EQUIVARIANT POINT THEOREMS

(Dedicated to Professor A. Komatu on his 70th birthday)

By

Minoru Nakaoka

1. Introduction.

This paper is a continuation of my previous paper [13], and is concerned with generalizations of the following two classical theorems on a continuous map $f$ of an $n$-sphere $S^n$ to itself.

**Theorem 1.1.** If the degree of $f$ is even then there exists $x \in S^n$ such that $f(-x) = f(x)$.

**Theorem 1.2.** If the degree of $f$ is odd then there exists $x \in S^n$ such that $f(-x) = -f(x)$.

Throughout this paper, a prime $p$ is fixed, and $G = \{1, T, \cdots, T^{p-1}\}$ will denote a cyclic group of order $p$.

Generalizing the situation in the above theorems, we shall consider the following problems.

**Problem 1.** Let $f : N \rightarrow M$ be a continuous map between $G$-spaces. Under what conditions does $f$ have an equivariant point, i.e., a point $x \in N$ such that

\[(1.1) \quad f(T^i x) = T^i f(x)\]

for $i = 1, 2, \cdots, p-1$?

**Problem 2.** Let $f : L \rightarrow M$ and $g : L \rightarrow N$ be continuous maps of a space $L$ to $G$-spaces $M$ and $N$. Under what conditions do there exist $p$ points $x_1, \cdots, x_p \in L$ such that

\[(1.2) \quad f(x_{i+1}) = T^i f(x_1), \quad g(x_{i+1}) = T^i g(x_1)\]

for $i = 1, 2, \cdots, p-1$?

We shall denote by $A(f)$ the set of points $x \in N$ satisfying (1.1), and by $A(f, g)$ the set of points $(x_1, \cdots, x_p) \in L^p$ satisfying (1.2).

If $L = N$ in Problem 2, then $A(f, \text{id})$ may be identified with $A(f)$. Therefore
Problem 2 is more general than Problem 1; still Problem 2 can be reduced to Problem 1. In fact, if we define \( h : L^p \rightarrow M \times N \) by
\[
\begin{align*}
h(x_1, \ldots, x_p) &= (f(x_1), g(x_1)) \quad (x_i \in L),
\end{align*}
\]
and regard \( L^p \) and \( M \times N \) as \( G \)-spaces by cyclic permutations and the diagonal action respectively, then we have \( A(h) = A(f, g) \).

Throughout this paper, a manifold will always mean a compact connected topological manifold which is assumed to be oriented if \( p \) is odd. The dimension of manifolds \( M, N, \ldots \) will be denoted by \( m, n, \ldots \). By a \( G \)-manifold is meant a manifold on which \( G \) acts topologically.

In this paper we shall consider Problems 1 and 2 in case \( M \) and \( N \) are \( G \)-manifolds. Some answers have been obtained by Conner-Floyd [3], Munkholm [10], Fenn [5], Lusk [8] and others with respect to generalizations of Theorem 1.1, and by Milnor [9] and the author [13] with respect to generalizations of Theorem 1.2. By pushing the line of [13] we shall prove in this paper more general results.

Throughout this paper the cohomology stands for the Čech cohomology and it takes coefficients from \( \mathbb{Z}_p \), the group of integers mod \( p \).

2. Theorems

In this section we shall state our main theorems answering to Problem 2 and then corollaries answering to Problem 1. The main theorems will be proved in §5 and §6.

Let \( o_k \in H^k(BG) \ (k=0,1,\ldots) \) denote the usual generators, where \( BG \) is the classifying space for \( G \). If \( X \) is a paracompact space on which \( G \) acts freely, \( H^*(X/G) \) can be regarded as an \( H^*(BG) \)-module via the homomorphism induced by a classifying map of \( X \); in particular we have \( o_k = o_k \cdot 1 \in H^k(X/G) \).

The first main theorem is stated as follows, and it generalizes Theorem 1.1 (see Remark 1 below).

**Theorem A.** Let \( f : L \rightarrow M \) and \( g : L \rightarrow N \) be continuous maps of a compact space \( L \) to \( G \)-manifolds \( M \) and \( N \). Suppose that

1) the action on \( M \) is trivial;
2) the action on \( N \) is free and \( o_m \in H^m(N/G) \) is not zero;
3) \( n \geq (p-1)m \);
4) \( f^* : H^q(M) \rightarrow H^q(L) \ (q > 0) \) is trivial;
5) \( g^* : H^n(N) \rightarrow H^n(L) \) is not trivial.

Then we have \( A(f, g) \neq 0 \); if \( L \) is moreover a manifold, we have
\[
\dim A(f, g) \geq pl - (p-1)(m+n) \geq 0,
\]
Equivariant Point Theorems

where \( \dim A \) denotes the covering dimension of \( A \).

Putting \( L=N \) and \( g=\text{id} \), we get

**Corollary.** Let \( f : N \rightarrow M \) be a continuous map of a \( G \)-manifold \( N \) to a manifold \( M \). Suppose that

i) the action on \( N \) is free and \( \omega \in H^n(N|G) \) is not zero;

ii) \( f^* : H^q(M) \rightarrow H^q(N) \quad (q>0) \) is trivial.

Then we have

\[
\dim A(f) \geq n - (p-1)m,
\]

where \( M \) is regarded as a \( G \)-manifold by the trivial action.

**Remark 1.** Taking

\[
N = a \mod p \text{ homology } n\text{-sphere}
\]

in the above corollary, we have the results due to Conner-Floyd [3], Munkholm [10] and the author [12], which are direct generalizations of Theorem 1.1.

**Remark 2.** Taking

\[
L=N = a \mod p \text{ homology } n\text{-sphere},
\]

\[
M = S^m, \quad \deg f = 0, \quad \deg g \equiv 0 \mod p
\]

in Theorem A, we have the results due to the to Fenn [5] and Lusk [8].

To state the second main theorem and its corollaries, we shall make some preparations.

For any indexing set \( I \), consider the complement \( I_\pi = \mathcal{I}^\pi - dI \) of the diagonal in \( \mathcal{I}^\pi \), and define \( (i_1, \ldots, i_p), (i'_1, \ldots, i'_p) \in I_\pi \) to be equivalent if \( (i_1, \ldots, i_p) \) is a cyclic permutation of \( (i'_1, \ldots, i'_p) \). We denote by \( R(I_\pi) \) a set of representatives of the equivalent classes.

Let \( f : L \rightarrow M \) and \( g : L \rightarrow N \) be continuous maps of a manifold \( L \) to \( G \)-manifolds \( M \) and \( N \). Given homogeneous bases \( \{\alpha_i\}_{i \in I}, \{\beta_j\}_{j \in J} \) of \( H^*(M) \), \( H^*(N) \) and sets \( R(I_\pi), R(J^\pi) \), we define \( \lambda(f, g), \lambda'(f, g) \in \mathbb{Z}_p \) as follows.

Define \( \Delta : M \rightarrow M^p \) by

(2.1) \[
\Delta(x) = (x, Tx, \ldots, T^{p-1}x) \quad (x \in M),
\]

and put

(2.2) \[
\Delta_1(1) = \sum_{(i_1, \ldots, i_p) \in \mathcal{I}^\pi} c_{i_1i_2} \alpha_{i_1} \times \cdots \times \alpha_{i_p} \quad (c_{i_1i_2} \in \mathbb{Z}_p)
\]

for the Gysin homomorphism \( \Delta_1 : H^*(M) \rightarrow H^*(M^p) \).

Similarly, put
for the homomorphism \( \Delta_1 : H^*(N) \to H^*(N^p) \).

We define

\[
\lambda(f, g) = \langle (f \ast^n_{(1)} \sum_{(i_1, \ldots, i_p) \in R(I^p)} c_{i_1} \cdots i_p \alpha_{i_1} \times \cdots \times \alpha_{i_p}) (g \ast^n_{(1)}) \delta(1), [L]^p \rangle,
\]

\[
\lambda'(f, g) = \langle (f \ast^n_{(1)} \sum_{(j_1, \ldots, j_p) \in R(I^p)} d_{j_1} \cdots i_p \beta_{j_1} \times \cdots \times \beta_{j_p}) , [L]^p \rangle.
\]

Obviously we have \( \lambda(f, g) = \lambda(g, f) \).

If \( L = N \) and \( g^* = \text{id} \), we write \( \lambda(f, g) = \lambda(f, g) \). It follows that

\[
\lambda(f) = \langle (f \ast^n_{(1)} \sum_{(i_1, \ldots, i_p) \in R(I^p)} c_{i_1} \cdots i_p \alpha_{i_1} \times \cdots \times \alpha_{i_p}) (T \ast^n_{(1)} f \ast^n_{(1)}) \cdots (T \ast^n_{1} f \ast^n_{1}) \delta(1), [N]^p \rangle.
\]

Remark 3. By the definition of \( \Delta_1 \) we have

\[
\langle \Delta^n_{(1)} (\alpha_{i_1} \times \cdots \times \alpha_{i_p}), [M]^p \rangle = \langle (\alpha_{i_1} \times \cdots \times \alpha_{i_p}) \delta(1), [M]^p \rangle.
\]

From this we get

\[
y_{\pm k} = \sum_{(i_1, \ldots, i_p) \in I^p} (-1)^{\varepsilon(i_1, \ldots, i_p)} c_{i_1} \cdots i_p z_{h_1, \ldots, 2h_1, \ldots, 2h_p},
\]

where

\[
y_{\pm k} = \langle \alpha_{i_1} (T \ast^n_{(1)} \alpha_{i_1}) \cdots (T \ast^n_{1} \alpha_{i_p}) , [M] \rangle,
\]

\[
z_{k} = \langle \alpha_{\alpha i}, [M] \rangle,
\]

\[
\varepsilon(i_1, \ldots, i_p, k_1, \ldots, k_p) = \sum_{i=1}^{p-1} \alpha_i (| \alpha_{i_{1}}| + \cdots + | \alpha_{k_p}|),
\]

being \( | \alpha_i | = \deg \alpha_i \). The relations (2.3) for \( (i_1, \ldots, i_p) \in I^p \) characterize the coefficients \( c_{i_1} \cdots i_p \) ([6]). In particular, if \( p = 2 \) we see that the matrix \( (c_{ij}) \) is the inverse of the matrix \( (y_{ij}) \).

Now the second main theorem is stated as follows, and it generalizes Theorem 1.2 (see Remark 5 below).

Theorem B. Let \( f : L \to M \) and \( g : L \to N \) be continuous maps of a manifold \( L \) to \( G \)-manifolds \( M \) and \( N \). Suppose that

i) \( i^* : H^q(M) \to H^q(M^G) \) is trivial for \( q \geq m|p \), where \( M^G \) is the fixed point set of \( M \), and \( i \) is the inclusion;

ii) the action on \( N \) is free;

iii) \( pl = (p-1)(m+n) \).

Then \( \lambda(f, g) \) and \( \lambda'(f, g) \) are independent of the choices of \( (\alpha_i)_{i_1}, (\beta_{j})_{j_1}, R(I^p) \),...
Equivariant Point Theorems

Let \( f : N \to M \) be a continuous map between \( G \)-manifolds, and suppose that

i) \( i^* : H^q(M) \to H^q(M^o) \) is trivial for \( q \equiv m/p \);

ii) the action on \( N \) is free;

iii) \( n = (p-1)m \).

Then \( \lambda(f) \) is independent of the choices of \( \{a_i\}_{i=1}^p \) and \( R(f) \), and if \( \lambda(f) \neq 0 \) we have \( A(f) \neq \phi \).

Putting \( L = N \) and \( g = \text{id} \) in Theorem B we have

\[ R(I^o) \], and we have \( \lambda(f, g) = \lambda'(f, g) \). If \( \lambda(f, g) \neq 0 \) we have \( A(f, g) \neq \phi \).

**Corollary 1.** Let \( f : N \to M \) be a continuous map between \( G \)-manifolds, and suppose that

i) \( i^* : H^q(M) \to H^q(M^o) \) is trivial for \( q \equiv m/p \);

ii) the action on \( N \) is free;

iii) \( n = (p-1)m \).

Then \( \lambda(f) \) is independent of the choices of \( \{a_i\}_{i=1}^p \) and \( R(f) \), and if \( \lambda(f) \neq 0 \) we have \( A(f) \neq \phi \).

Put \( L = M \) and \( f = \text{id} \) in Theorem B, and replace the notations \( M, N, g \) by \( N, M, f \) respectively. Then we get

**Corollary 2.** Let \( f : N \to M \) be a continuous map between \( G \)-manifolds, and suppose that

i) \( i^* : H^q(N) \to H^q(N^o) \) is trivial for \( q \equiv n/p \);

ii) the action on \( M \) is free;

iii) \( n = (p-1)m \).

Then the same conclusions as in Corollary 1 hold.

**Remark 4.** The above two corollaries for \( p = 2 \) have been obtained in [13]. The following proposition will be proved in §4 (see p. 407 of [2] for \( p = 2 \)).

**Proposition 2.1.** If \( M \) is a \( G \)-manifold such that \( i^* : H^{m/p}(M) \to H^{m/p}(M^o) \) is trivial, then

\[ \langle \alpha(T^*\alpha) \cdots (T^{*p-1}\alpha), [M] \rangle = 0 \quad (\alpha \in H^k(M)) \].

Let \( M \) be the one in Proposition 2.1 for \( p = 2 \). Then, the proposition and the Poincaré duality show that \( H^*(M) \) has a homogeneous basis \( \{\mu_1, \ldots, \mu_r, \mu_1', \ldots, \mu_r'\} \) such that

\[ \langle \mu_i(T^*\mu_k), [M] \rangle = 0, \langle \mu_i'(T^*\mu_k'), [M] \rangle = 0, \langle \mu_i(T^*\mu_k'), [M] \rangle = \delta_{ik} \].

In terms of this basis we see that

\[ \lambda(f) = \sum_{i=1}^r \langle f^*\mu_i(T^*f^*\mu_i), [N] \rangle \]

if \( p = 2 \). In particular, if \( M = N \) and \( f^* = \text{id} \) then \( \lambda(f) \) equals the **semi-characteristic**

\[ \chi_{1/2}(M) = \dim H^*(M)/2 \mod 2 \].

Thus, for \( p = 2 \) we have the following
COROLLARY 3. Let $M$ be a manifold with a free involution $T$, and assume $\chi_{1/2}(M) \neq 0$. Let $f, g : M \rightarrow M$ be continuous maps such that $f^* = g^* = \text{id} : H^*(M) \rightarrow H^*(M)$. Then there exist $x, x' \in M$ such that $f(x') = T(f(x))$ and $g(x') = T(g(x))$. In particular, there exists a point $x \in M$ such that $fT(x) = Tf(x)$.

REMARK 5. Taking $M = a \mod 2$ homology $m$-sphere

in Corollary 3, we have the result due to Milnor [9], which is a direct generalization of Theorem 1.2.


In this section we shall explain how to prove Theorems A and B.

Let $M$ be a $G$-manifold. If we regard $M^p$ as a $G$-manifold by cyclic permutations, the map $\Delta : M \rightarrow M^p$ in (2.1) is an equivariant embedding. Regard $S^{2k+1}$ as a $G$-manifold by the standard free action. Then we have a pair $(S^{2k+1} \times M^p, S^{2k+1} \times \Delta M)_G$ of manifolds, and hence the Thom isomorphism

$$\theta_k : H^q(S^{2k+1} \times \Delta M)_G \cong H^{q+(p-1)m}(S^{2k+1} \times (M^p, M^p - \Delta M)_G)$$

which is the composite of the duality isomorphisms for $S^{2k+1} \times \Delta M$ and for $(S^{2k+1} \times M^p, S^{2k+1} \times \Delta M)_G$ (see p. 353 of [14]). We denote the Thom class $\theta_k(1)$ by $\hat{U}_k^G$.

The isomorphisms $\theta_k$ for sufficiently large $k$ define the Thom isomorphism

$$\theta : H^q(\Delta M) \cong H^{q+(p-1)m}(M^p, M^p - \Delta M)$$

of the equivariant cohomology. The element $\theta(1)$ is denoted by $\hat{U}_M$, and is called the equivariant fundamental cohomology class of $M$.

The image of $\hat{U}_M$ in $H^m_m(p-1)(M^p)$ is denoted by $\hat{U}_M^G$, and is called the equivariant diagonal cohomology class of $M$.

If the action of $G$ on $M$ is free, the diagonal set $dM$ is in $M^p - \Delta M$. In this case the image of $\hat{U}_M$ in $H^m_m(p-1)(M^p, dM)$ is denoted by $\hat{U}_M^G$, and is called the modified equivariant diagonal cohomology class of $M$.

LEMMA 3.1. Let $M$ and $N$ be $G$-manifolds, and regard $M \times N$ as a $G$-manifold by the diagonal action. If the action on $N$ is free, we have

$$\hat{U}_{M \times N} = \pm (q_1^* \hat{U}_N^G)(q_2^* \hat{U}_N^G),$$

where $q_1^* : H^\ast_G(M^p) \rightarrow H^\ast_G((M \times N)^p)$ and $q_2^* : H^\ast_G(N^p, dN) \rightarrow H^\ast_G((M \times N)^p, d(M \times N))$ are induced by the projections $q_1 : M \times N \rightarrow M$, $q_2 : M \times N \rightarrow N$. 


Equivariant Point Theorems

PROOF. There are the following natural inclusions of manifolds:

\[(S^{2k+1} 	imes \Delta M) \times (S^{2k+1} \times \Delta N) \subset (S^{2k+1} \times M^p) \times (S^{2k+1} \times N^p)\]

\[\bigcup_{G} S^{2k+1} \times \Delta (M \times N) \subset \bigcup_{G} S^{2k+1} \times (M \times N)^p\]

From properties of the Thom class (see 325 of [4]), it follows that the Thom class for the pair in the upper line equals \(\pm \hat{O}^{(k)}_M \times \hat{O}^{(k)}_N\), and that it is sent to \(\pm \hat{O}^{(k)}_{M \times N}\) by the homomorphism \(i^*\) induced by the natural inclusion of the lower line to the upper. Therefore we have

\[\hat{O}^{(k)}_{M \times N} = \pm i^* (\hat{O}^{(k)}_M \times \hat{O}^{(k)}_N) = \pm i^* \left( p_1^* \hat{O}^{(k)}_M \cdot p_2^* \hat{O}^{(k)}_N \right) = \pm (q_1^* \hat{O}^{(k)}_M) \cdot (q_2^* \hat{O}^{(k)}_N),\]

where \(p_1, p_2\) are the projections of \((S^{2k+1} \times M^p) \times (S^{2k+1} \times N^p)\) to \(S^{2k+1} \times M^p, S^{2k+1} \times N^p\). This fact proves immediately the desired result.

**Lemma 3.2.** Let \(f : N \to M\) be a continuous map of a \(G\)-space \(N\) to a \(G\)-manifold \(M\), and define an equivariant map \(\tilde{f} : N \to M^p\) by

\[\tilde{f}(x) = (f(x), fT(x), \cdots, fT^{p-1}(x)) \quad (x \in N).\]

If the action on \(M\) is free, and if \(\tilde{f}^*(\hat{O}^p_M) \neq 0\) for the homomorphism \(\tilde{f}^* : H^p_\Sigma (M^p, dM) \to H^p_\Sigma (N, N^0)\), then we have \(A(f) \neq \phi\). If \(N\) is moreover a \(G\)-manifold, we have

\[\dim A(f) \geq n - (p-1)m \geq 0.\]

PROOF. In virtue of a commutative diagram

\[\begin{array}{ccc}
H^p_\Sigma (M^p, dM - \Delta M) & \xrightarrow{i^*} & H^p_\Sigma (M^p, dM) \\
\tilde{f}^* & \downarrow & \tilde{f}^* \\
H^p_\Sigma (N, N - A(f)) & \xrightarrow{i^*} & H^p_\Sigma (N, N^0),
\end{array}\]

\(\tilde{f}^*(\hat{O}^p_M) \neq 0\) implies \(H^m_\Sigma (N, N - A(f)) \neq 0\). Therefore \(A(f) \neq \phi\). If \(N\) is a \(G\)-manifold, we have isomorphisms

\[H^{n-m(p-1)}(A(f))/G \cong H^{m(p-1)}(N'/G, (N' - A(f))/G)\]

\[\cong H^{m(p-1)}(N'/G, (N' - A(f))/G) \cong H^{m(p-1)}(N', N' - A(f))\]

\[\cong H^{m(p-1)}_\Sigma (N, N - A(f)),\]

where \(N' = N - N^0\). Therefore \(H^{n-m(p-1)}(A(f))/G \neq 0\), and so \(\dim A(f) \geq n - m(p-1) \geq 0\). This completes the proof.

**Proposition 3.3.** Let \(f : L \to M\) and \(g : L \to N\) be continuous maps of a space
L to G-manifolds M and N. Suppose that the action on N is free. Then if
\[(f^p*\bar{U}_M)(g^p*\bar{U}_N) \in H_{G}^{p+n}(L^p, dL)\]
is not zero, we have \(A(f, g) \neq \phi\). If L is moreover a manifold, we have
\[\dim A(f, g) \geq pl - (p-1)(m+n) \geq 0.\]

**Proof.** Consider \(h : L^p \to M \times N\) defined by (1.3). Then, for the map \(h : L^p \to (M \times N)^p\) we have \(q^p_i \circ h = f^p_i, q^p_i \circ h = g^n_i\). Therefore by Lemma 3.1 we have
\[h^*(\bar{U}_{M \times N}) = \pm h^*((q^p_1 \circ \bar{U}_M)(q^p_2 \circ \bar{U}_N))\]
\[= \pm (f^p \circ \bar{U}_M)(g^p \circ \bar{U}_N).\]
This proves the desired result by Lemma 3.2.

We shall prove Theorems A and B by making use of Proposition 3.3. For this purpose we are asked to examine the following:

(i) structure of the equivariant cohomologies \(H_G^*(X^p)\) and \(H_G^*(X^p, dX)\) for a compact space \(X\).

(ii) the equivariant diagonal cohomology class \(\bar{U}_M\) and the modified equivariant diagonal cohomology class \(\bar{U}_N\) for a G-manifold \(M\).

As for (i) we have the results due to Steenrod and Thom, which are stated in § 4. Thus Theorems A and B will be proved by examining (ii), as seen in § 5 and § 6.

4. Preparations

In this section we shall recall some facts needed later.

Let \(X\) be a paracompact \(G\)-space. Then we have
\[H^*(X) = \lim_{\longrightarrow} H^*(K),\]
\[H_G^*(X, X^0) = \lim_{\longrightarrow} H^*(K/G, K^0/G),\]
where \(K\) ranges over the nerves of \(G\)-coverings of \(X\) (see Chap III, § 6 and Chap VII, § 1 of [2]). For each \(K\) a cochain map
\[\varphi_K : C^*(K) \to C^*(K/G, K^0/G)\]
is defined by
\[\langle \varphi_K(u), \pi(s) \rangle = \sum_{i=0}^{p-1} u(T^is),\]
where \(u \in C^*(K), s\) is a simplex of \(K\), and \(\pi : K \to K/G\) is the projection. Thus
we have a homomorphism

\[ \pi_1 : H^*(X) \to H^*_G(X, X^o) \]

defined by the cochain maps \( \varphi_K \).

We define

\[ \pi_1 : H^*(X) \to H^*_G(X) \]

to be the composite \( j^* \circ \pi'_1 \), where \( j^* : H^*_G(X, X^o) \to H^*_G(X) \) is induced by the inclusion. It follows that \( \pi_1 \) is the composite of the usual transfer \( H^*(X) \to H^*(X/G) \) and the canonical homomorphism \( H^*(X(G)) \to H^*_G(X) \).

We call \( \pi_1 \) in (4.2) the \textit{transfer}, and \( \pi'_1 \) in (4.1) the \textit{modified transfer}.

Put

\[ \sigma^* = \sum_{i=0}^{p-1} T^i \ast : H^*(X) \to H^*(X). \]

Then it is easily seen that

\[ \pi^* \circ \pi_1 = \sigma^* \]

for the canonical homomorphism \( \pi^* : H^*_G(X) \to H^*(X) \), and that

\[ \pi_1(\alpha_1) \cdot \pi_1(\alpha_2) = \pi'_1(\alpha_1 \cdot \sigma^* \alpha_2) = \pi'_1(\sigma^* \alpha_1 \cdot \alpha_2) \]

\((\alpha_1, \alpha_2 \in H^*(X))\). We have also

\[ \pi_1(\alpha) \cdot \delta^*(\beta) = 0 \quad (\alpha \in H^*(X), \beta \in H^*_G(X^o)) \]

for the coboundary homomorphism \( \delta^* : H^*_G(X^o) \to H^*_G(X, X^o) \).

In fact

\[ (-1)^{\alpha_1} \pi_1(\alpha) \cdot \delta^*(\beta) = \delta^*(i^* \pi_1(\alpha) \cdot \beta) \]

\[ = \delta^*(i^* j^* \pi'_1 (\alpha) \cdot \beta) = 0, \]

where \( i^* : H^*_G(X) \to H^*_G(X^o) \).

If \( X \) is a paracompact \( G \)-space, the Smith special cohomology groups \( H^*_G(X) \) are defined for \( \rho = \sigma = \sum_{i=0}^{p-1} T^i \) and \( \rho = \tau = 1 - T \), and we have the exact sequences

\[ \cdots \to H^q(X) \to H^q(X) \oplus H^q(X^o) \to \]

\[ H^q(X) \to H^q(X) \oplus H^q(X^o) \]

for \((\rho, \bar{\rho}) = (\sigma, \tau) \) and \((\tau, \sigma) \). We have also an isomorphism

\[ H^*_G(X) \cong H^*_G(X, X^o). \]
(See p. 143 of [2].)

It follows that

\[(4.6) i^*_\pi = (\pi^*, i^*): H^*(X) \rightarrow H^*_G(X, X^0) \oplus H^*(X^0).\]

**Lemma 4.1.** If $M$ is a $G$-manifold such that the action is not trivial, then it holds

\[\pi^*: H^m(M) \cong H^m_G(M, M^0).\]

**Proof.** In the exact sequence

\[H^m(M) \rightarrow H^m_G(M, M^0) \oplus H^m(M^0) \rightarrow H^m_G(M),\]

we have $H^m_G(M, M^0) = 0$, $H^m(M) \cong \mathbb{Z}_p$, $H^m(M^0) \cong H_0(M, M - M^0) = 0$, and moreover $H^m_G(M) \neq 0$ is proved as follows. Therefore we get the desired result by (4.6).

Suppose $H^m_G(M) = 0$. Then, by the Smith cohomology exact sequence, we see that $i^*_\pi: H^m(M) \cong H^m_G(M)$ and $\tau^*: H^m_G(M) \rightarrow H^m(M)$ is onto. This implies that $\tau^*: H^m(M) \rightarrow H^m_G(M)$ is onto and so $H^m_G(M)$ is equal to $0$, which is a contradiction.

For a paracompact space $X$, consider the equivariant cohomology $H^*_G(X^p)$, where $G$ acts on $X^p$ by cyclic permutations. Then we have the external Steenrod $p$-th power operation

\[P: H^q(X) \rightarrow H^*_G(X^p),\]

which is related to the Steenrod square $Sq^i$ if $p = 2$, and to the reduced $p$-th power $\mathcal{S}^i$ and the Bockstein operation $\beta^*$ if $p \neq 2$ as follows ([15]):

\[(4.7) d^*P(\alpha) = \begin{cases} \sum_{i=0}^{\lfloor q/2 \rfloor} \alpha(l_{i-1}) \times Sq^i \alpha & \text{if } p = 2, \\ h_q \sum_{i=0}^{\lfloor q/2 \rfloor} (-1)^i \alpha(l_{i-1}) \times \mathcal{S}^i \alpha - \alpha(l_{i-2}) \times \mathcal{S}^i \alpha & \text{if } p \neq 2, \end{cases}\]

where $d^*: H^*_G(X^p) \rightarrow H^*_G(BG \times X)$ is induced by the diagonal map, and

\[(4.8) h_q = \begin{cases} (-1)^{q/2} & \text{if } q \text{ is even}, \\ (-1)^{(q-1)/2}((p-1)/2)! & \text{if } q \text{ is odd}. \end{cases}\]

$P$ is natural, and it satisfies also

\[(4.9) \pi^*P(\alpha) = \alpha^p\]

for the canonical homomorphism $\pi^*: H^*_G(X^p) \rightarrow H^*(X^p)$. 
Lemma 4.2. Let \( M \) be a \( G \)-manifold, and let \( \alpha \in H^*(M) \) satisfy \( i^*(\alpha) = 0 \) for \( i^* : H^*(M) \to H^*(M^G) \). Then \( \Delta^*P(\alpha) \) is in the image of \( j^* : H^*_G(M, M^G) \to H^*_G(M) \) induced by the inclusion.

Proof. Consider a diagram

\[
\begin{array}{ccc}
H^*_G(M, M^G) & \xrightarrow{j^*} & H^*_G(M) \\
\downarrow \Delta^* & & \downarrow (id \times i)^* \\
H^*_G(M^G) & \xrightarrow{i^*} & H^*(BG \times M^G)
\end{array}
\]

in which the rectangle is commutative and the lower sequence is exact. Then it follows from (4.7) that \( i^* \Delta^*P(\alpha) = (id \times i)^*d^*P(\alpha) = 0 \). Therefore \( \Delta^*P(\alpha) \in \text{Im}j^* \).

Proof of Proposition 2.1. We may assume that the action is non-trivial and \( |\alpha| = m/p \). Consider a commutative diagram

\[
\begin{array}{ccc}
H^*_G(M, M^G) & \xrightarrow{j^*} & H^*_G(M) \\
\downarrow \pi^* & & \downarrow \pi^* \\
H^m(M) & \xrightarrow{\sigma^*} & H^m(M)
\end{array}
\]

By Lemmas 4.1 and 4.2, we see

\( \pi^* \Delta^*P(\alpha) \in \text{Im} \sigma^* \).

Since \( \sigma^* H^m(M) = 0 \) and

\( \pi^* \Delta^*P(\alpha) = \Delta^*(\alpha^p) = \alpha(T^* \alpha) \cdots (T^p - 1^* \alpha) \)

by (4.9), the proof completes.

The following theorem is due to Steenrod [15] (see also [12]).

Theorem 4.3. Let \( X \) be a compact space, and \( \{\alpha_i\}_{i \in I} \) be a homogeneous basis of \( H^*(X) \). Then the totality of elements

\[
\omega_I P(\alpha) \quad (i \in I, j \geq 0), \\
\pi_I (\alpha_{i_1} \times \cdots \times \alpha_{i_p}) \quad ((i_1, \ldots, i_p) \in R(I^*_G))
\]

is a homogeneous basis of \( H^*_G(X^p) \).

The following is due to Thom [16] (see also [1], [11], [17]).
Theorem 4.4. Let $X$ be a compact space, and $(\alpha_i)_{i \in I}$ be a homogeneous basis of $H^\bullet(X)$. Then the totality of elements

$$
\partial^* (\omega_j \times \alpha_i) \quad (i \in I, \, 0 \leq j < (p-1)|\alpha_i|),
$$

$$
\pi_1(\alpha_1 \times \cdots \alpha_i) \quad ((i_1, \ldots, i_p) \in R(I^*_p))
$$

is a homogeneous basis of $H^\bullet_\bullet(X^p, dX)$, where $\partial^* : H^\bullet(BG \times X) = H^\bullet(dX) \longrightarrow H^\bullet_\bullet(X^p, dX)$ is the coboundary homomorphism. Furthermore we have

$$
\pi_1(\alpha \times \alpha) = \sum_{i=0}^{[\frac{q}{2}]} \partial^* (\omega_{|\alpha| - i - 1} \times Sq^i \alpha)
$$

if $p=2$, and

$$
\pi_1(\alpha \times \cdots \times \alpha) = \sum_{i=0}^{[\frac{q}{2}]} \varepsilon_i \partial^* (\omega_{|\alpha| - 2i - 1} \times \Sp^i \alpha)
$$

with some $\varepsilon_i \neq 0 \mod p$ if $p \neq 2$.

Remark. Theorems 4.3 and 4.4 are proved in the literatures for a compact polyhedron. However we can extend them to compact spaces by the device seen in [2].

5. Proof of Theorem A.

The equivariant diagonal cohomology class $\hat{U}'_M$ in case the action on $M$ is trivial has been studied by Haefliger. By Theorem 3.2 in his paper [6] and

(5.1)

$$
\pi^*(\hat{U}') = A_1(1),
$$

we have the following (see the proof of Theorem 9.1 in [13]).

Proposition 5.1. If the action on $M$ is trivial, then

$$
\hat{U}'_M = \sum_{k=0}^{[m/2]} \omega_{m-2k} P(V_k) + \sum_{i_j}^{c_{ij}} (c_{ij} - c_{i_2i_j}) \pi_1(\alpha_i \times \alpha_j)
$$

if $p=2$, and

$$
\hat{U}'_M = h_m \sum_{k=0}^{[m/2]} (-1)^k \omega_{(p-1)(m-2k)p} P(V_k)
$$

$$
+ \sum_{(i_1, \ldots, i_p) \in R(I^*_p)} (c_{i_1 \cdots i_p} - c_{i_1 \cdots i_p}) \pi_1(\alpha_1 \times \cdots \alpha_p)
$$

if $p \neq 2$, where $(\alpha_i)_{i \in I}$ is a homogeneous basis of $H^\bullet(M)$, $c_{i_1 \cdots i_p}$, $h_m$ are those in (2.2), (4.8), and $V_k \in H^\bullet(M)$ are the Wu classes given by

$$
\langle V_k \alpha, [M] \rangle = \begin{cases} 
\langle Sq^k \alpha, [M] \rangle & \text{if } p = 2, \\
\langle \Sp^k \alpha, [M] \rangle & \text{if } p \neq 2. 
\end{cases}
$$
We shall next prove

**Proposition 5.2.** If the action on $M$ is free and $\omega_m \in H^p(M/G)$ is not zero, it holds

$$\omega_m \mathcal{U}_M = \delta^* (\omega_{p-1} \times \mu),$$

where $\mu$ is a generator of $H^m(M)$.

**Proof.** Let $V$ be an equivariant open neighbourhood of $dM$ in $M_p$, and put

$$W = M^p - \Delta M - dM, \quad C = M^p - \Delta M - V.$$

Then $C/G$ is a closed connected and non-compact subset of $W/G$, and hence we have $H^{mp}(W/G, W/G - C/G) = 0$ (see p. 260 of [4]). Therefore it follows that

$$H^{mp}_G(M^p - \Delta M, V) \cong H^{mp}_G(W, W - C) \cong H^{mp}(W/G, W/G - C/G) = 0.$$

This shows that $i^*: H^{mp}_G(M^p, M^p - \Delta M) \longrightarrow H^{mp}_G(M^p, dM)$ is onto, and so is

$$i^* \circ \theta: H^p_G(\Delta M) \longrightarrow H^{mp}_G(M^p, dM).$$

It follows from Lemma 4.1 and the assumptions that $H^{mp}_G(\Delta M) \cong \mathbb{Z}_p$ is generated by $\omega_m$. By Theorem 4.4, $H^{mp}_G(M^p, dM) \cong \mathbb{Z}_p$ is generated by $\delta^*(\omega_{p-1} \times \mu)$. Since $i^* \circ \theta$ is a homomorphism of $H^*(BG)$-modules and it sends 1 to $\mathcal{U}_M$, we have the desired result.

We shall now give

**Proof of Theorem A.** By the assumption ii) and Proposition 5.2, it holds

$$\omega_n \mathcal{U}_M = \delta^* (\omega_{p-1} \times \nu),$$

where $\nu$ is a generator of $H^*(N)$. Therefore we have

$$\omega_n (g^p \mathcal{U}_M) = \delta^* (\omega_{p-1} \times g^* \nu),$$

and this is not zero by the assumption v) and Theorem 4.4. Since $n \geq (p-1)m$ by the assumption iii), it holds

$$\omega_{p-1} (g^p \mathcal{U}_M) \neq 0.$$

On the other hand, it follows from the assumptions i), iv) and Proposition 5.1 that

$$f^p (\mathcal{U}_M) = h_m \omega_{p-1}$$

with $h_m \neq 0 \mod p$. Consequently we have

$$f^p (\mathcal{U}_M) \cdot g^p (\mathcal{U}_M) = h_m \omega_{p-1} \neq 0,$$

which completes the proof by Proposition 3.3.
6. Proof of Theorem B and an example.

The following proposition has been proved in [13] if \( p = 2 \). By the similar method we shall prove it for any \( p \).

**Proposition 6.1.** If \( i^* : H^q(M) \rightarrow H^q(M^G) \) is trivial for \( q \geq m/p \), then we have

\[
\hat{U}'_m = \sum_{(i, \ldots, i_p) \in \mathbb{N}^p} c_{i, \ldots, i_p} \pi_1(a_i, \ldots, a_p),
\]

where \( (a_i)_{i=1}^n \) is a homogeneous basis of \( H^*(M) \), and \( c_{i, \ldots, i_p} \) are those in \((2.2)\).

Before we proceed to proof we make some preparations.

The equivariant homology group \( H^G_*(X^p) = H_*(EG \times X^p) \) is canonically identified with \( H_*(G; H_*(X)^p) \), the homology group of the group \( G \) with coefficients in \( H_*(X)^p = H_*(X) \otimes \cdots \otimes H_*(X) \) on which \( G \) acts by cyclic permutations. Taking the standard \( G \)-free acyclic complex \( W \), we have an element of \( H_*(G; H_*(X)^p) \) represented by \( w_0 \otimes a \otimes \cdots \otimes a \), where \( w_0 \in W \) is the basis of degree \( k \) and \( a \in H_*(X) \). The corresponding element in \( H^G_*(X^p) \) will be denoted by \( P_k(a) \).

**Lemma 6.2.** Suppose that \( i^* : H^q(M) \rightarrow H^q(M^G) \) is trivial for \( q \geq m/p \). Then, for any \( k \geq 0 \) and for any \( a \in H^*(M) \), we have

\[
\langle w_0 \hat{U}'_m, P_{k+1}(\alpha \wedge [M]) \rangle = 0
\]

if \( p = 2 \), and

\[
\langle \hat{U}'_m, P_{2k+1}(\alpha \wedge [M]) \rangle = 0,
\]

\[
\langle w_0 \hat{U}'_m, P_{2k+1}(\alpha \wedge [M]) \rangle = 0
\]

if \( p \neq 2 \).

**Proof.** Similarly to Lemma 4.4 in [13], the result for \( p \neq 2 \) is proved as follows.

It follows that \( P_{2k+1}([M]) \) is in the image of

\[
i_k : H_{2k+1+m^p}(S^{2k+1} \times M^p) \rightarrow H_{2k+1+m^p}^G(M^p)
\]

induced by the inclusion, and that \( i_k(\hat{U}'_m) \) is the image of \( 1 \) under the homomorphism

\[
(id \times \Delta)^! : H^*(S^{2k+1} \times M) \rightarrow H^*(S^{2k+1} \times M^p).
\]

From these facts we see that \( \hat{U}'_m - P_{2k+1}([M]) \) is in the image of
Equivariant Point Theorems

\[ H_{2k+1+m}(S^{2k+1} \times M) \xrightarrow{i_{k*}} H^G_{2k+1+m}(M) \xrightarrow{\Delta_*} H^G_{2k+1+m}(M^g). \]

Therefore it follows that

\[ \langle \hat{U}_{l}', P_{2k+1}(\alpha - [M]) \rangle = \langle \hat{U}_{l}', P(\alpha) - P_{2k+1}([M]) \rangle = \langle P(\alpha), \hat{U}_{l}' - P_{2k+1}([M]) \rangle = \varepsilon_k \langle P(\alpha), \Delta_{g*}i_{k*} [S^{2k+1} \times M] \rangle = \varepsilon_k \langle a*P(\alpha), i_{k*} [S^{2k+1} \times M] \rangle \quad (\varepsilon_k \in \mathbb{Z}_p), \]

and similarly

\[ \langle \omega_1 \hat{U}_{l}', P_{2k+1}(\alpha - [M]) \rangle = \varepsilon_k \langle \omega_1 a*P(\alpha), i_{k*} [S^{2k+1} \times M] \rangle. \]

To prove the desired two equalities, we may suppose \( p|\alpha| \geq m+1 \) in the first, and \( p|\alpha| \geq m \) in the second. Consequently it suffices to prove that

\[ \Delta_* P(\alpha) = 0 \quad \text{if} \quad p|\alpha| \geq m+1, \]

\[ \omega_1 a*P(\alpha) = 0 \quad \text{if} \quad p|\alpha| \geq m. \]

By Lemma 4.2, \( \Delta_* P(\alpha) \) and \( \omega_1 a*P(\alpha) \) are in the image of \( j_* : H^*_G(M, M^g) \rightarrow H^*_G(M) \), and the Smith cohomology exact sequence implies \( H^*_G(M, M^g) = 0 \) \((q > m)\). Therefore we have the desired results, and the proof completes.

**Proof of Proposition 6.1.** In virtue of Theorem 4.3 it can be written uniquely that

\[ \hat{U}_{l}' = \sum_i \xi_{ij} \omega_j P(\alpha_i) + \sum_{(i_1, \ldots, i_p) \in \mathcal{R}(I_p^g)} \eta_{i_1 \cdots i_p} \pi_1(\alpha_{i_1} \times \cdots \times \alpha_{i_p}) \]

with some \( \xi_{ij}, \eta_{i_1 \cdots i_p} \in \mathbb{Z}_p \). Since it is easily seen that

\[ \text{Im} \pi_1 \cap \pi_k(a') = 0, \]

\[ \langle \omega_j P(\alpha), P_k(a) \rangle = \delta_{jk} \langle \alpha, a' \rangle \]

\((\alpha \in H^*(M), a' \in H^*_G(M))\), it follows from Lemma 6.2 that \( \xi_{ij} = 0 \). We see from (5.1) that \( \eta_{i_1 \cdots i_p} = c_{i_1 \cdots i_p} \) if \((i_1, \ldots, i_p) \in \mathcal{R}(I_p^g)\) and \( c_{i_1 \cdots} = 0 \) for any \( i \in I \). This completes the proof.

**Remark 1.** Working in the smooth category, Hattori [7] has given formulae for \( \hat{U}_{l}' \) with no assumption on \( M^g \).

The following is immediate from Proposition 6.1 and Theorem 4.5.

**Proposition 6.3.** If the action on \( M \) is free, then it can be written uniquely that
\[ \tilde{U}_N = \sum_{(i_1, \ldots, i_p) \in R(P)} c_{i_1} \cdots c_{i_p} \alpha_{i_1} \times \cdots \times \alpha_{i_p} \]
\[ + \sum_{|\alpha_i| \geq m-m/p} \varepsilon_i \delta^*(\omega_{(p-1)m-|\alpha_i|-1} \times \alpha_i) \]

with some \( \varepsilon_i \in \mathbb{Z}_p \).

Remark 2. The author does not know how to determine \( \varepsilon_i \) in the above. If \( M \) is a mod \( p \) homology sphere, it follows from Propositions 5.2 and 6.3 that

\[ \tilde{U}_N = \begin{cases} \pi_i(1 \times \mu) & \text{if} \quad p=2, \\ \pi_i(1 \times \mu \times \cdots \times \mu) + \varepsilon \delta^*(\omega_{(p-2)m-1} \times \mu) & \text{if} \quad p \neq 2, \end{cases} \]

where \( \varepsilon \neq 0 \mod p \), and \( \mu \in H^m(M) \) is a generator such that \( \langle \mu, [M] \rangle = 1 \).

We shall now give

Proof of Theorem B. By the assumption i) and Proposition 6.1 we have

\[ f^p \tilde{U}_M = \pi_i f^p \left( \sum_{(i_1, \ldots, i_p) \in R(P)} c_{i_1} \cdots c_{i_p} \alpha_{i_1} \times \cdots \times \alpha_{i_p} \right), \]

and by the assumption ii) and Proposition 6.3 we have

\[ g^p \tilde{U}_N = \pi_i g^p \left( \sum_{(i_1, \ldots, i_p) \in R(P)} d_{i_1} \cdots d_{i_p} \beta_{i_1} \times \cdots \times \beta_{i_p} \right) \]
\[ + \sum_{|\beta_j| \geq n-p} \varepsilon_j \delta^*(\omega_{(p-1)m-1} \times \beta_{i_1} \times \cdots \times \beta_{i_p}). \]

It follows from (5.1) and Proposition 6.1 that

\[ \sigma^* \left( \sum_{(i_1, \ldots, i_p) \in R(P)} c_{i_1} \cdots c_{i_p} \alpha_{i_1} \times \cdots \times \alpha_{i_p} \right) = \sigma(1), \]
\[ \sigma^* \left( \sum_{(i_1, \ldots, i_p) \in R(P)} d_{i_1} \cdots d_{i_p} \beta_{i_1} \times \cdots \times \beta_{i_p} \right) = \sigma(1). \]

Thus, by (4.4), (4.5) and the assumption ii), we have

\[ (f^p \tilde{U}_M) \cdot (g^p \tilde{U}_N) = \pi_i \left( f^p \sum_{(i_1, \ldots, i_p) \in R(P)} c_{i_1} \cdots c_{i_p} \alpha_{i_1} \times \cdots \times \alpha_{i_p} \right) \left( g^p \sigma(1) \right) \]
\[ = \pi_i \left( f^p \sigma(1) \right) \left( g^p \sum_{(i_1, \ldots, i_p) \in R(P)} d_{i_1} \cdots d_{i_p} \beta_{i_1} \times \cdots \times \beta_{i_p} \right) \]

in \( H^k_0(L, dL) \).

It follows from Theorem 4.4 that \( H^k_0(L, dL) \equiv \mathbb{Z}_p \) is generated by \( \delta^*(\omega_{p-1} \times \cdots \times \rho) \) or \( \pi_i(\rho \times \cdots \times \rho) \), where \( \rho \in H^L(L) \) is a generator such that \( \langle \rho, [L] \rangle = 1 \).

Consequently we have

\[ (f^p \tilde{U}_M) \cdot (g^p \tilde{U}_N) = \lambda(f, g) \pi_i(\rho \times \cdots \times \rho) \]
\[ = \lambda(f, g) \pi_i(\rho \times \cdots \times \rho), \]
which completes the proof by Proposition 3.3.

Theorem B for \( p=2 \), particularly corollary 3 in § 2, has interesting applications as is seen in [13]. The author does not know so interesting applications of Theorems B for \( p \neq 2 \). However there is the following example for which Theorem B for \( p=3 \) is applicable.

Let \( n=1,3 \) or 7, and take in Theorem B

\[ L = S^n \times S^n, \quad M = S^n \times S^n, \quad N = S^n, \]

where the action on \( N \) is any free \( G \)-action, and action on \( M \) is given as follows:

\[ T(x, y) = (y, y^{-1}x^{-1}), \]

\( x, y \) being complex numbers, quaternions or Cayley numbers according as \( n=1,3 \) or 7. It follows that the fixed point set of \( M \) is homeomorphic to \( S^{n-1} \)-point. Thus the assumptions i), ii), iii) in Theorem B are satisfied.

Let \( \nu \in H^*(S^n) \) denote a generator, and put \( \nu_1 = \nu \times 1, \quad \nu_2 = 1 \times \nu \in H^*(S^n \times S^n) \). Then, by Remark 3 in § 2, it can be seen that

\[ A_1(1) = \sigma^*(1 \times \nu_1 \nu_2 - \nu_1 \times \nu_1 \nu_2 - \nu_2 \times \nu_2 \nu_1 - \nu_2 \times \nu_1 \nu_2) \]

for the homomorphism \( A_1 : H^*(M) \to H^*(M^3) \), and

\[ A_1(1) = \sigma^*(1 \times \nu \nu) \]

for the homomorphism \( A_1 : H^*(N) \to H^*(N^3) \). Therefore, if continuous maps \( f : L \to M, \ g : L \to N \) satisfy

\[ f^*(\nu_1) = a_1 \nu_1 + a_2 \nu_2, \quad g^*(\nu) = b_1 \nu_1 + b_2 \nu_2 \]

\((a_{ij}, b_i \in \mathbb{Z}_2)\), simple calculation shows

\[ \lambda(f, g) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ b_1 & b_2 \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ b_1 & b_2 \end{vmatrix} = 0 \]

This yields by Theorem B the following

**Theorem 6.4.** Let \( n=1,3 \) or 7, and let \( f_1, f_2, g : S^n \times S^n \to S^n \) be continuous maps of type \((a_{11}, a_{12}), (a_{21}, a_{22}), (b_1, b_2)\) respectively. Let \( T : S^n \to S^n \) be a homomorphism of period 3 without fixed points. Then, if

\[ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ b_1 & b_2 \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ b_1 & b_2 \end{vmatrix} \equiv 0 \mod 3, \]

there exist \( x, y, z \in S^n \times S^n \) such that

\[ (f_2(x), f_2(y), f_2(z)) = (f_1(y), f_1(z), f_1(x)), \]
$$(Tg(x), Tg(y), Tg(z)) = (g(y), g(z), g(x)),
\quad f_1(x)f_1(y)f_1(z) = 1.$$  

In particular, taking $f_i =$ projection to the $i$-th factor, we have

**Corollary.** If $b_1 + b_2 \neq 0$ then there exist $x, y, z \in S^n$ such that
$$Tg(x) = g(y), \quad Tg(y) = g(z), \quad x y z = 1,$$

where $n, g$ and $T$ are those in Theorem 6.4.

**References**


M. Nakaoka  
Department of Mathematics  
Osaka University  
Toyonaka 560 Japan