.

Our assertion may be regarded as a refinement of Miech's work (1)(2), and must be compared with a conditional bound (3). Within the frame of Circle method, we appeal to the large sieve [7, 8] and R.C. Vaughan's method [11; Chap. 4] on Weyl sums.

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2. **Singular series.**

In this section we collect the facts of $\mathcal{S}(n)$. For the proof, see [1, 5, 9, 12].

For integers $q$ and $n$, let $\rho(q, n)$ be the number of solutions of the congruence $x^2 \equiv n \pmod{q}$, and $\rho_1$ be the convolution inverse of $\rho$ with respect to $q$. Define, for $Q \geq 3$,

$$\mathcal{S}(n, Q) = \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \rho_1(q, n).$$

Then, uniformly for $n$,

$$\mathcal{S}(n, Q) \ll \log Q.$$ 

Let $\mathcal{D}$ be the set of fundamental discriminants. An integer $n$ may be uniquely written as $n = n_1 n_2^2$ with a square-free $n_1$. Put

$$\delta(n) = \begin{cases} n_1 & \text{if } n_1 \equiv 1 \pmod{4} \\ 4n_1 & \text{otherwise.} \end{cases}$$

Thus, if $n \neq k^2$ then $\delta(n) \in \mathcal{D}$. For $d \in \mathcal{D}$, the Kronecker symbol $(d/\cdot)$ is a primitive character to modulus $d$. Let $\mathcal{L} = \mathcal{L}(T)$, $T \geq 3$, denote the set of $d \in \mathcal{D}$ for which $L(s, (d/\cdot))$ the Dirichlet $L$-function has no zero in the region:
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Re(s) ≥ 29/30 and |Im(s)| ≤ T. Suppose x ≫ T. If δ(n) ∈ L then there exists a constant η > 0 such that

\( θ(n, Q) = θ(n) + O(Q^{-\eta} \exp(\sqrt{\log x})) \)
uniformly for \( n \leq x \). Moreover,

\( \pm \{ \delta : \delta \in \mathcal{D} \setminus \mathcal{L}, \delta \leq 4x \} \ll x^{1/4}(\log x)^{14}. \)

Finally, for \( n \neq k^2 \),

\( \frac{\varphi(n)}{n} \ll θ(n) L\left( 1, \left( \frac{\delta(n)}{\varphi(n)} \right) \right) \ll \frac{n}{\varphi(n)}. \)

3. A conditional estimate.

In this section we illustrate our device with the proof of Proposition. We employ the Circle method [11].

Let \( x \) be a large parameter. We divide the unit interval by the Farey dissections of order \( Q = x^{1/2}(\log x)^5 \).

For \( (a, q) = 1 \), write

\( I_{q, a} = \left[ \frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right]. \)

Put

\( M = \bigcup_{q \in \mathcal{P}} \bigcup_{0 \leq a < q} I_{q, a}, \quad P = x/100Q, \)

\( m = [Q^{-1}, 1 + Q^{-1}] \setminus M. \)

We define the exponential sum

\( W(\alpha) = \sum_{m \leq x} e(\alpha m^2) \)

where \( e(t) = e^{2\pi it} \). By Weyl's inequality, we see that

\( |W(\alpha)|^2 \ll \left( \frac{x}{q} + x^{1/2} + q \right) \log qx, \)

for \( |\alpha - (a/q)| \leq q^{-1} \) with \( (a, q) = 1 \). When \( \alpha \in I_{q, a} \), \( W(\alpha) \) is approximated by

\( V(\alpha) = q^{-1}g(a, q)v\left( \alpha - \frac{a}{q} \right) \)

where

\( g(a, q) = \sum_{m \leq x} e\left( \frac{a}{q} m^2 \right) \) and \( v(\beta) = \sum_{m \leq x} \frac{e(\beta m)}{2\sqrt{m}}. \)

Actually it follows from [11; Theorem 4.1] and [12; p. 38] that
(3.2) \[ |W(\alpha) - V(\alpha)|^2 \ll q^{\log q} \]
for \( |\alpha - (a/q)| \leq (4q\sqrt{x})^{-1} \). In addition, we note that
(3.3) \[ |g(a, q)|^2 \ll q, \]
(3.4) \[ |v(\beta)|^2 \ll \min(x, \|\beta\|^{-1}) \]
where \( \|t\| = \min_{n \neq 0} |t - n|. \)

Put
\[ S(\alpha) = \sum_{n \in \mathbb{Z}} A(n) e(\alpha n) \]
where \( A \) is the von Mangoldt function. It is expected that, for \( \alpha \in I_{q, a} \), \( S(\alpha) \)
is nearly equal to
\[ T(\alpha) = \frac{\mu(q)}{\phi(q)} t\left(\alpha - \frac{a}{q}\right) \]
where
\[ t(\beta) = \sum_{n \in \mathbb{Z}} e(\beta n) \ll \min(x, \|\beta\|^{-1}). \]

In order to show this, define
(3.6) \[ J(q) = \sum_{q^k \leq x} |S(\alpha) - T(\alpha)|^2 d\alpha \]
where * in \( \sum^* \) stands for \( (a, q) = 1 \). If \( \alpha \in I_{q, a} \),
\[ S(\alpha) - T(\alpha) = -\sum_{n \in \mathbb{Z}} \chi(n) A(n) e\left(\alpha - \frac{a}{q}\right) n + O((\log x)^2). \]

Here \# in \( \sum^\# \) means that if \( \chi \) is principal then \( \chi(n) A(n) \) should be replaced by \( A(n) - 1 \). When \( q \leq P \), by [2; Lemma 1], we have
\[ J(q) \ll \sum_{n \in \mathbb{Z}} |\chi(n) A(n) e(\beta n)|^2 d\beta + Q^{-1}(\log x)^4. \]

On noting \( qQ \leq PQ = x/100 \), it is easy to show that the above integral is
\[ \ll qQx(\log x)^4, \]
under the extended Riemann hypothesis. Therefore, uniformly for \( q \leq P \),
(3.7) \[ J(q) \ll Q^{-1}x(\log x)^4. \]

Now, for \( n \leq x \),
(3.8) \[ \sum_{t + m^r = n} A(n) = \int_{q-1}^{t+q-1} S(\alpha) W(\alpha) e(-na) d\alpha = \int_{q} + \int_{m}. \]
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By Bessel's inequality and (3.1),

\[ \sum_{n \leq x} \left| \int_{m} S(\alpha)W(\alpha)e(-na)d\alpha \right|^2 \leq \left( \int_{m} |S(\alpha)W(\alpha)|^2 d\alpha \right)^{1/2} \leq \sup_{\alpha \in \mathbb{M}} |W(\alpha)| |\int_{m} |S(\gamma)|^2 d\gamma \leq Qx(\log x)^2. \]

On \( \alpha \in \mathbb{M} \) we first exchange \( S(\alpha) \) for \( T(\alpha) \). Thus, by (3.1) and (3.7),

\[ \sum_{n \leq x} \left| \int_{\mathbb{M}} (S(\alpha) - T(\alpha))W(\alpha)e(-na)d\alpha \right|^2 \leq \sum_{q \in \mathbb{P}} \sum_{d=1}^{\phi(q)} \left| \int_{I_q,a} |S(\alpha) - T(\alpha)|^2 |W(\alpha)|^2 d\alpha \right| \leq \sum_{q \in \mathbb{P}} \frac{x}{q} (\log x) \sum_{q} \left( \sum_{\beta} |t(\beta)|^2 \right)^{1/2} \leq x^2 Q^{-1}(\log x)^4. \]

Next we replace \( W(\alpha) \) by \( V(\alpha) \). On using (3.2) and (3.5),

\[ \sum_{n \leq x} \left| \int_{\mathbb{M}} T(\alpha)V(\alpha)e(-na)d\alpha \right|^2 \leq \sup_{\alpha \in \mathbb{M}} |V(\alpha)| |\int_{\mathbb{M}} |T(\gamma)|^2 d\gamma \leq \sum_{q \in \mathbb{P}} \sum_{d=1}^{\phi(q)} \left( \sum_{\beta} \mu(q) \frac{\mu(q)}{q \phi(q)} |t(\beta)|^2 \right)^{1/2} \leq P \frac{x}{\log P} \left( \sum_{q \in \mathbb{P}} \frac{x}{q} \right)^{1/2} \leq Px(\log x)^3. \]

Finally we extend the Farey arc \( I_{q,a} \) to \( I_{q,a} = [(a/q)-(1/2), (a/q)+(1/2)] \). The resulting remainder is then equal to

\[ r_n = \sum_{q \in \mathbb{P}} \sum_{d=1}^{\phi(q)} \int_{I_{q,a}} T(\alpha)V(\alpha)e(-na)d\alpha \]

\[ = \sum_{q \in \mathbb{P}} \sum_{d=1}^{\phi(q)} \int_{1/q < \beta < 1/3} \frac{\mu(q)}{q \phi(q)} t(\beta)q^{-1}g(a, q)v(\beta)\left(-n\left(\frac{a}{q} + \beta\right)\right) d\beta \]

\[ = \int_{1/q < \beta < 1/3} t(\beta)v(\beta)e(-\beta n) \sum_{q \in \mathbb{P}} \sum_{\beta > 1/3} \frac{\mu(q)}{q \phi(q)} g(a, q) e\left(-\frac{a}{q} n\right) d\beta. \]

On using Cauchy's inequality and (3.4), we have

\[ \sum_{n \leq x} \left| r_n \right| \leq \sum_{n \leq x} \left| \int_{1/q < \beta < 1/3} |v(\beta)|^2 d\beta \int_{1/q < \beta < 1/3} |t(\beta)|^2 \sum_{\beta > 1/3} \frac{\mu(q)}{q \phi(q)} g(a, q) e\left(-\frac{a}{q} n\right) d\beta \]

\[ \leq (\log x) \int_{1/q < \beta < 1/3} |t(\beta)|^2 \left( \sum_{n \leq x} \left| \sum_{\beta > 1/3} \frac{\mu(q)}{q \phi(q)} g(a, q) e\left(-\frac{a}{q} n\right) \right|^2 \right)^{1/2} d\beta. \]
The large sieve inequality [7, 8] yields that
\[
\sum_{n \leq x} |r_n|^2 \ll (\log x)^2 \int_{1/2 < \beta < 1/2} |t(\beta)|^2 \left( \sum_{q \leq x} \sum_{P \leq q} (x+qP) \left| \frac{\mu(q)}{q \phi(q)} g(a, q) \right|^2 \right) d\beta
\]
\[
\ll (\log x) \sum_{q \leq x} (x+qP) \frac{\mu(q)}{q \phi(q)} \int_{1/2 < |\beta| < 1/2} |t(\beta)|^2 d\beta
\]
\[
\ll (\log x) (x+P^2) Q \sum_{q \leq x} \frac{\mu(q)}{q \phi(q)}
\]
\[
(3.13) \quad \ll Qx(\log x)^6.
\]
Here we used the bounds (3.3) and (3.5).

It remains to calculate
\[
(3.14) \quad \sum_{q \leq x} \sum_{\alpha \in \mathcal{I}(\alpha)} T(\alpha) V(\alpha) e(-n\alpha) d\alpha
\]
\[
= \sum_{q \leq x} \sum_{\alpha \in \mathcal{I}(\alpha)} \frac{\mu(q)}{q \phi(q)} g(a, q) e\left(-\frac{a}{q}\right) \left( \frac{1+1}{2} \right)^{1-1/2} \sum_{m \leq x} e\left(\frac{\beta(t+m-n)}{2 \sqrt{m}}\right) d\beta.
\]
The above sum is \( \mathcal{E}(n, P) \) with the definition (2.1). The integral is equal to
\[
\sum_{t \leq n} \frac{1}{2\sqrt{n-t}} = \sqrt{n} + O(1).
\]
Hence, by (2.2), (3.14) becomes
\[
(3.15) \quad \mathcal{E}(n, P) \sqrt{n} + O(\log x).
\]
On summing up the above argument (3.9)-(3.15), we obtain
\[
(3.16) \quad \left| \sum_{n \leq x} \log p - \mathcal{E}(n, P) \right|^2 \ll Qx(\log x)^6 + x^2Q^2(\log x)^6 + Px(\log x)^5
\]
Now, the extended Riemann hypothesis implies that \( \mathcal{L} = \emptyset \) the sets introduced in section 2, and that \( \mathcal{E}(n) \gg (\log \log n)^{-2} \) for \( n \neq k^2 \) and \( n > 1 \). By (2.3) we then have that, for \( n \neq k^2 \) (\( \gg 1 \)),
\[
\mathcal{E}(n, P) \gg (\log \log n)^{-2}.
\]
Consequently (3.16) leads that
\[
x^{3/2}(\log x)^5 \sum_{n \leq x} \left| \sum_{p \leq m \leq x} \log p - \mathcal{E}(n, P) \sqrt{n} \right|^2
\]
\[
\gg \sum_{s \in \mathcal{S} \cap [5, 10]} \frac{1}{\mathcal{E}(n, P)^{\delta n}}
\]
\[
\gg x(\log \log x)^{-4} \sum_{n \neq k^2} 1
\]
or

\[ E(x) \leq \sum_{\frac{x}{2} \leq n \leq x} 1 + \sum_{\frac{x}{2} \leq n \leq x} \left( \frac{x}{2^r} \right)^{1/2} (\log x \log \log x)^t + x^{1/3} \leq x^{1/2} (\log x)^b, \]

as required.

4. Proof of Theorems.

In this section we derive Theorems from the known results mentioned in section 2 and our main lemma below. Lemma will be verified in the next section.

**Lemma.** Let \( x \) be a large parameter. For given \( 1/2 < \Theta \leq 3/4 \) and \( 7/12 < \Xi \leq 1 \), put \( \Delta = y^b \) and \( y = x^{a} \). Write

\[ \nu(n, y) = \# \{ (p, m) : x - y < p \leq x, m^2 \leq y, p + m^2 = n \}, \]

and

\[ K(n, y) = \int_1^{n-(x-y)-3} \frac{dt}{2\sqrt{t \log(n-t)}}. \]

Then, for any \( A > 0 \), we have

\[ \sum_{x - x^a \leq n \leq x} |\nu(n, y) - \Xi(n, \sqrt{x})K(n, y)|^2 \leq \Delta y (\log x)^{-A} \]

where the implied constant depends on \( \Theta, \Xi \) and \( A \) only.

**Proof of Theorem 1.** Let \( \mathcal{L} = \mathcal{L}(x) \) in section 2. Choose \( \Xi = 1 \) in Lemma. Then, because of \( \nu(n, x) = \nu(n) \),

\[ \sum_{x - x^a \leq n \leq x} |\nu(n) - \Xi(n, \sqrt{x})K(n, x)|^2 \leq x^{6/4} (\log x)^{-A} \]

for any \( A > 0 \). We note that

\[ K(n, x) = \frac{\sqrt{\nu}}{\log n} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right) \]

with an absolute \( O \)-constant. Combining the above with (2.3) and (2.5) we have

\[ S = \sum_{x - x^a \leq n \leq x} |\nu(n) - \Xi(n)K(n, x)|^2 \]

\[ = \sum_{d(n)^2 \leq n \leq x} + \sum_{\frac{n}{x} \leq k \leq x} \left( \nu(n)^2 + \left( \frac{\Xi(n)}{\log n} \right)^2 n \right) + \sum_{\frac{n}{x} \leq k \leq x} \nu(k^2)^2 \]
\[
\ll x^{-\theta_{w}} \sum_{x-x^{\theta_{w}} < n \leq x} |\nu(n) - \mathcal{S}(n, \sqrt{x})K(n, x)|^{2} + x^{\theta_{w}+1} \sup_{\delta(n) \in \mathcal{L}} \left| \mathcal{S}(n) - \mathcal{S}(n, \sqrt{x}) \right|^{2} \\
+ x \left( 1 + \sup_{\delta(n) \in \mathcal{L}} \left( \frac{\mathcal{S}(n)}{\log n} \right)^{2} \right) \sum_{d \in \mathbb{Z}^{+}} \frac{1}{\delta(n) = d} \sum_{x-x^{\theta_{w}} < n \leq x} 1
\ll x^{\theta_{w}+1}(\log x)^{-A} + x \left( 1 + \sup_{d \in \mathbb{Z}^{+}} \left| L \left( 1, \left( \frac{d}{\mathcal{S}(n)} \right) \right) \right|^{-2} \right) \left( 1 + \sum_{d \in \mathbb{Z}^{+}} 1 \right) x^{\theta_{w}-1/8}.
\]

By (2.4) and Siegel's theorem [10; Kap. IV, § 8], \( S \) becomes

\[
\ll x^{\theta_{w}+1}(\log x)^{-A} + x^{\theta_{w}-1/2}(x^{\theta})(1 + x^{1/4}(\log x)^{14})
\ll x^{\theta_{w}+1}(\log x)^{-A}.
\]

Hence we obtain Theorem 1 in case \( 1/2 < \theta \leq 3/4 \). If \( 3/4 < \theta \leq 1 \), Theorem 1 follows from the case of \( \theta = 2/3 \), by splitting up the interval \( (x-x^{\theta}, x] \) into the sum of smaller intervals of type \( (u-u^{2/3}, u] \).

**Proof of Theorem 2.** Put \( \theta = \theta \mathcal{L} \) in Lemma. Then, \( 7/24 < \theta \leq 3/4 \). It is sufficient to prove Theorem 2 for \( \theta \) in the above range only. Since \( \nu(n) = 0 \) implies \( \nu(n, y) = 0 \), Lemma yields that

\[
\sum_{x-x^{\theta} < n \leq x} |\mathcal{S}(n, \sqrt{y})K(n, y)|^{2} \ll yx^{\theta}(\log x)^{-A-\delta}.
\]

Here, \( K(n, y)^{2} \ll y(\log x)^{-2} \). Thus,

\[
\sum_{x-x^{\theta} < n \leq x} 1 \ll x^{\theta}(\log x)^{-A-\delta}(\log x \log \log x)^{2}
\]

by (2.3) and (2.5) with \( \mathcal{L} = \mathcal{L}(x) \). Hence, by (2.4), we obtain

\[
E(x) - E(x-x^{\theta}) \ll \sum_{x-x^{\theta} < n \leq x} 1 + \sum_{x-x^{\theta} < n \leq x} 1
\ll x^{\theta}(\log x)^{-A} + \sum_{d \in \mathbb{Z}^{+}} \left( x^{\theta-1/2} + 1 \right)
\ll x^{\theta}(\log x)^{-A} + x^{1/4}(\log x)^{14}
\ll x^{\theta}(\log x)^{-A},
\]

as required.

5. **Proof of Lemma.**

Put

\[
S(\alpha) = \sum_{x-y < n \leq x} e(\alpha p).
\]
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And we define the exponential sums $W(a)$ and $V(a)$ by the similar way in section 3, except for changing the parameter $x$ in section 3 by $y$. The Farey arcs are determined as follows:

$$Q = y^{1/2} (\log x),$$

$$M = \bigcup_{q \geq P} \bigcup_{a \leq y} I_{q,a}, \quad L = \bigcup_{q \geq P} \bigcup_{a \leq y} I_{q,a}, \quad P = (\log x)^{\delta}$$

$$m = \bigcup_{P < q < R} \bigcup_{a \leq y} I_{q,a}, \quad R = y/Q$$

$$n = [Q^{-1}, 1 + Q^{-1}] \setminus (M \cup m).$$

Here $I_{q,a}$ and $I_{q,a}$ are similar to that in section 3. We then have

$$\psi(n, y) = \int_{Q^{-1}}^{1 + Q^{-1}} S(a)W(\alpha)e(-n\alpha)d\alpha$$

$$= \int_{L} SV - \int_{L \setminus M} SV + \int_{M \setminus m} SV + \int_{m} SV + \int_{n} SV$$

$$= J_1 - J_2 + J_3 + J_4 + J_5, \quad \text{say.}$$

First we evaluate $J_1$. An elementary calculation leads that

$$J_1(n) = \int_{L} S(a)V(\alpha)e(-n\alpha)d\alpha$$

$$= \sum_{q \leq P} \sum_{a = 1}^{\varphi(q)} \int_{-1/2}^{1/2} S\left(\frac{a}{q} + \beta\right) q^{-1} g(a, q)v(\beta)e\left(-n\left(\frac{a}{q} + \beta\right)\right)d\beta$$

$$= \sum_{q \leq P} \sum_{a = 1}^{\varphi(q)} q^{-1} g(a, q)e\left(-\frac{a}{q}n\right) \sum_{\frac{x}{y} \in P \cap \mathbb{Z}} \frac{\varphi(\frac{x}{y})p}{2\sqrt{m}} \int_{-1/2}^{1/2} \alpha(\beta \pm m - n)d\beta$$

$$= \sum_{q \leq P} q^{-1} \sum_{\frac{x}{y} \in P \cap \mathbb{Z}} \frac{\varphi(\frac{x}{y})p + m - n}{2\sqrt{n - p}}$$

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$$(5.2)$$

We now appeal to the well known result on primes in arithmetical progressions [10; Kap. IV. § 3]. It follows from [10; Kap. VIII. Satz 6.2, Kap. IV. Satz 8.1] zero free region and [4, 6; Theorem 12.1] zero density estimates for the Dirichlet $L$-functions that, for given positive constants $\varepsilon, E$ and $F$, 

$$(5.3) \quad \frac{K(n, y)}{\varphi(d)} + O\left(1 + \sup_{(b, d) = 1} \sup_{\lambda > 0} \int_{-\lambda}^{\lambda} \left| \sum_{p \leq X \leq Y} \log p - \frac{t}{\varphi(d)} \right| \right).$$

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$$(5.3) \quad \frac{K(n, y)}{\varphi(d)} + O\left(1 + \sup_{(b, d) = 1} \sup_{\lambda > 0} \int_{-\lambda}^{\lambda} \left| \sum_{p \leq X \leq Y} \log p - \frac{t}{\varphi(d)} \right| \right).$$
\[ \sum_{\substack{p \leq X^{1/2} \leq k \leq X \atop p \text{ prime}}} \log \frac{p}{q(k)} = O(\log X)^{-\varepsilon} \]

uniformly for \((l, k) = 1, k \leq (\log X)^{\varepsilon}\) and \(X^{1/2 + \varepsilon} \leq Y \leq X\). Hence the O-term in (5.3) is at most

\[ y^{1/4}(\log x)^{-1/4}, \]

and contributes to (5.2)

\[ \ll y^{1/4}(\log x)^{-1/4} \sum_{q \leq \sqrt{x}} \tau(q)q \]

\[ \ll y^{1/4}(\log x)^{-1/4} P^2(\log P) \]

\[ \ll y^{1/4} P. \]

On combining this with (5.2) and (5.3) we have

\[ J_1(n) = K(n, y) \sum_{q \leq \sqrt{x}} q^{-1} \sum_{m \leq y} \sum_{d \mid (d, m^2 - n)} \mu(q) \frac{d}{\varphi(d)} + O(\frac{y^{1/4} P}{\log P}). \]

Notice that the above sum is

\[ \sum_{q \leq \sqrt{x}} \frac{\mu(q)}{q \varphi(q)} \sum_{m \leq y} c_q(m^2 - n) = \Omega(n, P) = \sum_{q \leq \sqrt{x}} \frac{\mu(q)}{q \varphi(q)} g(a, q)e\left(\frac{a}{q} n\right). \]

We widen the range of \( q \) up to \( \sqrt{y} \). Let \( J_1(n) \) be the resulting cost. On employing the large sieve inequality [7, 8] and (3.3),

\[ \sum_{x - \Delta \leq n \leq x} |J_1(n)|^2 \ll K(n, y)^2 \sum_{x' \leq n \leq x} \left| \sum_{P \leq q \leq \sqrt{y}} \frac{\mu(q)}{q \varphi(q)} g(a, q)e\left(\frac{a}{q} n\right) \right|^2 \]

\[ \ll y(\log x)^{-\varepsilon} \sum_{P \leq q \leq \sqrt{y}} \frac{\mu(q)}{q \varphi(q)} g(a, q) \left| \frac{\mu(q)}{q \varphi(q)} \right|^2 \]

\[ \ll y(\log x)^{-\varepsilon} \left( \frac{\Delta}{P} + \sqrt{y} \right) \sum_{q \leq \sqrt{y}} \frac{\mu(q)}{q \varphi(q)} \]

\[ \ll y(\Delta P^{-1} + \sqrt{y}). \]

In conjunction with (5.4), (5.5) and (5.6) we obtain

\[ J_1(n) = K(n, y) \sum_{x - \Delta \leq n \leq x} e\left(\frac{a}{q} n\right) + \Omega(n, P) \ll \Delta y P^{-1} + y^{1/4}. \]

We proceed to \( J_2 \). On using Cauchy's inequality and (3.3),

\[ J_2(n) = \int_{L, M} S(\alpha)V(\alpha)e(-\alpha n) d\alpha \]

\[ = \sum_{q \leq \sqrt{y}} q^{-1} g(a, q)e\left(-\frac{a}{q} n\right) \int_{1/2 < \beta < 1} S\left(\frac{a}{q} + \beta\right) e(n \beta) v(\beta) d\beta \]

\[ \ll P^{1/2} \left( \sum_{q \leq \sqrt{y}} q^{-1} \sum_{d \mid (d, q)} |S\left(\frac{a}{q} + \beta\right) e(n \beta) v(\beta) d\beta| \right)^{1/2}. \]
By Bessel’s inequality and (3.4), we have

\[
\sum_{x - \Delta \leq n \leq x} |J_3(n)|^2 \ll P^2 \sum_{q \leq P} \sum_{\alpha = 1}^{q-1} \int_{1 + \beta \leq q \leq 1 + \frac{1}{x}} \left| S\left(\frac{a}{q} + \beta \right) \nu(\beta) \right|^2 d\beta \\
\ll P^2 Q \sum_{q \leq P} \sum_{\alpha = 1}^{q-1} \int_{1 + \beta \leq q \leq 1 + \frac{1}{x}} \left| S\left(\frac{a}{q} + \beta \right) \right|^2 d\beta \\
(5.8) \\
\ll P^4 Q (\log x)^{-1}.
\]

Next we consider \( J_4 \). Changing the order of summation and integration, we use Cauchy’s inequality and (3.4). Thus,

\[
J_4(n) = \int_{m} (S(\alpha)V(\alpha)e^{-n\alpha}) d\alpha \\
= \sum_{p \leq Q} \sum_{\alpha = 1}^{q-1} \int_{1 + \beta \leq r \leq 1 + \frac{1}{x}} S\left(\frac{a}{q} + \beta \right) q^{-1} g(\alpha, q) \nu(\beta) e\left(-n\left(\frac{a}{q} + \beta \right)\right) d\beta \\
= \int_{1 + \beta \leq r \leq 1 + \frac{1}{x}} \nu(\beta) e\left(-n\beta\right) \sum_{p \leq Q} \sum_{\alpha = 1}^{q-1} q^{-1} g(\alpha, q) \left| S\left(\frac{a}{q} + \beta \right) \right| e\left(-\frac{a}{q} \beta\right) d\beta \\
or
\sum_{x - \Delta \leq n \leq x} |J_4(n)|^2 \\
\ll (\log x) \left[ \int_{1 + \beta \leq r \leq 1 + \frac{1}{x}} \left| \sum_{p \leq Q} \sum_{\alpha = 1}^{q-1} q^{-1} g(\alpha, q) \left| S\left(\frac{a}{q} + \beta \right) \right| e\left(-\frac{a}{q} \beta\right) \right|^2 d\beta \right]^{1/2}.
\]

The large sieve \([7, 8]\) yields that

\[
\sum_{x - \Delta \leq n \leq x} |J_4(n)|^2 \ll (\log x) \left[ \sum_{1 + \beta \leq r \leq 1 + \frac{1}{x}} \left| \sum_{p \leq Q} \sum_{\alpha = 1}^{q-1} (\Delta + qR) \left| q^{-1} g(\alpha, q) \right| S\left(\frac{a}{q} \beta + \beta \right) \right|^2 d\beta \right]^{1/2} \\
\ll (\log x) \left( \frac{\Delta}{P} + R \right) \left[ \int_{m} |S(\alpha)|^2 d\alpha \right]^{1/2} \\
\ll (\Delta P^{-1} + R) y.
(5.9)
\]

We turn to \( J_5 \).

\[
x - \Delta \leq n \leq x \sum_{x - \Delta \leq n \leq x} |J_5(n)|^2 = \sum_{x - \Delta \leq n \leq x} \left| \int_{M \leq m} S(\alpha)(W(\alpha) - V(\alpha))e(-n\alpha) d\alpha \right|^2 \\
\ll \int_{M \leq m} |S(\alpha)|^2 |W(\alpha) - V(\alpha)|^2 d\alpha \\
\ll R(\log x)^2 \int_{M \leq m} |S(\alpha)|^2 d\alpha \\
(5.10) \\
\ll x^{3/2},
\]

by Bessel’s inequality and (3.2). Similarly, by (3.1),
\[ \sum_{x-d<n<x} |J_s(n)|^\alpha = \sum_{x-d<n<x} \left| \int \frac{S(\alpha)W(\alpha)e(-n\alpha)d\alpha}{\alpha^\beta} \right|^\alpha \leq \int_n |S(\alpha)W(\alpha)|^\beta d\alpha \]
\[ \ll \left( \frac{\delta}{R} + Q \right)(\log n) \int_n |S(\alpha)|^\beta d\alpha \]
\[ \ll y^{\beta/3}(\log x). \]

(5.11)

In conjunction with (5.7)-(5.11) and (5.1), we have that
\[ \sum_{x-d<n<x} |V(n, y)|^\beta \ll \Delta yP^{-1} + P^\gamma Q y(\log x)^{-1} + R y + y^{\beta/3}(\log x) \]
\[ \ll \Delta yP^{-1} + P^\gamma y^{\beta/3} \]
\[ \ll \Delta y(\log x)^{-A}, \]
as required.

This completes our proof.

References


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