ON THE RESIDUAL TRANSCENDENTAL EXTENSIONS
OF A VALUATION. KEY POLYNOMIALS
AND AUGMENTED VALUATION

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Let $K$ be a field and $v$ a valuation on $K$. The problem of extending $v$ to $K(X)$ (the field of rational functions of one indeterminate) has been previously considered in some works as [7] and [10]. Particularly in [7], MacLane studied the case when $v$ is discrete and rank one. In solving the problem in this case, MacLane used some notions as key polynomial and augmented valuation.

An extension $w$ of $v$ to $K(X)$ is called residual transcendental (briefly, an r.t. extension) if the residue field of $w$ is a transcendental extension of the residue field of $v$ (MacLane called these extensions “inductive value”). Some aspects of r.t. extensions have been considered in [5, Ch. VI], [9], [1], [2], [3] and [11]. Particularly in [2] and [11] all r.t. extensions of $v$ to $K(X)$ were described using the notion of “minimal pair” (see definition in Section 1). Although in [3] some results on minimal pairs were given, the problem of finding minimal pairs in the general setting seems to be difficult.

In this work we follow, for arbitrary r.t. extensions, MacLane’s ideas of key polynomial and augmented valuation and show that these give a powerful tool in the study of all extensions of $v$ to $K(X)$. In particular, the key polynomials over an r.t. extension give us the possibility of defining some new minimal pairs (Theorem 5.1).

Now we briefly describe the content of the paper. Section 1 contains notation, definitions and the main results from [2] and [11], Theorem 1.2 and some consequences of this theorem will be used in this paper.

In section 2, we give some technical results related to the domination of valuations on $K(X)$, which was also introduced by MacLane in [7]. This notion has been used in [4] to describe all valuations on $K(X)$. In Section 3 (after MacLane [7]) key polynomial and augmented valuation are defined.

The key polynomials over an r.t. extension are studied in Section 4. The main results are given in Theorems 4.4 and 4.6. We remark that Theorem 4.6

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and the proof are inspired by MacLane’s work [7].

In Section 5 it is proved that a (“comensurable”) augmented valuation over a given r.t. extension is also an r.t. extension (Theorem 5.1). By this theorem we can define new minimal pairs, starting from a given one. Theorem 5.5 shows how the augmented valuations are closely related to the domination. Finally in Section 6, using the results of previous sections, we give another proof of a result which asserts that there exist r.t. extensions with given residue field and value group (Theorem 6.4, see [3, Theorem 4.4]).

In a forthcoming paper we will use the results developed here to study all valuations on \( K(X) \) and related topics.

1. Notations and definitions

In this section we recall notations, definitions and the main results of [2] and [11] (see Theorem 1.2), which will be used in the rest of this paper. Also some new consequences of Theorem 1.2 are given.

Let \( K \) be a field and \( v \) a valuation on \( K \). We sometimes emphasize this situation saying that \( (K, v) \) is a valuation pair. \( G_v, O_v, k_v \) and \( \rho_v: O_v \rightarrow k_v \) represent the value group of \( v \), the valuation ring of \( v \), its residue field and the residue homomorphism, respectively. If \( x \in O_v \), we usually write by \( x^* \) the image \( \rho_v(x) \) of \( x \) in \( k_v \). We refer the reader to [5], [6], [12], [13] for general notions and definitions.

Let \( K'/K \) be an extension of fields. A valuation \( v' \) on \( K' \) will be called an extension of \( v \) if \( v'(x) = v(x) \) for all \( x \in K \). When \( v' \) is an extension of \( v \), we shall identify canonically \( k_v \) with a subfield of \( k_{v'} \) and \( G_v \) with a subgroup of \( G_{v'} \).

Throughout this paper, we fix a valuation pair \( (K, v) \), an algebraic closure \( \overline{K} \) of \( K \) and an extension \( \overline{v} \) of \( v \) to \( \overline{K} \). Then \( k_{\overline{v}} = k_v \) an algebraic closure of \( k_v \) and \( G_{\overline{v}} = G_v = QG_v \), i.e. \( G_v \) is the smallest divisible group which contains \( G_v \).

As usual we denote by \( K[X] \) and \( K(X) \) the polynomial ring and the field of rational functions of an indeterminate \( X \) over \( K \), respectively. If \( r = f/g, f, g \in K[X] \) and \( f, g \) are relatively prime, we define the order of \( r \) by the equality: \( \text{ord } r = \max(\deg f, \deg g) \). It is easy to see that \( \text{ord } r = [K(X): K(r)] \).

2. Let \( w \) be an extension of \( v \) to \( K(X) \). According to [8] (see also [1] and [2]), \( w \) is called a residual transcendental (r.t.)-extension of \( v \) if \( k_w/k_v \) is a transcendent extension. An element \( (a, \delta) \) of \( \overline{K} \times \overline{G_v} \) is usually called a pair. If \( (a, \delta) \) is a pair, we define the valuation \( w_{(a, \delta)} \) of \( \overline{K}(X) \)(see [5, Ch. VI, par.
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10]:

\[ w_{(a, \delta)}(f) = \inf (\delta(a_i) + i\delta) \text{ when } f(x) = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n \in \overline{K}[X]. \]

Usually one says that \( w_{(a, \delta)} \) is defined by \( \inf, \delta, a \in \overline{K} \) and \( \delta \in \mathcal{C}_v \). It is easy to see that \( w_{(a, \delta)} \) is an r.t. extension of \( \nu \) to \( \overline{K}(X) \) ([5, Ch. VI, par. 10], or [1]).

**Proposition 1.1.** ([1]). Every r.t. extension \( w \) of \( \nu \) to \( \overline{K}(X) \) is of the form \( w = w_{(a, \delta)} \) for a suitable pair \( (a, \delta) \). Moreover two pairs \( (a, \delta) \) and \( (a', \delta') \) define the same valuation on \( \overline{K}(X) \), i.e. \( w_{(a, \delta)} = w_{(a', \delta')} \), if and only if \( \delta = \delta' \) and \( \nu(a - a') \geq \delta \).

A pair \( (a, \delta) \in \overline{K} \times \mathcal{C}_v \) will be called minimal with respect to \( K \) if, for every \( b \in \overline{K} \) such that \( [K(b): K] < [K(a): K] \), one has \( \nu(a - b) < \delta \). In [2], [3], [4] and [11] it is shown that the minimal pairs play a prominent part in the definition and in the study of r.t. extensions.

**Theorem 1.2.** ([2], [11]). Let \( w \) be an r.t. extensions of \( \nu \) to \( K(X) \). Then there exists a pair \( (a, \delta) \) minimal with respect to \( K \) such that \( w \) coincides with the restriction of \( w_{(a, \delta)} \) to \( K(X) \). Moreover one has:

a) Denote by \( f \) the monic minimal polynomial of \( a \) with respect to \( K \) and put \( \gamma = w(f) \). Then

\[ w(F) = \inf (\delta(F(a)) + i\gamma), \quad \text{where } F = F_0 + F_1f + \cdots + F_nf^n \in K[X], \]

\[ \deg F_i < n = \deg f. \]

b) Let \( \nu \) be the restriction of \( \nu \) to \( K(a) \). If \( e \) is the smallest natural number such that \( e\gamma \in G_v \) one has:

\[ G_w = G_v + Z\gamma \quad \text{and} \quad [G_w : G_v] = e[G_v : G_v]. \]

c) Let \( h \in K[X] \). If \( \deg h < n = \deg f \) and \( \nu(h(h(a))) = e\gamma = ew(f) \), then \( r = f^{e}/h \) is the element of \( O_w \) of the smallest order such that \( r^* \in k_w \) is transcendental over \( k_v \).

d) The field \( k_w \) can be canonically identified with the algebraic closure of \( k_v \) in \( k \). Moreover, on has \( k_w = k_v(r^*) \).

**Notation 1.3.** If \( w \) is an r.t. extension of \( \nu \) to \( K(X) \), a minimal pair \( (a, \delta) \) in the previous theorem is called a minimal pair of definition of \( w \). In what follows, for every r.t. extension \( w \) of \( \nu \) to \( K(X) \), we fix a minimal pair of definition \( (a, \delta) \). Also, the symbols \( f, \gamma, \nu, e, h, r \) and \( r^* \) are used as in
Theorem 1.2.

Now we give some consequences of Theorem 1.2, under the same notations and the hypothesis.

**Corollary 1.4.** (cf. [3, Proposition 1.1]) Let \( A_1, \ldots, A_s, B_1, \ldots, B_t \) be elements of \( K[X] \) such that \( \deg A_i < n \), and \( \deg B_j < n \) for all \( 1 \leq i \leq s \), \( 1 \leq j \leq t \).

If \( w(A_1, \ldots, A_s) = w(B_1, \ldots, B_t) \), then \( y = \left( \frac{A_1 \cdots A_s}{B_1 \cdots B_t} \right)^* = \left( \frac{A(a) \cdots A_s(a)}{B(a) \cdots B_t(a)} \right)^* \in k_\ast \)

**Corollary 1.5.** If \( g \in K[X] \) satisfies \( w(g) \in G_\ast \) then for \( q \in K[X] \) such that \( \deg q < n \) and that \( w(q) = v(q(a)) = w(g) \), \( (g/q)^* \in k_\ast[r^*] \). In particular, if \( w(g) = 0 \) and \( g^* \in k_\ast \), \( v(g(a)) = 0 \) and \( g^* = (g(a))^* \).

**Proof.** Let \( g = g_0 + g_1f + \cdots + g_t f^t \) be the \( f \)-expansion of \( g \) with \( \deg g_i < n \), \( 0 \leq i \leq t \). Then, by definition \( w(g) = \inf(v(g_i(a)) + i\gamma) \). Since \( w(g) \in G_\ast \), \( v(g(a)) + i\gamma > w(g) = w(q) \), if \( i \equiv 0 (\text{mod } e) \).

Since \( w(g) = w(q) \), one has \( w(g_i f^i/q) \geq 0 \), for all \( i \). Hence \( w\left( \frac{g_0 h^0}{q} \cdots \frac{f^t}{h^t} \right) \geq 0 \) and \( w\left( \frac{g_0 h^0}{q} \right) \geq 0 \) because \( w\left( \frac{f^t}{h^t} \right) = w(r^*) = 0 \). Therefore,

\[
(g/q)^* = (g_0/q)^* + (g_s h/q)^* r^* + \cdots \in k_\ast[r^*].
\]

Finally, if \( w(g) = 0 \) and \( g^* \in k_\ast \), then we may take \( q = 1 \), and one has necessarily that \( (g_s h)^* = (g_0 h)^* = \cdots = 0 \). Hence \( g^* = g_\ast = (g(a))^* \), as claimed.

**Corollary 1.6.** The assignment: \( F \rightarrow F^* = \rho(F) \) defines an onto ring-homomorphism \( \rho: O_\ast \cap K[X] \rightarrow k_\ast[r^*] \).

**Proof.** According to Corollary 1.5, it is enough to show that there exists \( F \in K[X] \cap O_\ast \) such that \( F^* = r^* \). Indeed, take \( t \in K[X] \) such that \( \deg t < n \), \( w(t) = \inf(v(t(a)) = -w(h) = -v(h(a)) \), and that \( (h)^* = 1 \). Then \( (f^t)^* = (th \cdot f^t)^* = (th)^* r^* = r^* \), as claimed.

3. **Notation 1.7.** Let \( G = u_0 + u_1 r^* + \cdots + r^{s*} \) be a monic polynomial of \( k_\ast[r^*] \). For every \( i(0 \leq i < s) \), choose a polynomial \( g_i \in K[X] \) such that \( \deg g_i < n \), \( w(g_i) \geq 0 \) and that \( g_i^* = u_i \). Let

\[
A = g_0 + g_1 r^* + \cdots + r^{s*} = \frac{g_0 h^0 + g_1 h^{s-1} f^t + \cdots + f^{se}}{h^t}.
\]

Then \( A \in K(X), w(A) = 0 \), and \( A^* = G \). We shall say that the polynomial \( g = g_0 h^s + g_1 h^{s-1} f^t + \cdots + f^{se} \) is a lifting in \( K[X] \) of the polynomial \( G \) in \( k_\ast[r^*] \).
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Note that $G$ has many liftings in $K[X]$.

2. Domination of r.t. extensions of $v$ to $K(X)$

Let $w_1, w_2$ be two r.t. extensions of $v$ to $K(X)$. Let $(a_i, \delta_i)$ be a minimal pair of definition of $w_i, i=1, 2$. As in Notation 1.3, let $f_i$ be the monic minimal polynomial of $a_i$, with $\gamma_i = w_i(f_i)$, $\bar{v}_i$ the restriction of $\bar{v}$ to $K(a_i)$, $\epsilon_i$ the smallest positive integer such that $\epsilon_i \gamma_i \in G_{\bar{v}_i}$, $h_i \in K[X]$ the polynomial such that $\deg h_i < n_i = \deg f_i$ and that $w(h_i) = \epsilon_i \gamma_i, r_i = f_i^i / h_i$. Let $r_i^* = \rho_{w_i}(r_i), i=1, 2$.

According to [7] (see also [4]), one says that $w_i$ dominants $w_j$ if $w_i(g) \leq w_j(g)$ for all $g \in K[X]$, and $w_i(G) < w_j(G)$ for at least one $G \in K[X]$. This inequality should be understood in $G_\bar{v}$ because $G_{w_i}$ and $G_{w_j}$ are of finite index over $G_\bar{v}$ (see [1] or [2]).

If $w_1 < w_2$, then $O_{w_1} \cap K[X] \subseteq O_{w_2} \cap K[X]$ and there exists a unique ring homomorphism $\varphi: k_{\bar{v}_1}[r_1^*] \rightarrow k_{\bar{v}_2}[r_2^*]$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
O_{w_1} \cap K[X] & \longrightarrow & O_{w_2} \cap K[X] \\
\rho_1 \downarrow & & \rho_2 \downarrow \\
k_{\bar{v}_1}[r_1^*] & \varphi & k_{\bar{v}_2}[r_2^*] 
\end{array}
$$

(1)

For the sake of simplicity we write $\rho_i = \rho_{w_i}, i=1, 2$ (cf. Corollary 1.6).

**Proposition 2.1.** Let $w_1, w_2$ be two r.t. extensions of $v$ such that $w_1 < w_2$. Consider the diagram (1), then

a) $\varphi(y) \neq 0$, whenever $y \in k_{\bar{v}_1}, y \neq 0$,

b) $\ker \varphi \neq 0$,

c) $\varphi(\rho_i(F)) = \rho_i(F(a_2))$ for any $F \in O_{w_1} \cap K[X]$.

**Proof.** a) Clear because $k_{\bar{v}_1}$ is a field.

b) Indeed, since $w_1 < w_2$, there exists $g \in K[X]$ such that $w_1(g) < w_2(g)$. Let $m$ be a positive integer such that $w_1(g^m) \neq w_2(g^m)$. According to (2) it follows that $\varphi(\rho_i(F))$ is algebraic over $k_{\bar{v}}$, so it belongs to $k_{\bar{v}_1}$. If $w_2(F) = 0$, $\varphi(\rho_i(F)) = \rho_i(F) = \rho_i(F(a_2))$ by Corollary 1.5. Assume that $w_2(F) > 0$. Write $F = F_0 + F_1 f_2 + \ldots + F_r f_r^r, \deg F < \deg f_2, 0 \leq i \leq r$. Then $w_2(F) \leq w_2(F_0) = \bar{v}(F_0(a_3)) = \bar{v}(F(a_3))$. Hence $\varphi(\rho_i(F)) = \rho_i(F) = 0 = \rho_i(F(a_3))$. 


Corollary 2.2. With the notation and hypothesis in Proposition 2.1, we have:

a) \( w_1(f_2) < w_2(f_2) \) and \( n_1 \leq n_2 \),

b) if \( h \in K[\mathcal{X}] \) and \( \deg h < n_1 \), \( \bar{v}(h(a_1)) = \bar{v}(h(a_2)) \).

Proof. a) Let \( t \) be a natural number such that \( w_1(f_2) = -\bar{v}(b) \), \( b \in K \). Then \( \rho_1(b f_2) / b \neq 0 \). According to Proposition 2.1 c), \( \varphi(\rho_1(b f_2)) = \rho_2(b f_2) = \rho_2(b f_2(a_2)) = 0 \). This means that \( w_2(f_2) > w_1(f_2) \). Furthermore, since \( \varphi(\rho_1(b f_2)) = 0 \), \( \rho_1(b f_2) \) is transcendental over \( k_\mathcal{X} \). Then, according to [3, Proposition 1.1], there exists a root \( a'_2 \) of \( f_2 \) such that \( (a'_2, \delta_1) \) is a pair of definition of \( w_1 \). Now since \( (a_1, \delta) \) is a minimal pair of definition of \( w_1 \) and \( \bar{v}(h(a_1)) = \bar{v}(h(a_2)) \), by definition of \( w_1 \), \( w_1(h) = \bar{v}(h(a_2)) \).

Remark 2.3. Now we make some remarks on the relation of domination between r.t. extensions.

a) Let \( w_i = w_{\mathcal{X}_i, \mathcal{Y}_i}, i = 1, 2 \), be two r.t. extensions of \( \bar{v} \) to \( \bar{K}(\mathcal{X}) \). In [4, Proposition 2.1], it is proved that \( w_1 < w_2 \) if and only if \( \bar{v}(a_1 - a_2) \geq \delta_1 \) and \( \delta_2 < \delta_1 \). When \( K \) is not necessarily algebraically closed and \( w_1, w_2 \) are two r.t. extensions of \( v \) to \( K(\mathcal{X}) \) such that \( w_1 < w_2 \), we say that \( w_2 \) dominates \( w_1 \). It is clear that if \( w_2 \) dominates \( w_1 \), then \( w_2 \) dominates \( w_1 \). Actually, we do not know if in general the domination implies the well domination. However, this is the case when \( v \) is Henselian or of rank one.

b) The relation of domination may be defined also between (not necessarily r.t.) extensions of \( v \) to \( K(\mathcal{X}) \) in the same manner. It is easy to see that the diagram (1) may be defined for any extensions \( w_1 \) and \( w_2 \) of \( v \) to \( K(\mathcal{X}) \). However, the results in Proposition 2.1 and Corollary 2.2 are valid only when \( w_1 \) and \( w_2 \) are r.t. extensions.

3. Definitions of key polynomials

1. Let \( (K, v) \) be a valuation pair. According to MacLane [7], one says that two elements \( a, b \in K \) are of the same order of magnitude or equivalent in \( v \) and writes \( a \sim b \) (in \( v \)), when:

\[ v(a - b) = v(b). \]
It is clear that \( \sim \) is an equivalence relation on \( K \). Moreover, if \( a \sim b \) and \( a' \sim b' \) then \( aa' \sim bb' \).

Let \( A \) be a suitable subring of \( K \). An element \( b \in A \) is said to be equivalence divisible in \( A \) by \( a \in A \) relative to \( v \) when there exists \( c \in A \) such that \( b \sim ca \) (in \( v \)). It is easy to see if \( a \sim a', b \sim b', c \sim c' \) and \( b \sim ca \) then \( b' \sim c' a' \).

Let \( \omega \) be a valuation on \( K(X) \). According to [7], a key polynomial over \( \omega \) is a non-constant polynomial \( g(X) \in K[X] \) which satisfies the following:

(i) Irreducibility: If \( F, G \in K[X] \) and \( FG \) is equivalence divisible in \( K[X] \) by \( g \) relative to \( \omega \), then one of the factors is equivalence divisible in \( K[X] \) by \( g \).

(ii) Minimal degree: Any non-zero polynomial equivalence divisible in \( K[X] \) by \( g \) has the degree in \( X \) not less than \( \deg g(X) \).

(iii) The leading coefficient of \( g \) is 1, i.e. \( g \) is monic.

A polynomial \( g \) with condition (i) is said to be equivalence irreducible in \( \omega \).

**Proposition 3.1.** Let \( f \in K[X] \) be equivalence irreducible in \( \omega \). Assume that a product \( FG \) of polynomials in \( K[X] \) is equivalence divisible by \( f^i, i \geq 1 \), and \( F \) is not equivalence divisible by \( f \). Then \( G \) is equivalence divisible by \( f^i \).

The proof follows by induction over \( i \) and is left to the reader.

2. Let \( \omega \) be a valuation on \( K(X) \) and let \( g \) be a polynomial in \( K[X] \). Suppose an ordered group \( G \) contains \( G_\omega \) as an ordered subgroup and take \( \gamma \subseteq G \). Then a new valuation \( \omega_1(F) \) may be defined as follows:

\[
\omega_1(F) = \inf(\omega(F_i) + i\gamma)
\]

where \( F = F_0 + F_1 \gamma + \cdots + F_s \gamma^s \), \( \deg F_i < \deg g \), \( 0 \leq i < s \) is the \( g \)-expansion of \( F \in K[X] \).

For the proof of the following result, we send the reader to [7; Theorems 4.2 and 5.1].

**Theorem 3.2.** (MacLane [7]) If \( g \) is a key polynomial over \( \omega \) and \( \gamma > \omega(g) \), then the function \( \omega_1 \) defined above is also a valuation on \( K[X] \) (and on \( K(X) \)), which dominates \( \omega \).

According to MacLane's terminology, \( \omega_1 \) will be called the augmented valuation over \( \omega \), associated with \( g \) and \( \gamma \). If \( \gamma \in QG_\omega \), i.e. there exists a positive integer \( e \neq 0 \) such that \( e\gamma \in G_\omega \), we shall say that \( \omega_1 \) is a commensurable augmented valuation.
4. Key polynomials over r.t. extension

In this section we study key polynomials over an r.t. extension. The main results are Theorems 4.4 and 4.6. We remark that Theorem 4.6 and its proof are inspired by MacLane's work [7, Theorem 9.4].

Throughout this section $w$ is an r.t. extension of $v$ to $K(X)$ and $(a, \delta)$ is a minimal pair of definition of $w$. We use the notation in Notation 1.3.

Proposition 4.1. If $F$ is a key polynomial over $w$, then $\deg F \geq n$.

Proof. It is enough to show that if $g \in K[X]$ is of $\deg g < n$ then $g$ cannot be a key polynomial over $w$. Indeed, take $q \in K[X]$ such that $\deg q < n$ and that $w(g) + w(q) = 0$. Then by Corollary 1.4, $(gq)^* \in k$. Hence there exists $t \in K[X]$ with $\deg t < n$, such that $w(t) = 0$, and that $t^* = (g^q)^*$. Therefore $(t^q)^* = 1$ and so $w(tqg - 1) > 0$. Hence the condition (ii) is not satisfied by $g$.

Proposition 4.2. For $g \in K[X]$, let $g = qf + g_0$, with $\deg g_0 < n$. The following are equivalent:

a) $g$ is equivalence divisible in $K[X]$ by $f$ (relative to $w$).

b) $w(g - qf) = w(g) > w(g)$.

Proof. The implication b) $\Rightarrow$ a) is obvious.

a) $\Rightarrow$ b) Suppose there exists $q_i \in K[X]$ such that $w(q - q_i f) > w(g)$. Then $w(g_0 + (q - q_i f)) > w(g)$. By definition of $w$, it follows that $w(g) \leq w(g_0)$.

Assume that $w(g) = w(g_0)$. Then $w((q - q_i f)) = w(g_0) \in G_k$. Let $(q - q_i f) = h_i f + \cdots + h_t f^t$ deg $h_i < n$, $1 \leq i \leq t$.

Then $w((q - q_i f)) = \inf \{ \bar{v}(h_i(a)) + ir \}$. Since $w((q - q_i f)) \in G_k$,

$b(h_i(a)) + ir > w(g_0) = w(q - q_i f)$, if $i \equiv 0(\mod e)$.

Hence $w\left(1 - \left(\frac{h_i}{g_0} f + \cdots + \frac{h_t}{g_0} f^t\right)\right) > 0$, or $w\left(1 - \left(\frac{h_i}{g_0} f + \cdots + \frac{h_t}{g_0} f^t + \cdots\right)\right) > 0$.

Thus $1 = \left(\frac{h_i h_t}{g_0}\right)^* h^* + \cdots$. But this equality is impossible, because $\left(\frac{h_i h_t}{g_0}\right)^*$ belongs to $k$ and $r^*$ is transcendental over $k$. Therefore $w(g) < w(g_0)$ as claimed.

Corollary 4.3. The polynomial $f$ (used in the definition of $w$) is a key polynomial over $w$.

Proof. We show that the conditions (i)-(iii) in the definition of a key
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polynomial are fulfilled.

(i) Let $A, B \subseteq K[X]$ be such that $AB$ is equivalence divisible relative to $w$ by $f$. Let $A=A'+A_0$, $\deg A_0 < n$ and $B=B'+B_0$, $\deg B_0 < n$. According to Proposition 4.2, we must prove that $w(A) < w(A_0)$ or $w(B) < w(B_0)$. Assume that $w(A) \geq w(A_0)$ and $w(B) \geq w(B_0)$. If we write $A_0B_0 = Cf + C_0$ with $\deg C_0 < n$, then $w(AB) \geq w(A_0B_0) = v(A_0(a)B_0(a)) = v(C_0(a)) = w(C_0)$. But this is a contradiction. The condition (ii) results by Proposition 4.2 and (iii) is obvious.

Now we try to give a characterization of key polynomials over $w$. According to Proposition 4.1, we shall treat key polynomials of degree just $n = \deg f$ and key polynomials whose degrees are greater than $n$ separately.

**Theorem 4.4.** Let $g \in K[X]$ be a monic polynomial. Consider the following:
1) $g$ is a key polynomial over $w$ and equivalence divisible by $f$.
2) $g$ is a key polynomial over $w$ and of $\deg g = n = \deg f$.
3) $g$ is irreducible and there exists a root of $g$ such that $(b, d)$ is also a minimal pair of definition of $w$.

Then we always have 1) $\Rightarrow$ 2) $\iff$ 3). Moreover, 2) $\Rightarrow$ 1) when $\gamma = w(f)$ does not belong to $G_\delta$.

**Proof.** 1) $\Rightarrow$ 2) Let $g = qf + g_0$, $\deg g_0 < n$. According to Proposition 4.2 b), one has $w(g-qf) = w(g_0) > w(g)$. Now since $g$ is also a key polynomial, $q$ or $f$ is equivalence divisible by $g$. Being $\deg q < \deg g$, $f$ is equivalence divisible by $g$. So $\deg f = \deg g$. Hence $\deg f = \deg g$ by Proposition 4.1.

2) $\Rightarrow$ 3) By 2) one has $g = f + g_0$, $\deg g_0 < n$. So $w(g) = \inf(w(f), w(g_0)) = \inf(\gamma, w(g_0))$. Thus $w(g) \leq \gamma$. Now we remark that $w(g) = \gamma$. Assume $w(g) < \gamma$. Then $w(f) = w(g) = \gamma > w(g)$. But this is impossible, because $g$ is a key polynomial over $w$ and $\deg g_0 < n = \deg g$.

Let $b_1, \ldots, b_n$ be all roots of $g$ in $\bar{K}$ and $g = \prod_{i=1}^n (X-b_i)$. We assert that $w(X-b_i) \geq \delta$ for at least one index $i$ (here $w = w_{(a, b)}$). Indeed, assume that $\bar{v}(a-b_i) < \delta$. Then

$$w(g) = \sum_i w(X-b_i) = \sum_i v(a-b_i) = v(g(a)).$$

Then $e = 1$, and we may choose $h = g_0$ (see Theorem 1.2, c)). Therefore if we put $r = f/g_0$ then $w(r) = 0$ and $r^*$ is transcendental over $k_{e_\delta}$. Consequently, $(g/g_0)^* = r^* + 1$ is also transcendental over $k_{e_\delta}$. Hence by [3, Proposition 2.1],
there exists a root $b$ of $gg_0=0$ such that $(b, \delta)$ is a pair of definition of $\bar{w}$.

Now since $(a, \delta)$ is a minimal pair and $\deg g_0 < n$, it follows that $g_0(b) \neq 0$. In conclusion, one has necessarily $g(b) = 0$ and $\bar{b}(b-a) \geq \delta$. This is a contradiction.

3) $\Rightarrow$ 2) Since $(b, \delta)$ is also a minimal pair of definition of $w$, $g$ is a key polynomial over $w$ and $\deg g = \deg f = n$.

Now let us assume that $\gamma \notin G_\delta$. Then one has the implication 2) $\Rightarrow$ 1). Indeed, we have remarked that $w(g-f) = w(g_0)$, i.e. $g$ is equivalence divisible by $f$.

Remark 4.5. Now we give an example which shows that the implication 2) $\Rightarrow$ 1) in Proposition 4.4 is not necessarily valid if $\gamma \in G_\delta$. For that take an algebraically closed field $K$ and $a, b \in K$ such $v(a-b) = \delta$. Let $w = w(a, b)$. Then $(a, \delta)$ and $(b, \delta)$ are both minimal pairs of definition of $w$.

Hence, $X-a$ and $X-b$ are both key polynomials over $w$. But since $w(X-b) = v(a-b) = \delta$, $X-b$ is not equivalence divisible in $K[X] (\text{with respect to } w)$ by $X-a$.

For key polynomials over $w$ whose degrees are greater than $n = \deg f$, one has:

Theorem 4.6. Take $g \in K[X]$ such that $\deg g > n = \deg f$ and consider the $f$-expansion of $g$:

$$g = g_0 + g_1 f + \cdots + g_t f^t, \quad \deg g_i < n, 0 \leq i \leq t.$$  

Then the following are equivalent:

1) $g$ is a key polynomial over $w$.

2) $g$ satisfies the following:

$\alpha)$ $w(g) = w(g_0)$,

$\beta)$ $t \equiv 0 \pmod{\epsilon}$, $g_i = 1$, and $w(g) = w(f^t) = se\gamma$,

$\gamma)$ $g$ is equivalence irreducible in $w$.

3) $t \equiv se, w(g) = se\gamma$, $g$ is monic of degree $tn$ and $(g/h^s)^* \in k[\epsilon^*]$ is a monic and irreducible polynomial of degree $s$ in $k_\epsilon[\epsilon^*]$ whose constant term is not zero.

Proof. 1) $\Rightarrow$ 2). By definition of $w$, $w(g) \leq w(g_0)$. If $w(g) = w(g_0)$, $w(g-qf) > w(g)$ for a suitable $q$ of $\deg q < \deg g$. Hence $q$ or $f$ is equivalence divisible by $g$. But this is impossible. Thus has $w(g) = w(g_0)$.

Further, $w(g) = w(g_0)$ shows that $w(g) \in G_\delta$. Now we note that $w(g) = w(g_1 f^t)$. Assume that $w(g) < w(g_1 f^t)$. Then $w(g-(g_0 + \cdots + g_{t-1} f^{t-1})) > w(g)$. But this is also impossible, because $g$ is a key polynomial over $w$. Hence two remarks $w(g) = w(g_1 f^t)$ and $w(g) \in G_\delta$ imply that $t \equiv 0 \pmod{\epsilon}$. 

Let us show that \( g_t=1 \). If \( g_t \neq 1 \) then, since \( g \) is monic, \( \deg g_t > 0 \). Take \( u \in K[X] \) such that \( \deg u < n, w(ug_t) = 0 \) and \( (ug_t)^* = 1 \). This means \( w(ug_t - 1) > 0 \). So \( w(ug_t f^i - f^i) > w(f^i) = w(ug) \). Now since \( f \) is a key polynomial and \( \deg g_{t-1} < n, ug_t f^i = u' g - d, \) where \( \deg d < \deg q \) and \( w(u') = w(u) \). Thus \( w(ug_t f^i - f^i) = w(u' g - d - f^i) > w(f^i) = w(u' g) \). But this is impossible, because \( g \) is also a key polynomial over \( w \) and \( \deg (d + f^i) < \deg g_t - 1 \).

2) \( \Rightarrow 1) \) By \( \beta \) and \( \gamma \) conditions (i) and (iii) of a key polynomial are fulfilled. Now let \( d \in K[X] \) be equivalence divisible by \( g \). Hence there exists \( q \in K[X] \) with \( w(qg - d) > w(d) \). We must show \( \deg d \geq \deg g \). Writing \( q = \sum g_t f_t \), with \( \deg g_t < n \), let \( j \) be the greatest index \( t \) such that \( w(q) = w(q_{j+1}) + j \gamma \). Thus \( \deg w(qg) \) one has the term \( A = (q_{j+1} g_{t-1} + \ldots) f_{j+t}\). Then \( w(A) = w(q_{j+1}) + w(f_{j+t}) = w(qg) - w(d) \). Hence if \( \deg d < \deg g \) the term \( A \) whose degree is at least \( \deg g \) must appear in \( qg - d \). So the inequality \( w(qg - d) > w(d) = w(qg) \) is impossible. Therefore \( \deg d \geq \deg g \), as claimed.

2) \( \Rightarrow 3) \) By \( \beta \) it results

\[
t = se = 0 (mod e), w(g) = se\gamma = w(h^s) \quad \text{and} \quad g \text{ is monic}
\]

(remind that \( w(h) = e\gamma \) and \( \deg h < n \)). Hence one has \( w(g/h^s) = 0 \), and

\[
g/h^s = g_0/h^s + g_t f/h^s + \ldots + f^{se}/h^s,
\]

\( w(g, f^i/h^s) \geq 0 \) and \( w(g, f^i/h^s) > 0 \) if \( j \neq 0 (mod e) \).

If \( j = ie \), \( w(g, f^i e/h^s) = w(\frac{g, f^i e}{h^s e-t^i}) \geq 0 \). Since \( w(f^e/h) = 0 \), \( w(g, f^i e/h^s) \geq 0 \). So \( (g, f^i e/h^s) \in k \) (see Corollary 1.4). Therefore

\[
(g/h^s)^s = A_0 + A_1 r^s + \ldots + r^{se}, A_t \in k, 0 \leq i < s.
\]

Now we show that this is an irreducible polynomial of \( k[r^s] \). Indeed, assume that \( A', B', C' \) are polynomials of \( k[r^s] \) such that \( (g/h^s)^s C' = A' B' \). Let \( A, B \) and \( C \) be the liftings of \( A' \), \( B' \) and \( C' \), respectively (see Notation 1.7). Then \( (g/h^s)^s (C/h^s)^s = (A/h^s)^s (B/h^s)^s \). Hence

\[
w(\frac{gC}{h^{s+u}} - \frac{AB}{h^{s+t}}) > 0.
\]

Let \( i = q + t - s - u \). If \( i \geq 0 \), then \( w(g Ch^t - AB) > w(h^{s+t}) = w(AE) \) (see Notation 1.7). Then by condition \( \gamma \) it follows that, say, \( A \) is equivalence divisible by \( g \). Thus for a suitable polynomial \( D \in K[X] \), one has

\[
w(gD - A) = w(A) = w(gD), \quad w\left(\frac{g/h^2 + D}{h^{s+t} - A/h^s}\right) > 0, \quad \text{or} \quad (g/h^s)^s (D/h^{s-t})^s = (A/h^s)^s = A'.
\]
Therefore, since \((D/h^{*})\in k_{t}[r^{*}]\), \(A'\) is divisible by \((g/h^{*})\). If \(i<0\), \(w(gC-h^{*}AB)>w(ABh^{*})=w(gC)\). Now because \(g\) is a key polynomial and \(\deg h<n<\deg g\), it results that, say, \(A\) is equivalence divisible by \(g\). Thus as above \(A'=(A'/h^{*})\) is divisible by \((g/h^{*})\) in \(k_{t}[r^{*}]\). In conclusion \((g/h^{*})\) is irreducible in \(k_{t}[r^{*}]\) and since \(w(go)=w(g)=w(h^{*})\), its constant terms is not zero.

3) \(\Rightarrow 2)\) By 3) it results that \(t=sr\), \(w(g)=sr=\gamma\), and \(g_{t}=1\). Hence \(\beta\) is accomplished. The condition \(\beta\) is also satisfied because the constant term of \((g/h^{*})\) is not zero.

Now we are only to show that \(\gamma\) is also true. For this take \(A, B\in K[X]\) such that \(AB\) is equivalence divisible by \(g\). Then there exists \(D\in K[X]\) such that

\[
w(gD-AB)>w(AB)=w(gD).
\]

Let \(i\) and \(j\) be the smallest non-negative integers such that

\[
w(A)+iy=v(\omega(a)), \quad \text{and} \quad w(B)+jy=v(\sigma(a)),
\]

where \(\omega, \sigma\in K[X]\), \(\deg \omega<n\) and \(\deg \sigma<n\). Then

\[
w(A)+iy=v(\omega(a)), \quad \text{and} \quad w(B)+jy=v(\sigma(a)),
\]

where \(\omega, \sigma\in K[X]\), \(\deg \omega<n\) and \(\deg \sigma<n\). Then

\[
w\left(\frac{g}{h^{*}}\cdot \frac{Dh^{*}f^{*}}{\omega} \cdot \frac{Af^{*}}{\omega} \cdot \frac{Bf^{*}}{\sigma}\right)>0, \quad \text{i.e.,}
\]

\[
(g/h^{*})\left(\frac{Dh^{*}f^{*}}{\omega} \right)^{*} = \left(\frac{Af^{*}}{\omega}\right)^{*} \left(\frac{Bf^{*}}{\sigma}\right)^{*}.
\]

Here according to Corollary 1.6, all factors are polynomials of \(k_{t}[r^{*}]\). So, since \((g/h^{*})\) is irreducible by hypothesis, it results that it divides, say, \((A^{f^{*}})\).

Hence one has the equality

\[
(g/h^{*})\cdot G'=(A^{f^{*}}), \quad G'\in k_{t}[r^{*}].
\]

According to Notation 1.7, one may write \(G'=(G/h^{*})\) with \(G\in K[X]\) and a suitable non-negative integer \(p\). Then

\[
w\left(\frac{gG}{h^{*}}\cdot \frac{Af^{*}}{\omega}\right)>0, \quad \text{or} \quad w(gGa-Af^{*}h^{*})>w(h^{*}A)=w(gGa).
\]

Furthermore, by 3) and Proposition 4.2, \(g\) is not equivalence divisible by \(f\). Then by Proposition 3.1, it results that \(G\omega\) is equivalence divisible by \(f^{4}\), i.e. \(w(G\omega-f^{4}H)>w(G\omega)=w(f^{4}H), H\in K[X]\). Hence

\[
w(gHf^{*}-Af^{*}h^{*})>w(gHf^{*}), \quad \text{or} \quad w(gH-Ah^{*})>w(gH).
\]

Now let \(d\in K[X]\) be such that \(w(dh^{*})=0\), \(\deg d<n\), and that \(w(dh^{*}-1)>0\). Thus
On the residual transcendental extensions

\[ w(gHd - A(dh^{p-1+1})) > w(gHd), \quad \text{or} \quad w(gHd - A) > w(gHd) = w(A). \]

This means that \( A \) is equivalence divisible by \( g \). In conclusion \( g \) is equivalence irreducible. The proof of Theorem 4.6 is complete.

**Corollary 4.7.** Let \( G \) be a monic and irreducible polynomial of \( k = \mathbb{F}_r \) whose constant term is not zero. Let \( g \) be a lifting of \( G \) (see Notation 1.7). Then \( g \) is a key polynomial over \( w \). In particular, \( g \) is an irreducible polynomial of \( K[X] \).

**Proof.** Let \( s = \deg G \). Then \( \deg g = s = e + \deg A \), and \( g = g_0 + g_1 + \cdots + g_s \), \( g_i \in K[X] \), \( \deg g_i < n \). The condition that \( G \) has a non-zero constant term shows that \( w(g) = w(g_0) = w(f)^e = s e^r \). Thus, since \( G = (g/h)^e \), by condition 3) in Theorem 4.6 it results that \( g \) is a key polynomial over \( w \).

5. Valuation defined by a key polynomial

In this section we show that key polynomials over an r.t. extension of \( v \) to \( K(X) \) give new r.t. extensions of \( v \) to \( K(X) \). In particular, we show that key polynomials may be used to yield minimal pairs.

**Theorem 5.1.** Let \( w \) be an r.t. extension of \( v \) to \( K(X) \) and let \( f_i \) be a key polynomial over \( w \). Take \( \gamma_i > G_0 \) such that \( \gamma_i > w(f_i) \). Let \( w_i \) be the augmented valuation over \( w \) associated with \( f_i \) and \( \gamma_i \). Then \( w_i \) is an r.t. extension of \( v \) to \( K(X) \). Moreover there exists a root \( a_i \) of \( f_i \) and \( \delta_i \in \mathcal{O}_0 \) such that \( (a_i, \delta_i) \) is a minimal pair of \( w_i \) with respect to \( K \) and \( v_i \) well dominates \( w \).

**Proof.** As usual we keep the notations stated in Notation 1.3. Let \( (a, \delta) \) be a minimal pair of definition of \( w \). Then two cases are possible \( \deg f_i = n = \deg f \) or \( \deg f_i > n \) (see Proposition 4.1). We shall consider each case separately.

A) First assume that \( \deg f_i = n \). Then according to condition 3) in Theorem 4.4, there exists a root \( a_i \) of \( f_i \) such that \( (a_i, \delta) \) is also a minimal pair of definition of \( w \). Hence we may assume that \( f_i = f \) and \( a_i = a \). Since \( (a, \delta) \) is a minimal pair of definition of \( w \), one has

\[ w(f) = \gamma = \inf (\theta(A_i) + i \delta), \quad \text{where} \quad f = \sum_{i=1}^n A_i (X - a_i), \quad A_i \in \overline{K} \]

and \( \delta = \sup \frac{\gamma - \theta(A_i)}{i} \).

Now let us define
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Since by hypothesis $\gamma_1 > \gamma$, it results that $\delta_1 > \delta$. Therefore $(a, \delta)$ is also a minimal pair because $(a, \delta)$ is a minimal pair with respect to $K$. Let $w'$ be the restriction of $w_{(a, \delta)}$ to $K(X)$. Then by Theorem 1.2, for $F \in K[X]$

$$w'(F) = \inf(\bar{v}(F_i(a)) + i^*(\delta_i)) = \inf(w(F_i) + i^*(\delta_i)),$$

where $F = F_0 + F_1 f + \cdots + F_s f^s$, $\deg F_i < n$, $0 \leq i \leq s$. Hence $w_i = w'$ by definition of an augmented valuation. Therefore $w_1$ well dominates $w$ because $w_{(a, \delta)} < w_{(a, \delta)}$ by [4, Proposition 2.1].

B) Next assume that $\deg f_1 = n_1 > n$. Then by assertion 3) in Theorem 4.6, there exists a positive integer $s$ such that $u(f_1/h^s) = 0$ and that $(f_1/h^s)$ is an irreducible polynomial of $k_\delta[r^s]$. Then, according to [3, Proposition 1.1] there exists an element $a_i \in K$ such that $f_1(a_i) h(a_i) = 0$ and that $(a, \delta)$ is a pair of definition of $w_{(a, \delta)}$, or equivalently $\bar{v}(a_i - a) \geq \delta$. Now since $\deg h < n$ and $(a, \delta)$ is a minimal pair with respect to $K$, one has necessarily $f_1(a_i) = 0$. Writing $f_1 = \sum_{i=1}^{n_1} A_i(X - a_i)^i$, $A_i \in K$, define

$$\delta_1 = \sup_{i=1}^{n_1} \frac{\gamma_1 - \bar{v}(A_i)}{i}.$$  

In what follows we shall show that $w_1$ is an r.t. extension of $v$ to $K(X)$ and that $(a_1, \delta_1)$ is a minimal pair of definition of $w_1$ with $\delta < \delta_1$, or $w_1$ well dominates $w$. We shall divide the proof in several steps.

B1) At this point we introduce an useful notation. Let us denote by $P$ the subring of $K[X]$ whose elements are fractions $p = F/G$ such that $w(p) \geq 0$, and that every irreducible factor of $G$ has the degree smaller than $n$. According to Corollaries 1.4 and 1.6 it results that for every $p \in P$ the mapping $p \mapsto p^*$ gives a surjective ring homomorphism $\rho : P \to k_\delta[r^s]$.

If $p \in O_{w_1}$, let us denote by $p^{**}$ the image of $p$ into the residue field $k_{w_1}$. According to MacLane’s Theorem (see Theorem 3.2), one has $w_1 < w_1$. So if $p \in P$, then $p \in O_{w_1}$. Hence the mapping $p \mapsto p^{**}$ gives a ring homomorphism $\rho_1 : P \to k_{w_1}$. Finally, it is easy to see that the mapping $p \mapsto p^{**}$ gives a $k_\delta$-algebras homomorphism $\varphi : k_\delta[r^s] \to k_{w_1}$, which makes the following diagram commutative

$$\begin{array}{ccc}
p & \mapsto & p^* \\
\downarrow & \downarrow & \downarrow \\
k_\delta[r^s] & \to & k_{w_1}
\end{array}$$
B2) Since \( r_i > w(f) \) and \( w(h) = w(h) \), we note that the kernel of \( \varphi \) is generated by \( (f, h^t)^* \). This implies that for every \( z \in k^* \), \( \varphi(z) \) is algebraic over \( k_v \).

B3) Let \( e_i \) be the smallest positive integer such that \( e_i r_i \in G_w \). We claim that there exists a polynomial \( h_i \in K[X] \) such that \( \deg h_i < \deg f = n_i \) and that \( w(h_i) = w(h_i) = e_i r_i \).

According to Theorem 1.2, since \( G_w = G_\theta + Z^f \) and \( e_i r_i \in G_\theta \), one has \( e_i r_i = w(g f^i) \) for suitable \( g \) and \( i \) with \( \deg g < n_i \), \( 0 \leq i < e \). Then, if \( g f^i = q f^i + h_i \), \( \deg h_i < n_i \), we have \( w(g f^i) = e_i r_i = w(h_i) \). Indeed, assume that \( e_i r_i > w(h_i) \). Then \( w(q f^i + h_i) > w(h_i) \). But this is impossible, because \( f_i \) is a key polynomial over \( w \) and \( \deg h_i < n_i \). Further, if \( e_i r_i < w(h_i) \), then \( w(g f^i - q f^i) > w(g f^i) = e_i r_i \). Since \( f_i \) is a key polynomial over \( w \), it results that one of the polynomials \( g \) or \( f \) is equivalence divisible by \( f_i \). But this is also impossible since \( \deg n < n_i \). Hence \( w(g f^i) = e_i r_i = w(h_i) \).

B4) Now we shall prove that, if we put \( r_i = f^i / h_i \), with \( h_i \) as above (see B3)), \( w_i(r_i) = 0 \) and \( r_i^* \in k_w \), is transcendental over \( k_v \). Moreover, \( r_i \) is the element of \( K(X) \) of the smallest degree with these properties.

For the sake of simplicity, in the rest of this proof we shall express \( r_i^* \) by \( y \). Assume that \( y \in k_w \) is algebraic over \( k_v \). Then there exists \( b_t, \ldots, b_{-1} \in K \) such that \( v(b_i) \geq 0 \), \( 0 \leq i < t \), and that
\[
b_t^* + \cdots + b_{-1}^* y^{t-1} + y^t = 0.
\]
Let us consider the polynomial \( G = b_t h_i^t + b_{t-1} h_i^{t-1} f_i^t + \cdots + b_{-1} h_i f_i^{t-i} + f_i^t \). Then \( w_i(G) = w(h_i) = t_i e_i \).

On the other hand, since \( \deg h_i < n_i \), \( \deg b_i h_i^{t-i} f_i^t < t_i e_i n_i \). So, in the \( f_i \)-expansion of \( G \), the term \( f_i^t \) must appear. But, then, according to the definition of \( w_i \), one has \( w_i(G) = t_i e_i \). This is a contradiction. Therefore \( y \) is transcendental over \( k_v \).

Furthermore, suppose \( p = F/H \in K(X) \) satisfies \( w_i(p) = 0 \) and \( \deg p = [K(X) : K(p)] = \max(\deg F, \deg H) \), \( \deg r_i = e_i n_i \). Let
\[
F = F_0 + F_1 f_1 + \cdots + F_m f^m \quad \text{and} \quad H = H_0 + H_1 f_1 + \cdots + H_q f^q
\]
be the \( f_i \)-expansions of \( F \) and \( H \) respectively. Since \( w_i(p) = 0 \), one has
\[
w_i(F) = \inf_i (w(F_i) + i r_i) = \inf_j (w(H_j) + i r_i).
\]
Now since \( \deg p < e_i n_i \), it follows that \( m < e_i, q < e_i \). So there exists only one index, say \( i \), such that
\[
w_i(F) = w(F_i) + i r_i = w(H_i) = w(H_i) + i r_i.
\]
But then $p^{**} = \left( \frac{F_t}{H_t} \right)^{**}$, because

$$p = \frac{F}{H} = \frac{F_t}{H_t} \cdot \frac{F_t/F_t H_t + \ldots + 1 + \ldots}{H_t/H_t H_t + \ldots + 1 + \ldots}.$$ 

To end the proof of B4) it is enough to show that $p^{**}$ is algebraic over $k_v$. Indeed, since $\deg F_t < n_1$, $\deg C_t < n_1$, $w_t(F_t) = w(H_t) = w_t(H_t) = w(H_t)$. Let $d$ be a positive integer such that $dw(F_t) = v(c)$ for a suitable $c \in K$. Then we have (see (2)):

$$p^{**} = \left( \frac{F_t}{H_t} \right)^{**} = (\frac{F_t^d/c}{(H_t^d/c)})^{**} = (\frac{F_t^d/c}{(H_t^d/c)})^{**},$$

$$\phi(p^{**}) = \phi(F_t^d/c)^{**} = \phi(H_t^d/c)^{**} = p^{**}.$$ 

Hence $p^{**}$ is also algebraic over $k_v$ (see B1).

B5) Finally we shall prove that the pair $(a_i, \delta_i)$ defined above (see B) is a minimal pair of definition of $w_i$ (with respect to $K$). For this we show that $\lbrack K(b) : K \rbrack \geq n_1$, whenever $(b, \delta_i)$ is a pair of definition of $w_i$ with respect to $K$. We shall prove that $\deg g \geq n_1$ if $g$ is the minimal polynomial of $b$ over $K$.

Indeed, let us assume that $\deg g < n_i$. According to the definition of an augmented valuation, one has $w_i(g) = w(g)$. Take a suitable positive integer $t$ such that $tw(g) = v(c)$, $c \in K$. Then $w(g^t/c) = w_i(g^t/c) = 0$, and by diagram (2) one has $0 = (g^t/c)^{**} = \phi((g^t/c)^{*})$. Hence $(g^t/c)^{**}$ is algebraic over $k_v$. But this contradicts the assumption that $(b, \delta_i)$ is a pair of definition of $w_i$ (see [1]).

Furthermore, since $(f^t H_1)^{**}$ is transcendental over $k_v$, according to [3, Proposition 1.1] it follows that there exists a root $a'_i$ of $f_1 H_1 = 0$ such that $(a'_i, \delta_i)$ is a pair of definition of $w_i$. Since $\deg H_1 < n_1 = \deg f_1$ it follows that $a'_i$ is necessarily a root of $f_1$ and that $(a'_i, \delta_i)$ is a minimal pair of definition of $w_i$.

And we have the inequality $\delta < \delta_i$, because $w(f_1) < \gamma_i$ (see B)). The proof of Theorem 5.1 is complete.

**Remark 5.2.** Let $w_1, w_2$ be two r.t. extensions of $v$ to $K(X)$ such that $w_1 < w_2$. In general, we do not know if $w_2$ well dominates $w_1$. However, according to Theorem 5.1, if $w_2$ is an augmented valuation over $w_1$, then $w_2$ well dominates $w_1$. Now we shall give an example which shows that the "well domination" is not a special property of an augmented valuation, i.e. it is possible that $w_2$ well dominates $w_1$ even if $w_2$ is not an augmented valuation over $w_1$.

Let $K$ be the field of 3-adic numbers and $v$ the 3-adic valuation on $K$. Let $a_3 = \sqrt[3]{3}$. The minimal polynomial of $a_3$ is $f_3 = X^3 - 3$. Let $\omega$ be a primitive cube-root of 1. Then, $a_3, \omega a_3, \omega^2 a_3$ are all roots of $f_3$. One has $\sup(\phi(a_3 - \omega a_3))$,
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\[ \overline{v}(a_2 - \omega^3 a_2) = 4/3. \] Hence according to [3, Proposition 3.2, b)], \( \langle a_2, 2 \rangle \) is a minimal pair. Let \( w_2 \) be the restriction of \( w_{\langle a_2, 2 \rangle} \) to \( K(X) \) and \( w_1 \) the restriction of \( w_{\langle a, 3 \rangle} \) to \( K(X) \). It is easy to see that \( w_2 \) well dominates \( w_1 \). However, \( w_2 \) is not an augmented valuation over \( w_1 \). Indeed, if \( w_2 \) would be an augmented valuation over \( w_1 \), then for every polynomial \( g \) of \( \deg g < 3 \) we have \( w_1(g) = w_2(g) \).

But it is easy to see that \( w_1(X^2 - 3) < w_2(X^2 - 3) \).

On the other hand, Theorem 5.5 gives a characterization of r.t. valuations which are augmented valuations over another r.t. extensions of \( v \) to \( K(X) \).

First we prove the following:

**Lemma 5.3.** Let \( w_1, w_2 \) be r.t. extensions of \( v \) to \( K(X) \) such that \( w_1 < w_2 \). Let \( g \in K[X] \) be a monic polynomial of the smallest degree such that \( w_1(g) < w_2(g) \). Then \( g \) is a key polynomial over \( w_1 \).

**Proof.** Let \( \langle a_i, \delta_i \rangle \) be a minimal pair of definition of \( w_i \) and let \( f_i \) be the monic minimal polynomial of \( a_i \), \( i = 1, 2 \). According to Corollary 2.2 b) it follows that, if \( h \in K[X] \) is of \( \deg h < \deg f_1 \), \( w_1(h) = \overline{v}(h(a_1)) = \overline{v}(h(a_2)) = w_2(h) \). Hence \( \deg g \geq \deg f_1 \). Let \( t \) be a positive integer such that \( w_1(g^t) = -v(c), c \in K \). Then \( w_1(cg^t) = 0 \), and \( (cg^t)^* \) is a non-zero element of \( k_v[X^+] \). The hypothesis \( w_1(g) < w_2(g) \) yields \( \varphi((cg^t)^*) = 0 \). But then, according to Proposition 2.1 a), it results that \( (cg^t)^* \) is transcendental over \( k_v \). Therefore, according to [3, Proposition 1.1] there exists a root \( b \) of \( g \) such that \( \langle b, \delta_1 \rangle \) is a pair of definition of \( w_1 \).

Now we consider the cases \( \deg g = \deg f_1 \) and \( \deg g > \deg f_1 \) separately.

Suppose \( \deg g = \deg f_1 \). Then \( \langle b, \delta \rangle \) is also a minimal pair of definition for \( w_1 \). Hence according to Corollary 4.3, \( g \) is a key polynomial over \( w_1 \).

Now let us assume that \( \deg g > \deg f_1 \) and

\[ g = A_0 + A_1 f_1 + \cdots + A_t f_1, \quad \deg A_i < \deg f_i, \quad 0 \leq i \leq t. \]

To show that \( g \) is a key polynomial over \( w_1 \) we shall prove that \( g \) satisfies the condition 3) in Theorem 4.6.

a) First we claim that \( w_1(g) = w_1(A_0) \). Since \( w_1(g) \leq w_1(A_0) \) we show that \( w_1(g) < w_1(A_0) \) implies a contradiction. Indeed, assume that \( w_1(g) < w_1(A_0) \). Let \( g = A_0 + f_1 q, \deg q < \deg g \) be the \( f_1 \)-expansion of \( g \). Then \( w_1(g - f_1 q) = w_1(A_0) < w_1(g) \). Hence \( w_1(g) = w_1(f_1 q) < w_1(A_0) \). Since \( \deg q < \deg g \) then, by hypothesis on \( g \), one has

\[ w_1(f_1 q) = w_1(f_1 q) = w_1(g) < w_2(g). \]

Thus \( w_1(g) = w_1(f_1 q) = w_2(g - f_1 q) = w_2(A_0) = w_1(A_0) \). So we get a desired contradiction.
b) Next we show that \( t = s_\ell \) and \( w_1(g) = s_\ell \gamma_1 \). Note that by a) it follows that \( w_1(g) = w_1(A_\ell) = \overline{v}(A_\ell(a_1)) \subseteq G_\ell \), where \( \overline{v} \) is the restriction of \( v \) to \( K(a_1) \). Now we remark that \( w_1(g) = w_1(A_\ell f_1) \). Indeed, if \( w_1(g) < w_1(A_\ell f_1) \), then \( w_1(B) = w_1(g) \), where \( B = g - A_\ell f_1 \). Hence \( w_1(A_\ell f_1) > w_1(B) \). Further, since \( \deg B < \deg g \), one has \( w_1(B) = w_1(g) \). And \( w_1(A_\ell f_1) = w_1(A_\ell f_1) \). On the other hand, since \( w_\ell(g) > w_\ell(B) = w_\ell(g - B) = w_\ell(B) = w_\ell(B) \) a contradiction. Therefore \( w_1(g) = w_1(A_\ell f_1) \). Since \( w_1(g) \in G_\ell \), it follows that \( w_1(f_1) \in G_\ell \), or \( t = s_\ell \).

Now we shall prove that \( A_1 = 1 \), or \( A_1 \) is of degree 0. Since \( w_1(g) \in G_\ell \), there exists \( h \in K[X] \), \( \deg h < \deg f_1 \) such that \( w_1(g) = w_1(h) = \overline{v}(h(a_1)) \). Hence \( 0 = (g/h)^* \in k_\ell[\gamma_1^*] \). We show that \( (g/h)^* \) is in fact an irreducible polynomial of \( k_\ell[\gamma_1^*] \). Note that by hypothesis \( \varphi((g/h)^*) = 0 \). Hence to prove that \( (g/h)^* \) is irreducible it is enough to show that \( (g/h)^* \) is the kernel of \( \varphi \).

Let \( m = \gamma_1^* \) be the monic generator of the kernel of \( \varphi \), i.e.

\[
    m = u_0 + u_1 r_1^* + \cdots + u_p r_1^* + r_1^* \gamma_1^*.
\]

Since \( w_1(g) \in G_\ell \), \( w_1(A_\ell f_1) > w_1(g) = w_1(h) \), for every \( i \equiv 0 \pmod{e_i} \). Thus

\[
    (g/h)^* = (A_\ell/h)^* + (A_\ell h_1/h)(f_1/h_1)^* + \cdots + (A_\ell h_1/h)^*(f_1/h_1)^* = u_0 + u_1 r_1^* + \cdots + u_p r_1^* + \gamma_1^* \in k_\ell, \quad 0 \leq i \leq s.
\]

Now since \( m \) is the kernel of \( \varphi \), it follows that \( p \leq s \). Let \( M = m_0 + m_1 f_1^* + \cdots + f_1^* \) be a lifting of \( m \) in \( K[X] \). Since \( \varphi(m) = 0 \), it follows that \( w_1(M) < w_1(M) \). Thus \( \deg M \geq \deg g \), or \( p e_i, \deg f_i \geq s e_i, \deg f_i + \deg A_i \). This inequality together with the inequality \( p \leq s \) implies that \( s = p \) and \( A_1 = 1 \). Therefore it results that \( w_1(g) = w_1(f_1) = s_\ell \gamma_1 \) as claimed.

c) Finally we shall prove that \( (g/h_1)^* \) is an irreducible polynomial of \( k_\ell[\gamma_1^*] \), with non-zero constant term. Indeed, by a) and b) one has \( w_1(g) = w_1(A_\ell) = s_\ell \gamma_1 = w_1(h_1) \). On the other hand, since \( w_1(g) \in G_\ell, w_1(A_\ell f_1) > w_1(g) \) if \( i \equiv 0 \pmod{e_i} \). Hence

\[
    (g/h_1)^* = (A_\ell/h_1)^* + (A_\ell h_1/h_1)^* r_1^* + \cdots + r_1^* = (A_\ell/h_1)^* \neq 0.
\]

In the same way as for \( (g/h)^* \), we see that \( (g/h_1)^* \in \text{Ker} \varphi \). So \( (g/h_1)^* \) is divisible by \( m \). But, since we have already proved that \( s = p \), it follows that \( (g/h_1)^* \) is also an irreducible polynomial of \( k_\ell[\gamma_1^*] \) whose constant term \( (A_\ell/h_1)^* \) is not-zero.

Remark 5.4. A) Note that the diagram (1) can be derived only by the hypothesis that \( u_\ell \) is an extension (but not necessarily an r.t. extension) of \( v \) to \( K(X) \) which dominates \( w_1 \). So it is easy to see that Lemma 5.3 is true
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without the hypothesis that \( w_2 \) is an r. t. extension of \( v \).

B) By the proof of Lemma 5.3 it follows that \((g/h_1)^{•}\) is the kernel of \( \varphi \).

**Theorem 5.5.** Let \( w_1, w_2 \) be r.t. extensions of \( v \) to \( K(X) \) such that \( w_1 < w_2 \).
Let \((a_i, \delta_i)\) be a minimal pair of definition of \( w_i \) and let \( f_i \) be the monic minimal polynomial of \( a_i \) (with respect to \( K \)), \( i=1, 2 \). The following assertions are equivalent:

1) \( f_2 \) is a key polynomial over \( w_i \).
2) \( f_2 \) is the polynomial in \( K[X] \) of the smallest degree such that \( w_i(f_2) < w_2(f_2) \).

In this case \( w_2 \) is an augmented valuation over \( w_1 \) and \( w_2 \) well dominates \( w_1 \).

**Proof.** 1) \( \Rightarrow \) 2) First, let us assume that \( \deg f_1 = \deg f_2 \). Then, according to Theorem 4.4 3), there exists a root \( b \) of \( f_2 \) such that \( (b, \delta) \) is also a minimal pair of definition of \( w_1 \). Hence we may assume that \( f_1 = f_2 \) and that \( b = a_1 \). The inequality \( w_i(f_2) < w_2(f_2) \) follows by Corollary 2.2.

Next, let us assume that \( n_1 = \deg f_1 < n_2 = \deg f_2 \). According to Theorem 4.6 3), there exists a suitable positive integer \( s \) such that \((f_2/h_s)^* = \rho_s(f_2/h_s)\) is an irreducible polynomial of \( k_s[\tau^*_s] \) (see Notation 1.3). And by diagram (1) and Corollary 2.2 a), it follows that \( \varphi((f_2/h_s)^*) = 0 \). Hence \((f_2/h_s)^* \) is an irreducible polynomial which generates \( \text{Ker } \varphi \).

Furthermore, let \( g \in K[X] \) be of the smallest degree such that \( w_i(g) < w_2(g) \). By Lemma 5.3, \( g \) is a key polynomial over \( w_i \). And, by Theorem 4.6 3) and Corollary 2.2, we may assume \( \deg g > n_1 \). By Theorem 4.6 3), it results that for a suitable \( t \), \((g/h_t)^* \) is an irreducible polynomial and that \((g/h_t)^* \in \text{Ker } \varphi \).

So \( \deg(g/h_t) = \deg(f_2/h_t) \). This means that \( \deg g = \deg f_2 \).

The implication 2) \( \Rightarrow \) 2) is a special case of Lemma 5.3.

Finally, it is clear that \( w_2 \) is the augmented valuation over \( w_1 \) associated with \( f_2 \) and \( \gamma_2 = w_2(f_2) \). So by Theorem 5.1, \( w_2 \) well dominates \( w_1 \).

6. Some applications

In this section we use the above results on key polynomials and augmented valuations over an r.t. extension to give a new proof of a result [3, Theorem 4.4]. We begin by a completion of Theorem 5.1.

Let \( w \) be an r.t. extension of \( v \) to \( K(X) \). Let \((a, \delta)\) be a minimal pair of \( w \) with respect to \( K \). As usual we shall use Notation 1.3. If \( g \) is a key polynomial over \( w \) such that \( \deg g > \deg f \), then according to Theorem 4.6 3), there
exists a positive integer s such that \((g/h^s)^*\) is an irreducible polynomial of \(k_v[r^s]\). The polynomial \((g/h^s)^*\) will be called the residue of the key polynomial g.

**Lemma 6.1.** Let \(w\) be an r.t. extension of \(v\) to \(K(X)\) and let \(w_1\) be the augmented valuation over \(w\) associated with a key polynomial \(g\) over \(w\) and \(\gamma_1 \subseteq G_v\) with \(\gamma_1 > w(g)\). Then

\[ \text{a) } k_{w_1} \text{ is canonically isomorphic to } k_w \text{ if } \deg g = \deg f. \]

If \(\deg g > \deg f\), then

\[ \text{b) } k_{w_1} \cong (k_v[r^s]/(g/h^s)^*)(t), \text{ } t \text{ transcendental over } k_v, \text{ and } \]

\[ \text{c) } G_{w_1} = G_w + \mathbb{Z}_{\gamma_1}. \]

**Proof.** a) According to Theorem 4.4 3), there exists a root \(b\) of \(g\) such that \(b, \delta\) is a minimal pair of definition of \(w\). Thus according to Theorem 1.2 d), the residue field \(k_w\) is canonically isomorphic to \(k_{v'}(t)\), where \(v'\) is the restriction of \(\delta\) to \(K(b)\) and \(t\) is transcendental over \(k_{v'}\). Now according to the step A in Theorem 5.1, the augmented valuation \(w_1\) has a minimal pair \((b, \delta_1)\), where \(\delta < \delta_1\). Also according to Theorem 1.2 d), it follows that \(k_{w_1} \cong k_{v'}(u)\), where \(u\) is a variable, i.e. \(k_{w_1} \cong k_{w_1}\) as claimed.

b) Let us consider the diagram (1). Since \(g\) is the polynomial of the smallest degree such that \(w(g) < w_1(g)\), according to the proof of Lemma 5.3 (see Remark 5.4 B)) it results that \((g/h^s)^*\) (the residue of the key polynomial \(g\)) is the kernel of \(\varphi\). Since according to Theorem 1.2 d), \(k_{w_1}\) is isomorphic to the field of the rational function of one variable over the algebraic closure of \(k_v\) in \(k_{w_1}\), we are only to prove that the image of \(\varphi\) in \(k_{w_1}\) coincides with the algebraic closure of \(k_v\) in \(k_{w_1}\). Indeed, according to Theorem 5.1, \(w_1\) has a minimal pair of definition \((a_1, \delta_1)\) where \(a_1\) is a root of \(g\). Hence if \(y \subseteq k_{w_1}\) is algebraic over \(k_v\) then, according to Theorem 1.2 d), there exists \(F \in K[X]\), such that \(\deg F < \deg g\), \(w_1(F) = \bar{0}(F(a_1)) = 0\), and that \(F^*\) is just \(y\). Now since \(\deg F < \deg g\), \(w(F) = w_1(F) = 0\) and \(\varphi(F^*) = F^* = y\), where \(F^*\) is the residue of \(F\) in \(k_w\). To complete the proof, it suffices to remark that the image of \(\varphi\) is included in the algebraic closure of \(k_v\) in \(k_{w_1}\) because the kernel of \(\varphi\) is not trivial.

The part c) results from the definition of an augmented valuation.

**Lemma 6.2.** Let \(w\) be an r.t. extension of \(v\) to \(K(X)\). Assume that there exists a subgroup \(G\) of \(G_v\) such that \(G_w < G\) and that the quotient group \(G/G_w\) is cyclic. Then there exists a key polynomial \(g\) over \(w\) and \(\gamma_1 \subseteq G_v\) with \(\gamma_1 > w(g)\) such that, \(G_{w_1} = G\) and \(k_{w_1}\) is \(k_v\)-isomorphic to \(k_w\), where \(w_1\) is the augmented valuation over \(w\) defined by \(g\) and \(\gamma_1\).
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Let \((a, \delta)\) be a minimal pair of definition of \(w\). Two cases are possible: \(e = 1\) or \(e > 1\).

If \(e = 1\), then \(G_w = G_{\delta}\). Take \(\gamma_1 \in G\) such that \(\gamma_1 > \gamma = w(f)\) and the coset \(\bar{\gamma}_1\) of \(\gamma_1\) modulo \(G_w\) generates \(G/G_w\). Let \(w_1\) be the augmented valuation over \(w\) defined by \(f\) and \(\gamma_1\). Then by Theorem 1.2 and Lemma 6.1 a), \(G_{w_1} = G_{\delta} + Z\gamma_1 = G_w + Z\gamma_1 = G\) and \(k_{w_1} \cong k_{\delta}(t)\) is \(k_{\delta}\)-isomorphic to \(k_w \cong k_{\delta}(r^*)\).

Now assume that \(e > 1\). Let \(g = f^e + u\), taking \(u \in K[X]\) such that \(\deg u < \deg f\) and that \(w(u) = w(u(a)) = w(f^e) = e\gamma\). Then by Theorem 4.6 3), \(g\) is a key polynomial over \(w\) and \((g/h)^* = r^* + y\) with \(0 \neq y \in k_{\delta}\). Take \(\gamma_1 \in G\) such that \(\gamma_1 > w(g)\) and that the coset \(\bar{\gamma}_1\) or \(\gamma_1\) modulo \(G_w\) generates \(G/G_w\). Let \(w_1\) be the augmented valuation over \(w\) associated with \(g\) and \(\gamma_1\). Then since, \((g/h)^*\) is of degree 1, according to Lemma 6.1 b) and c) it follows that \(k_{w_1} \cong k_{\delta}(t)\) is \(k_{\delta}\)-isomorphic to \(k_w\) and that \(G_{w_1} = G\).

**Lemma 6.3.** Let \(w\) be an r.t. extension of \(v\) to \(K(X)\) and let \(k_{w_1} = k'(t)\) where \(k'\) is a finite extension of \(k_v\) and \(t\) is transcendental over \(k_v\). Let \(k/k'\) be a finite simple extension, i.e. \(k = k'(\alpha)\). Then there exists a key polynomial \(g\) over \(w\) and \(\gamma_1 \in G_w\) with \(\gamma_1 > w(g)\) such that, if \(w_1\) is the augmented valuation over \(w\) associated with \(g\) and \(\gamma_1\),

\[
k_{w_1} \cong k(t) \quad \text{and} \quad G_{w_1} = G_w.
\]

**Proof.** Using Notation 1.3, we may assume that \(k' = k_{\delta}\) and that \(k = k_{\delta}(\alpha)\). Let \(G \subseteq k_{\delta}[r^*]\) be the monic minimal polynomial of \(\alpha\). We may assume that \(k \neq k_{\delta}\), or \(G\) is of degree greater than 1. Let \(g\) be a lifting of \(G\) in \(K[X]\). According to Corollary 4.7, we know that \(g\) is a key polynomial over \(w\).

Take \(\gamma_1 \in G_w\) such that \(\gamma_1 > w(g)\) and let \(w_1\) be the augmented valuation over \(w\) associated with \(g\) and \(\gamma_1\). The proof of Lemma 6.3 follows from Lemma 6.1 and b) and c).

**Theorem 6.4.** Let \((K, v)\) be a valuation pair, \(k\) a finite extension field of \(k_v\) and \(G\) an ordered group such that \(G/G_v\) is a finite group. Then there exists an r.t. extension \(w\) of \(v\) to \(K(X)\) such that \(G_w \cong G\) and \(k_w \cong k(t)\), \(t\) transcendental over \(k\).

**Proof.** Since \(G/G_v\) is finite we may assume that \(G_v \subseteq G \subseteq G_{\delta}\), and that there exists a chain of subgroups \(G_v = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_m = G\) such that \(G_{i+1}/G_i\) is a non-trivial cyclic group, \(i = 0, \ldots, m-1\).

Let \(w_0\) be the r.t. extension of \(v\) to \(K(X)\) defined by the minimal pair \((0, 0)\). Then \(k_{w_0} = k_v(X^*)\) (as usual \(X^*\) is the image of \(X\) in the residue field), and
By repeated application of Lemma 6.2 we can define, starting from $w_\alpha$, an r. t. extension $w'$ of $v$ to $K(X)$ such that $G_{w'}=G$ and $k_{w'}=k(v')$, where $t'$ is transcendental over $k_v$.

Furthermore, since $k/k_v$ is a finite extension, we can define a tower of fields $k_v=k_0\subset k_1\subset \cdots \subset k_n=k$ such that $k_{i+1}/k_i$ is a simple extension for all $i$, $0\leq i<n$. By repeated application of Lemma 6.3, we can define, starting from $w'$, an r. t. extension $w$ of $v$ to $K(X)$ such that $G_w=G_{w'}=G$, and $k_w=k(t)$, where $t$ is transcendental over $k$. The proof of Theorem 6.4 is complete.

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