THE GROTHENDIECK RING OF VECTOR SPACES
WITH TWO IDEMPOTENT ENDOMORPHISMS

By

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Introduction.

In this paper we are concerned with a particular bialgebra $A$ over a field $k$, which is generated as an algebra by $e_1, e_2$ with defining relations $e_1^2 = e_1$, $e_2^2 = e_2$, and whose comultiplication $\Delta: A \rightarrow A \otimes A$ and counit $\varepsilon : A \rightarrow k$ are given by the formulas

\[
\Delta(e_1) = e_1 \otimes e_1 + (1-e_1) \otimes (1-e_2)
\]
\[
\Delta(e_2) = (1-e_2) \otimes (1-e_1) + e_2 \otimes e_2
\]
\[
\varepsilon(e_1) = \varepsilon(e_2) = 1.
\]

The purpose of this paper is to compute the representation ring of $A$, namely the Grothendieck ring of finite dimensional $A$-modules with respect to $\oplus$ and $\otimes$, when $k$ is an algebraically closed field of characteristic zero. The classification of indecomposable $A$-modules is known and our main task is to decompose tensor product of indecomposable $A$-modules.

The results are summarized at the end of Section 1. Our computations involve the decomposition of tensor product of $\mathbb{Z}_e$-graded $k[x]$-modules. More generally we do this for $\mathbb{Z}_e(=\mathbb{Z}/e\mathbb{Z})$-graded $k[x]$-modules for any integer $e \geq 2$. Here, for $\mathbb{Z}_e$-graded $k[x]$-modules $A, B$, we give $A \otimes B$ the standard grading and let $x$ act on it by

\[
x(a \otimes b) = xa \otimes b + \omega^i a \otimes xb \quad \text{deg } a = i,
\]

where $\omega$ is a fixed primitive $e^{th}$ root of 1.

The bialgebra $A$ comes from a certain universal construction. In general, for $k$-algebras $A, B$ such that $\dim A < \infty$, there is a $k$-algebra $a(A, B)$ equipped with a $k$-algebra map $\rho : B \rightarrow A \otimes a(A, B)$ having the following property: For any $k$-algebra $C$, the map $\text{Hom}_{k-\text{alg}}(a(A, B), C) \rightarrow \text{Hom}_{k-\text{alg}}(B, A \otimes C)$ induced by $\rho$ is a bijection. The algebra $a(A, A)$ becomes naturally a bialgebra. The bialgebra $a(A, A)^*$ in the dual space $a(A, A)^*$ is the universal measuring bialgebra.
of $A$ in the terminology of Sweedler [3]. Our bialgebra $A$ is isomorphic to $a(A, A)$ with $A=k \times k$. General theory of such bialgebras will appear elsewhere.

1. Main results.

Throughout this paper $k$ is an algebraically closed field of characteristic zero, $\otimes$ is over $k$ and all modules are finite dimensional over $k$. Let $A$ be a $k$-algebra generated by $e_{ij}$, $i, j=1, 2$, with defining relations

$1=\sum_j e_{ij}$, \quad $i=1, 2$

$e_{ij}e_{ik}=\delta_{jk}e_{ij}$, \quad $i, j, k=1, 2$.

We make $A$ a bialgebra, defining comultiplication $\Delta: A \rightarrow A \otimes A$ and counit $\varepsilon: A \rightarrow k$ by the formulas

$\Delta(e_{ik})=\sum_j e_{ij} \otimes e_{jk}$

$\varepsilon(e_{ij})=\delta_{ij}$.

This bialgebra is identified with the one in Introduction by $e_{ii}=e_i$. For right $A$-modules $V, W$, we always regard $V \otimes W$ as a right $A$-module through the map $\Delta$. Our object is to decompose $A$-modules $V \otimes W$ for all indecomposable $A$-modules $V, W$.

We begin with a parametrization of indecomposable $A$-modules. Since a $A$-module structure on $V$ is determined by the subspaces $Ve_{ij}$ of $V$, the classification of $A$-modules is a special case of that of quadruples of subspaces in vector spaces, which was done by Gelfand and Ponomarev, and by Nazarova.

For vector spaces $V_{ij}$, $i, j=1, 2$, and an isomorphism $\alpha: V_{11} \oplus V_{12} \rightarrow V_{21} \oplus V_{22}$, define a $A$-module $M(\alpha)$ as the vector space $V_{11} \oplus V_{12}$ on which $e_{11}, e_{12}$ act as the projections to $V_{11}, V_{12}$, and $e_{21}, e_{22}$ act as the projections to $\alpha^{-1}(V_{21}), \alpha^{-1}(V_{22})$ respectively. We write the isomorphism $\alpha$ in a matrix form

$\alpha=\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$, \quad $\alpha_{ij}: V_{ij} \rightarrow V_{1i}$.

Let $\mathcal{C}$ be the category of $k[x]$-modules on which $x$ acts nilpotently. Indecomposable objects of $\mathcal{C}$ are $V_n:=k[x]/(x^{n+1})$, $n \geq 0$. By a $Z_2$-graded $k[x]$-module we mean a $k[x]$-module $A$ equipped with a $Z_2(=Z/2Z)$-grading $A=A_0 \oplus A_1$ such that $x(A_i) \subseteq A_{i+1}$ for $i \in Z_2$. A homomorphism of $Z_2$-graded $k[x]$-modules is a $k[x]$-linear map preserving grading. Let $\mathcal{D}$ be the category of $Z_2$-graded $k[x]$-modules on which $x$ acts nilpotently. For each $n \geq 0$ and $j=0, 1,$
let $V'_k$ be a $\mathbb{Z}_2$-graded $k[x]$-module which has a basis $v, xv, \ldots, x^nv$ such that $\deg v = j$ and $x^{n+1}v = 0$. The modules $V'_n$ for $n \geq 0, j = 0, 1$ form a complete list of indecomposable objects in $\mathcal{O}$.

For an object $A$ of $\mathcal{O}$, define $A$-modules $L_i(A), L_0(A)$ by

$$L_i(A) = M\begin{pmatrix} f_0 & 1_{A_1} \\ 1_{A_0} & f_1 \end{pmatrix}$$

$$L_0(A) = M\begin{pmatrix} 1_{A_0} & f_1 \\ f_0 & 1_{A_1} \end{pmatrix}$$

where $f_0 : A_0 \to A_1$, $f_1 : A_1 \to A_0$ are multiplication by $x$. For an object $A$ of $\mathcal{E}$ and $\lambda \in k - \{0, 1\}$, define a $A$-module $L_2(A)$ by

$$L_2(A) = M\begin{pmatrix} 1_A & 1_A \\ 1_A & f \end{pmatrix}$$

where $f : A \to A$ is the map $a \to (1 - \lambda)a + xa$. From the table of indecomposable representations of the $D_7^*$-graph in Dlab and Ringel [1], we see the following.

**Proposition 1.1.** The $A$-modules

$$L_i(V'_k), L_0(V'_k) \quad n \geq 0, j = 0, 1$$

$$L_2(V'_n) \quad n \geq 0, \lambda \in k - \{0, 1\}$$

form a complete list of indecomposable $A$-modules.

Obviously $L_i(V'_0) \cong k$, the trivial $A$-module. We define functors

$$\otimes : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$$

$$\otimes : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$$

$$\otimes' : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$$

$$p^* : \mathcal{O} \to \mathcal{E}$$

$$p_* : \mathcal{E} \to \mathcal{O}$$

$$(\sim) : \mathcal{O} \to \mathcal{O}$$

in the following way. If $A, B$ are $k[x]$-modules, the $k[x]$-module $A \otimes B$ is defined to be the vector space $A \otimes B$ on which $x$ acts as

$$x(a \otimes b) = xa \otimes b + a \otimes xb.$$  

If $A, B$ are $\mathbb{Z}_2$-graded $k[x]$-modules, the $\mathbb{Z}_2$-graded $k[x]$-modules $A \otimes B$ and
$A \otimes' B$ have the underlying space $A \otimes B$, and the grading and the action of $x$ are defined as

$$A \otimes B : (A \otimes B)_k = \bigoplus_{i+j=k} A_i \otimes B_j$$

$$x(a \otimes b) = xa \otimes b + (-1)^{i}a \otimes xb, a \in A_i, b \in B$$

$$A \otimes' B : (A \otimes' B)_k = A \otimes B_k$$

$$x(a \otimes b) = xa \otimes xb.$$ 

If we exhibit a $\mathbb{Z}_2$-graded $k[x]$-module $A = A_0 \oplus A_1$ and a $k[x]$-module $B$ by the diagrams

$$
\begin{array}{ccc}
A_0 & \xrightarrow{f_0} & A_1 \\
\downarrow{f_1} & & \downarrow{g} \\
B & & B \otimes g
\end{array}
$$

where $f_0, f_1, g$ are multiplication by $x$, the functors $p_*, p^*$, $(\_)$ are defined as

$$p^* : A_0 \xrightarrow{f_0} A_1 \quad \longrightarrow \quad A_0 \otimes f_1 f_0 \oplus A_1 \otimes f_0 f_1$$

$$p_* : B \otimes g \quad \longrightarrow \quad B \xrightarrow{1} B \oplus B \xrightarrow{g} B$$

$$(\_): A_0 \xrightarrow{f_0} A_1 \quad \longrightarrow \quad A_1 \xrightarrow{f_1} A_0.$$ 

**Theorem 1.2.** Let $\lambda, \mu \in \mathbb{k}$ and let $A, B$ be objects of $\mathcal{D}$ or $\mathcal{E}$. Then we have an isomorphism of $A$-modules

$$L_\lambda(A) \otimes L_\mu(B) \cong L_{\lambda \mu}(C)$$

where $C$ is an object of $\mathcal{D}$ or $\mathcal{E}$ defined as follows.

(i) \quad $C = A \otimes B$ \quad if $\lambda = \mu = 1$

(ii) \quad $C = p^* A \otimes B$ \quad if $\lambda = 1, \mu \neq 0, 1$

(iii) \quad $C = A \otimes p^* B$ \quad if $\lambda \neq 0, 1, \mu = 1$

(iv) \quad $C = A \otimes B \oplus A \otimes B$ \quad if $\lambda, \mu 
eq 0, 1, \lambda \mu 
eq 1$

(v) \quad $C = p_*(A \otimes B)$ \quad if $\lambda, \mu 
eq 0, 1, \lambda \mu = 1$

(vi) \quad $C = B \otimes \dim A$ \quad if $\lambda = 1, \mu = 0$

(vii) \quad $C = B \otimes \dim A$ \quad if $\lambda \neq 0, 1, \mu = 0$

(viii) \quad $C = A \otimes \dim \mathbb{B} \oplus A \otimes \dim \mathbb{B}_1$ \quad if $\lambda = 0, \mu = 1$
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\( C = A^\oplus \dim B \oplus \overline{A^\oplus \dim B} \) if \( \lambda = 0, \mu \neq 0, 1 \)

\( C = A \otimes B \) if \( \lambda = \mu = 0 \).

Proof will be given in Section 2.

We next describe the effect of the functors \( \otimes, \otimes', p^*, p_* \) on indecomposable modules in \( \mathcal{O} \) and \( \mathcal{E} \).

**PROPOSITION 1.3.** (i) We have isomorphisms in \( \mathcal{E} \)

\[ V_m \otimes V_n \cong \bigoplus_{j=0}^{\min(m,n)} V_{m+n-2j} \]

for all \( m, n \geq 0 \).

(ii) The Grothendieck ring \( S \) of \( (\mathcal{E}, \oplus, \otimes) \) is the polynomial ring on one generator \([V_1]\).

This is well-known and an immediate consequence of the Clebsch-Gordan rule for tensor product of simple \( \mathfrak{sl}_2 \)-modules. See also Littlewood [2, p. 195].

**PROPOSITION 1.4.** (i) We have isomorphisms in \( \mathcal{D} \)

\[ V_i^m \otimes V_i^n \cong \begin{cases} \bigoplus_{j=0}^{\min(m,n)} V_{i+j+j} \otimes V_{i+j-2j} & \text{if } mn \text{ is even} \\ \bigoplus_{j, \text{even}}^{\min(m,n)-1} (V_{i+j+j} \otimes V_{i+j-2j}) \oplus V_{i+j+1} \otimes V_{i+j+1} & \text{if } mn \text{ is odd} \end{cases} \]

for all \( m, n \geq 0, i, j \in \mathbb{Z}_2 \).

(ii) The Grothendieck ring \( R \) of \( (\mathcal{D}, \oplus, \otimes) \) is a commutative ring generated by the classes \([V_1], [V_2], [V_3] \) with defining relations

\[ [V_1]^2 = 1([V_2]) \]

\[ [V_2]^3 = [V_3]^2 + [V_1]^2 V_2 \].

We shall prove this in Section 3. In fact we shall determine decomposition of tensor product of \( \mathbb{Z}_e \)-graded \( k[x] \)-modules for any \( e \geq 2 \).

**PROPOSITION 1.5.** (i) We have isomorphisms in \( \mathcal{D} \)

\[ V_i^m \otimes V_i^n \cong \begin{cases} \bigoplus_{j=0}^{n-1} V_i^j \oplus V_i^{m+j-n} \otimes V_i^{m+j-n-1} & \text{if } m \leq n \\ \bigoplus_{j=0}^{m-1} V_i^j \oplus V_i^{n+j-m} \otimes V_i^{n+j-m-1} & \text{if } m > n \end{cases} \]

for all \( m, n \geq 0, i, j \in \mathbb{Z}_2 \).

(ii) The Grothendieck ring \( T \) (without 1) of \( (\mathcal{D}, \oplus, \otimes') \) has a \( \mathbb{Z} \)-basis \( \{e_n: n \geq 0, j \in \mathbb{Z}_2\} \), where

\[ e_n^j = [V_n^j] - [V_{n-1}^j] - [V_{n+1}^j] + [V_{n+1}^j] \]
with the convention $V^i_j = V^j_i = 0$ and we have
\[
e_n^n e_n^i = \begin{cases} e_n^i & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}
\]

**Proposition 1.6.** (i) We have isomorphisms
\[
p_n^* V^i_n = \begin{cases} V_{n/2} \oplus V_{n/2} & \text{if } n \text{ is even} \\ V_{(n-1)/2} \oplus V_{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}
\]
\[
p_n^* V^i_n = V^i_{n+1} \oplus V^i_{n+1}
\]
for all $n \geq 0$, $i \in \mathbb{Z}_n$.

(ii) The functor $p_n^* : \mathcal{D} \to \mathcal{E}$ induces a surjective ring homorphism $p_n^* : R \to S$ such that
\[
p_n^*[V^i_n] = 1, \quad p_n^*[V^i_n] = 2, \quad p_n^*[V^i_n] = 1 + [V^i_n]
\]
and the functor $p_n^* : \mathcal{E} \to \mathcal{D}$ induces an injective homomorphism $p_n^* : S \to R$ such that
\[
p_n^* p_n^*(a) = (1 + [V^i_n])[V^i_n] a
\]
for all $a \in R$.

Proofs of Propositions 1.5, 1.6 are easy and omitted.

Combining these results, we see that the representation ring of $A$ is isomorphic to the ring $K$ defined as follows. The additive group of $K$ is the direct sum
\[
K = \bigoplus_{\lambda \in \mathbb{Z}_n} K_\lambda
\]
where
\[
K_\lambda = \begin{cases} R & \text{if } \lambda = 1 \\ S & \text{if } \lambda \neq 0, 1 \\ T & \text{if } \lambda = 0 \end{cases}
\]
and
\[
R = \mathbb{Z}[\varepsilon, \phi, \phi^2] \text{ a commutative ring with defining relations} \varepsilon^2 = 1, \quad \phi(\phi^2 - 1 - \varepsilon) = 0,
\]
\[
S = \mathbb{Z}[\phi] \text{ a polynomial ring},
\]
\[
T = \bigoplus_{n \geq 0, j = 1} \mathbb{Z} e_n^j \text{ is a ring without 1 such that } e_n^m e_n^i = \delta_{m,n} e_n^i.
\]

$1 \in R$ is the identity element of $K$. For $a \in K_\lambda$, $b \in K_\mu$, the product $a \cdot b$ lies in
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\[ K_\mu \] and

\[
\begin{align*}
\lambda = \mu = 1 & \implies a \cdot b = ab \\
\lambda = 1, \mu \neq 0, 1 & \implies a \cdot b = p^*(a)b \\
\lambda \neq 0, 1, \mu = 1 & \implies a \cdot b = a p^*(b) \\
\lambda, \mu \neq 0, 1, \lambda \mu \neq 1 & \implies a \cdot b = 2ab \\
\lambda, \mu \neq 0, 1, \lambda \mu = 1 & \implies a \cdot b = p_\#(ab) \\
\lambda = \mu = 0 & \implies a \cdot b = ab \\
\lambda = 0 & \implies \varepsilon \cdot a = a, \quad a \cdot \varepsilon = \tilde{a} \\
\phi_1 \cdot a = 2a, \quad a \cdot \phi_1 = a + \tilde{a} \\
\phi_2 \cdot a = 3a, \quad a \cdot \phi_2 = 2a + \tilde{a} \\
\phi^t \cdot a = 2^{t+1}a, \quad a \cdot \phi^t = 2^t(a + \tilde{a})
\end{align*}
\]

where the multiplications in the right hand sides are those of the rings \( R, S \) or \( T \), and

\[ p^*: R \to S \] is a ring homomorphism such that \( \varepsilon \mapsto 1, \phi_1 \mapsto 2, \phi_2 \mapsto 1 + \phi \)

\[ p_\#: S \to R \] is an \( R \)-linear map such that \( 1 \mapsto (1 + \varepsilon) \phi_1 \)

\[ (\cdot): T \to T \] is an additive map interchanging \( e_n^0 \) and \( e_n^1 \) for all \( n \geq 0 \).

\section{2. Proof of Theorem 1.2.}

Let \( \lambda, \mu \in k - \{0\} \) and let

\[
A = \left( A_0 \xrightarrow{f_0} A_1 \right), \quad B = \left( B_0 \xrightarrow{g_0} B_1 \right)
\]

be \( \mathbb{Z}_2 \)-graded \( k[x] \)-modules with the notation in Section 1 and suppose that \( 1 - \lambda - f_0 f_1, 1 - \lambda - f_1 f_0, 1 - \mu - g_0 g_1, 1 - \mu - g_1 g_0 \) are nilpotent.

We restate Theorem 1.2 in terms of the functor \( M \) as follows:

\[ (2.1) \] If \( \lambda = \mu = 1 \), then

\[
M\left( \begin{array}{c} f_0 \\ f_1 \end{array} \right) \otimes M\left( \begin{array}{c} g_0 \\ g_1 \end{array} \right) \cong M\left( \begin{array}{c} l_0 \\ l_1 \end{array} \right)
\]

where

\[
A_0 \otimes B_0 \oplus A_1 \otimes B_1 \overset{l_0}{\underset{l_1}{\longrightarrow}} A_0 \otimes B_1 \oplus A_1 \otimes B_0
\]
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\[ l_{\varepsilon} = \begin{pmatrix} 1 \otimes g_0 & f_1 \otimes 1 \\ f_0 \otimes 1 & -1 \otimes g_1 \end{pmatrix} \]

\[ l_{i} = \begin{pmatrix} 1 \otimes g_1 & f_1 \otimes 1 \\ f_0 \otimes 1 & -1 \otimes g_0 \end{pmatrix}. \]

(2.2) If \( \lambda \mu \neq 1 \), then

\[
M \left( \begin{array}{cc} f_0 & 1 \\ 1 & f_1 \end{array} \right) \otimes M \left( \begin{array}{cc} g_0 & 1 \\ 1 & g_1 \end{array} \right) = M \left( \begin{array}{cc} 1 & 1 \\ 1 & l_{i} \end{array} \right) \oplus M \left( \begin{array}{cc} 1 & 1 \\ 1 & l_{\varepsilon} \end{array} \right)
\]

where

\[
1 - \lambda \mu - l_{\varepsilon} = (1 - \lambda - f_1) \otimes 1 \otimes 1 + 1 \otimes (1 - \mu - g_1 g_0) \in \text{End}(A_0 \otimes B_1)
\]

\[
1 - \lambda \mu - l_{i} = (1 - \lambda - f_1) \otimes 1 \otimes 1 + 1 \otimes (1 - \mu - g_1 g_0) \in \text{End}(A_1 \otimes B_0).
\]

(2.3) If \( \lambda, \mu \neq 1, A_0 = A_1, B_0 = B_1, f_0 = 1, g_0 = 1 \), then

\[
M \left( \begin{array}{cc} 1 & 1 \\ 1 & f_1 \end{array} \right) \otimes M \left( \begin{array}{cc} 1 & 1 \\ 1 & g_1 \end{array} \right) = M \left( \begin{array}{cc} 1 & 1 \\ 1 & l \end{array} \right) \oplus M \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)
\]

where

\[
- l = (1 - \lambda - f_1) \otimes 1 \otimes 1 + 1 \otimes (1 - \mu - g_1) \in \text{End}(A_1 \otimes B_1).
\]

(2.4) If \( \mu = 1 \), then

\[
M \left( \begin{array}{cc} f_0 & 1 \\ 1 & f_1 \end{array} \right) \otimes M \left( \begin{array}{cc} g_0 & 1 \\ 1 & g_1 \end{array} \right) = M \left( \begin{array}{cc} 1 \otimes 1 & 1 \otimes g_1 \\ 1 \otimes g_0 & 1 \otimes 1 \end{array} \right)
\]

where the left factor 1 in \( 1 \otimes 1, 1 \otimes g_0, 1 \otimes g_1 \) is the identity map on \( A_0 \oplus A_1 \).

(2.5) If \( \lambda = 1 \), then

\[
M \left( \begin{array}{cc} 1 & f_1 \\ f_0 & 1 \end{array} \right) \otimes M \left( \begin{array}{cc} g_0 & 1 \\ 1 & g_1 \end{array} \right) = M \left( \begin{array}{cc} 1 \otimes 1_{B_0} & f_1 \otimes 1_{B_0} \\ f_0 \otimes 1_{B_0} & 1 \otimes 1_{B_0} \end{array} \right) \oplus M \left( \begin{array}{cc} 1 \otimes 1_{B_1} & f_1 \otimes 1_{B_1} \\ f_0 \otimes 1_{B_1} & 1 \otimes 1_{B_1} \end{array} \right).
\]

(2.6) If \( \lambda = \mu = 1 \), then

\[
M \left( \begin{array}{cc} 1 & f_1 \\ f_0 & 1 \end{array} \right) \otimes M \left( \begin{array}{cc} g_0 & 1 \\ 1 & g_1 \end{array} \right) = M \left( \begin{array}{cc} 1 & f_0 \otimes g_1 \\ f_0 \otimes g_0 & 1 \end{array} \right) \oplus M \left( \begin{array}{cc} 1 & f_0 \otimes g_1 \\ f_0 \otimes g_0 & 1 \end{array} \right).
\]

Indeed, cases (2.1)-(2.6) correspond to cases (i)-(x) in Theorem 1.2 in the following way

(2.1) \( \iff \) (i)

(2.2) \( \iff \) (ii), (iii), (iv)

(2.3) \( \iff \) (v)

(2.4) \( \iff \) (vi), (vii)
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(2.5) \iff (viii), (ix)

(2.6) \iff (x)

Note that in some cases the present $A, B, \lambda, \mu$ are different from $A, B, \lambda, \mu$ in Theorem 1.2.

**Lemma 2.7.** Given isomorphisms

$$
\alpha = \begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{pmatrix} : V_{11} \oplus V_{12} \longrightarrow V_{21} \oplus V_{22}
$$

$$
\beta = \begin{pmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22}
\end{pmatrix} : W_{11} \oplus W_{12} \longrightarrow W_{21} \oplus W_{22}
$$

$$
\beta^{-1} = \begin{pmatrix}
\beta_{11}' & \beta_{12}' \\
\beta_{21}' & \beta_{22}'
\end{pmatrix} : W_{21} \oplus W_{22} \longrightarrow W_{11} \oplus W_{12}
$$

with $\alpha_{ij} : V_{ij} \rightarrow V_{2i}$, $\beta_{ij} : W_{ij} \rightarrow W_{2i}$, $\beta_{ij}' : W_{2j} \rightarrow W_{1j}$, we have an isomorphism of $A$-modules

$$M(\alpha) \otimes M(\beta) \cong M(\gamma)$$

where

$$
\gamma : Z_{11} \oplus Z_{12} \longrightarrow Z_{21} \oplus Z_{22}
$$

$$
Z_{jk} = \bigoplus_{ij} V_{ij} \otimes W_{jk}
$$

$$
\gamma = \begin{pmatrix}
\alpha_{11} \otimes 1 & \alpha_{12} \otimes \beta_{11}' & 0 & \alpha_{12} \otimes \beta_{12}' \\
\alpha_{21} \otimes \beta_{11} & \alpha_{22} \otimes 1 & \alpha_{21} \otimes \beta_{12} & 0 \\
0 & \alpha_{12} \otimes \beta_{21}' & \alpha_{11} \otimes 1 & \alpha_{12} \otimes \beta_{22}' \\
\alpha_{21} \otimes \beta_{21} & \otimes & \alpha_{21} \otimes \beta_{22} & \alpha_{22} \otimes 1
\end{pmatrix}
$$

The columns of this matrix correspond to $V_{11} \otimes W_{11}$, $V_{12} \otimes W_{21}$, $V_{11} \otimes W_{12}$, $V_{12} \otimes W_{22}$, and the rows correspond to $V_{21} \otimes W_{11}$, $V_{21} \otimes W_{21}$, $V_{22} \otimes W_{12}$, $V_{22} \otimes W_{22}$ in order.

Proof is straightforward. Now we shall prove (2.1)-(2.6).

(1) Let

$$
\alpha = \begin{pmatrix}
f_0 & 1 \\
1 & f_1
\end{pmatrix}, \quad \beta = \begin{pmatrix}
g_0 & 1 \\
1 & g_1
\end{pmatrix}
$$

Then

$$
\beta^{-1} = \begin{pmatrix}
(g_1 g_0 - 1)^{-1} g_1 & -(g_1 g_0 - 1)^{-1} \\
-(g_0 g_1 - 1)^{-1} g_0 & (g_0 g_1 - 1)^{-1} g_0
\end{pmatrix}
$$

so $M(\alpha) \otimes M(\beta) \cong M(\gamma)$ with
Multiplying an invertible matrix with $\gamma$ on the left, we have

$$\gamma = \begin{pmatrix}
1 \otimes g & f_i \otimes 1 & 1 \otimes 1 & 0 \\
1 \otimes g & 0 & 1 \otimes f_i & 0 \\
0 & 1 \otimes 1 & f_i \otimes 1 & 1 \otimes g \\
0 & 1 \otimes 1 & f_i \otimes (1 - g_1 g_i) & -1 \otimes g_0
\end{pmatrix} = \begin{pmatrix} h_0 & 1 \\
0 & h_1 \end{pmatrix},$$

where

$$h_0 = \begin{pmatrix} f_i \otimes 1 & 1 \otimes g \\
1 \otimes (1 - g_1 g_i) & -1 \otimes g_0
\end{pmatrix}, \quad h_1 = \begin{pmatrix} f_i \otimes 1 & 1 \otimes g \\
1 \otimes (1 - g_1 g_i) & -1 \otimes g_0
\end{pmatrix}.$$

(1a) We shall prove (2.1). Let $\lambda = \mu = 1$. Then $A, B \in \mathcal{D}$. Let $l_0, l_1$ be as in (2.1).

**Lemma 2.8.** The $\mathbb{Z}_2$-graded $k[x]$-modules

$$A_0 \otimes B_0 \oplus A_0 \otimes B_1 \xrightarrow{l_0} A_0 \otimes B_1 \oplus A_0 \otimes B_0$$

$$A_0 \otimes B_0 \oplus A_1 \otimes B_1 \xrightarrow{h_0} A_0 \otimes B_1 \oplus A_1 \otimes B_0.$$

are isomorphic.

From this we have

$$M(\gamma) \cong M(h_0 \ 1) \cong M(l_0 \ 1)$$

which proves (2.1).

**Proof of Lemma 2.8.** The both $\mathbb{Z}_2$-graded $k[x]$-modules have the common underlying graded space $A \otimes B$, and $x$ acts on the first module as

$$x(a \otimes b) = xa \otimes b + (-1)^i a \otimes xb \quad a \in A_i$$

and on the second module as

$$x(a \otimes b) =
\begin{cases}
xa \otimes (1 - x^4) b + a \otimes xb & \text{if } a \in A_i \\
xa \otimes b - a \otimes xb & \text{if } a \in A_1.
\end{cases}$$

We may assume that $A, B$ are indecomposable. Let $\dim A = m, \dim B = n$, and let $u \in A, v \in B$ be homogeneous generators. Let $G = k[s, t]$ be a graded $k$-algebra
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with defining relations \( s^n = t^n = 0 \), \( ts = -st \) and \( \deg s = \deg t = 1 \). \( G \) acts on the vector space \( A \otimes B \) in two different ways.

The first action:
\[
s(a \otimes b) = xa \otimes b, \quad a \in A_i.
\]

The second action:
\[
s(a \otimes b) = \begin{cases} xa \otimes (1-x^2)b & \text{if } a \in A_s \\ xa \otimes b & \text{if } a \in A_t \\ \end{cases},
\]
\[
t(a \otimes b) = (-1)^s a \otimes xb, \quad a \in A_i.
\]

To prove the lemma, it is enough to show that these two \( \mathbb{Z}_2 \)-graded \( G \)-modules \( A \otimes B \) are isomorphic. With respect to either action, \( s^i t^j (u \otimes v) \) (\( 0 \leq i < m \), \( 0 \leq j < n \)) form a basis of \( A \otimes B \). Hence the both \( G \)-modules are free on the generator \( u \otimes v \). This proves the lemma.

(1b) Suppose next that \( \lambda \mu \neq 1 \). We shall prove (2.2). Putting
\[
k_s = f_s f_i \otimes (1-g_0 g_i) + 1 \otimes g_0 g_i,
\]
\[
k_t = f_s f_i \otimes (1-g_0 g_i) + 1 \otimes g_0 g_i,
\]
we have
\[
h_s h_t = \begin{pmatrix} k_s & 0 \\ 0 & k_t \end{pmatrix}.
\]

Since \( 1 - k_s, 1 - k_t \) have the unique eigenvalue \( \lambda \mu \), \( h_s h_t \) is an isomorphism. Similarly \( h_t h_s \) is an isomorphism. Therefore
\[
\gamma \cong \begin{pmatrix} 1 & 1 \\ 1 & h_s h_t \end{pmatrix} \cong \begin{pmatrix} 1 & 1 \\ 1 & k_s \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & k_t \end{pmatrix}.
\]

**Lemmas 2.9**. Let \( s \in \text{End} V, t \in \text{End} W \) be nilpotent endomorphisms and \( \lambda, \mu \in k - \{0\} \). Then \( (\lambda + s)(\mu + t) - \lambda \mu, s \otimes 1 + 1 \otimes t \in \text{End}(V \otimes W) \) are conjugate.

The proof of the lemma is similar to that of Lemma 2.8. Let \( l_0, l_1 \) be as in (2.2). Applying the lemma to \( s = 1 - \lambda - f_s, t = 1 - \mu - g_0 g_i \), we see that \( k_s \) and \( l_0 \) are conjugate. Similarly \( k_t \) and \( l_1 \) are conjugate. Thus
\[
\gamma \cong \begin{pmatrix} 1 & 1 \\ 1 & l_0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ 1 & l_1 \end{pmatrix}
\]
which proves (2.2).

(1c) Suppose \( \lambda, \mu \neq 1, \lambda \mu = 1 \). Let \( A_s = A_1, f_s = 1, B_s = B_1, g_s = 1 \). Then
where $P, Q$ are some invertible matrices and $k = f_i \otimes (1-g_1)+1 \otimes g_1$. Let $l_o$ be as in (2.3). Using Lemma 2.9 with $s=1-\lambda-f_i, t=1-\mu-g_1$, we see that $k$ and $l$ are conjugate. Hence

$$\gamma \equiv \begin{pmatrix} 1 & 1 \\ k & 1 \end{pmatrix} \oplus \begin{pmatrix} -k & 1 \\ 1 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ l & 1 \end{pmatrix} \oplus \begin{pmatrix} l & 1 \\ 1 & 1 \end{pmatrix}.$$ 

This proves (2.3).

(2) We shall prove (2.4). Let

$$\alpha = \begin{pmatrix} f_i & 1 \\ 1 & f_i \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & g_1 \\ g_1 & 1 \end{pmatrix}, \quad \mu = 1.$$ 

Then

$$\beta^{-1} = \begin{pmatrix} (1-g_1g_1)^{-1} & -(1-g_1g_1)^{-1}g_1 \\ -g_1^{-1}g_1^{-1}g_1 & (1-g_1g_1)^{-1} \end{pmatrix}.$$ 

So

$$\gamma = \begin{pmatrix} f_i \otimes 1 & 1 \otimes (1-g_1 g_1)^{-1} & 0 & -1 \otimes (1-g_1 g_1)^{-1} g_1 \\ 0 & 1 \otimes g_1 & 0 & 0 \\ 0 & -1 \otimes (1-g_1 g_1)^{-1} & 1 \otimes 1 & f_i \otimes 1 \\ 1 \otimes g_1 & 0 & 1 \otimes 1 & f_i \otimes 1 \\ f_i \otimes (g_1g_1-1) & -1 \otimes 1 & 0 & 1 \otimes g_1 \\ 1 \otimes g_1 & 0 & 1 \otimes 1 & f_i \otimes 1 \\ 0 & -1 \otimes g_1 & f_i \otimes (g_1g_1-1) & -1 \otimes 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \otimes 1 & f_i \otimes 1 & 1 \otimes g_1 & 0 \\ 0 & 1 \otimes g_1 & 0 & 1 \otimes g_1 \\ 0 & 1 \otimes g_1 & f_i \otimes (g_1g_1-1) & -1 \otimes 1 \end{pmatrix}.$$ 

Put

$$h_0 = \begin{pmatrix} f_i \otimes 1 \\ 1 \otimes g_1 \end{pmatrix} \in \text{End}(A \otimes B, \otimes A \oplus B \otimes)$$

$$h_1 = \begin{pmatrix} 1 \otimes 1 \\ f_i \otimes 1 \end{pmatrix} \in \text{End}(A \otimes B, \otimes A \oplus B \otimes).$$

These are isomorphisms, so

$$\gamma \equiv \begin{pmatrix} 1 \otimes 1 \otimes 1 & (1 \otimes g_1)h_1^{-1} \\ (1 \otimes g_1)h_0^{-1} & 1 \otimes 1 \otimes 1 \end{pmatrix},$$

where $A = A \oplus A$. We claim that the following two objects of $\mathcal{D}$ are isomorphic.

$$A \otimes B_0 \stackrel{(1 \otimes g_0)h_0^{-1}}{\longrightarrow} A \otimes B$$

$$A \otimes B_1 \stackrel{(1 \otimes g_1)h_1^{-1}}{\longrightarrow} A \otimes B_1.$$
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\[ A \otimes B_0 \xrightarrow{1 \otimes g_0} A \otimes B_1. \]

Note that the isomorphism class of an object \( C = C_0 \oplus C_1 \) of \( \mathcal{D} \) is determined by the integers \( \text{dim Ker}(x^n : C_i \to C_{i+n}) \) for \( n > 0, i = 0, 1 \). Since

\[
(1 \otimes g_0) h_0 = h_1(1 \otimes g_0), \quad (1 \otimes g_1) h_1 = h_0(1 \otimes g_1),
\]

we have

\[
\text{dim Ker}(1 \otimes g_i) h_i^{-1} \cdots (1 \otimes g_{i+n}) h_{i+n}^{-1} = \text{dim Ker}(1 \otimes g_i) \cdots (1 \otimes g_{i+n}) h_{i+n}^{-1}
\]

where indices are taken modulo 2. Thus the above two objects are isomorphic. It follows that

\[
\begin{pmatrix}
1 & (1 \otimes g_i) h_i^{-1} \\
(1 \otimes g_0) h_0 & 1
\end{pmatrix}
\cong
\begin{pmatrix}
1 & 1 \otimes g_1 \\
1 \otimes g_0 & 1
\end{pmatrix}.
\]

This proves (2.4).

(3) Let

\[
\begin{align*}
\alpha &= \begin{pmatrix} 1 & f_1 \\ f_0 & 1 \end{pmatrix}, \quad \lambda = 1 \\
\beta &= \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \\
\beta^{-1} &= \begin{pmatrix} \beta'_{11} & \beta'_{12} \\ \beta'_{21} & \beta'_{22} \end{pmatrix}
\end{align*}
\]

Then \( M(\alpha) \otimes M(\beta) \cong M(\gamma) \), where

\[
\begin{pmatrix}
1 \otimes 1 & f_1 \otimes \beta'_{11} & 0 & f_1 \otimes \beta'_{12} \\
f_0 \otimes \beta_{11} & 1 \otimes 1 & f_0 \otimes \beta_{12} & 0 \\
0 & f_1 \otimes \beta'_{21} & 1 \otimes 1 & f_1 \otimes \beta'_{22} \\
f_0 \otimes \beta_{21} & 0 & f_0 \otimes \beta_{22} & 1 \otimes 1
\end{pmatrix}
\cong
\begin{pmatrix}
1 \otimes 1 - f_1 f_0 \otimes \beta'_{11} \beta_{11} & 0 & 0 & f_1 \otimes \beta'_{12} \\
0 & 1 \otimes 1 & f_0 \otimes \beta_{12} & 0 \\
0 & f_1 \otimes \beta'_{21} & 1 \otimes 1 - f_1 f_0 \otimes \beta'_{21} \beta_{22} & 0 \\
f_0 \otimes \beta_{21} & 0 & 0 & 1 \otimes 1
\end{pmatrix}
\cong
\begin{pmatrix}
h_0 & f_1 \otimes \beta'_{11} & 0 & 0 \\
f_0 \otimes \beta_{21} & 1 \otimes 1 & f_0 \otimes \beta_{12} & 0 \\
0 & 0 & f_1 \otimes \beta'_{21} & h_1
\end{pmatrix}
\]

with

\[
h_0 = 1 \otimes 1 - f_1 f_0 \otimes \beta'_{11} \beta_{11}.
\]
Since \( f_1 f_s \) is nilpotent, \( h_0, h_1 \) are isomorphisms. Hence

\[
γ \equiv \left( \begin{array}{cc}
1 & h_0^{-1}(f_1 \otimes β_{s1}) \\
\otimes β_{s1} & 1
\end{array} \right) \oplus \left( \begin{array}{cc}
1 & f_0 \otimes β_{s1} \\
h_1^{-1}(f_1 \otimes β_{s1}) & 1
\end{array} \right).
\]

(3a) To prove (2.5) we let

\[
β = \begin{pmatrix} g_0 & 1 \\ 1 & g_1 \end{pmatrix}.
\]

Then

\[
γ \equiv \left( \begin{array}{cc}
1 & k_0^{-1}(f_1 \otimes 1) \\
f_0 \otimes 1 & 1
\end{array} \right) \oplus \left( \begin{array}{cc}
1 & f_0 \otimes 1 \\
k_1^{-1}(f_1 \otimes 1) & 1
\end{array} \right),
\]

where

\[
k_0 = f_1 f_0 \otimes g_0 g_0 - 1 \otimes g_0 g_0 + 1 \otimes 1 \quad \text{and} \quad k_1 = f_1 f_0 \otimes g_0 g_0 - 1 \otimes g_0 g_0 + 1 \otimes 1.
\]

Put

\[
k_0' = f_0 f_1 \otimes g_0 g_0 - 1 \otimes g_0 g_0 + 1 \otimes 1 \quad \text{and} \quad k_1' = f_0 f_1 \otimes g_0 g_0 - 1 \otimes g_0 g_0 + 1 \otimes 1.
\]

These are isomorphisms and we have

\[
\begin{align*}
(f_0 \otimes 1) k_0 &= k_0'(f_0 \otimes 1) \\
(f_1 \otimes 1) k_1' &= k_1'(f_1 \otimes 1).
\end{align*}
\]

Then, by the same argument as in (2), we know that there are isomorphisms in \( \mathcal{D} \)

\[
\begin{array}{cccc}
A_0 \otimes B_0 & \xrightarrow{f_0 \otimes 1} & A_0 \otimes B_0 & \xrightarrow{f_0 \otimes 1} & A_0 \otimes B_0 \\
\| & k_0^{-1}(f_1 \otimes 1) & \| & f_0 \otimes 1 & \| \\
A_0 \otimes B_0 & \xleftarrow{f_0 \otimes 1} & A_0 \otimes B_0 & \xleftarrow{f_0 \otimes 1} & A_0 \otimes B_0,
\end{array}
\]

Thus

\[
γ \equiv \left( \begin{array}{cc}
1 & f_1 \otimes 1 \\
f_0 \otimes 1 & 1
\end{array} \right) \oplus \left( \begin{array}{cc}
1 & f_0 \otimes 1 \\
f_1 \otimes 1 & 1
\end{array} \right)
\]

which proves (2.5).

(3b) Finally we prove (2.6). Let

\[
β = \begin{pmatrix} 1 & g_1 \\ g_0 & 1 \end{pmatrix}, \quad \mu = 1.
\]
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Then

\[ γ \cong \begin{pmatrix} 1 & \kappa^{-1}(f_i \otimes g_i) \\ f_i \otimes g_{0} & 1 \end{pmatrix} + \begin{pmatrix} 1 & g_i \otimes f_i \\ k_i^{-1}(f_i \otimes g_{0}) & 1 \end{pmatrix}, \]

where

\[ k_0 = f_i f_i \otimes 1 + 1 \otimes g_{0} - 1 \otimes 1, \]

\[ k_1 = f_i f_i \otimes 1 + 1 \otimes g_{0} - 1 \otimes 1. \]

Put

\[ k'_0 = f_i f_i \otimes 1 + 1 \otimes g_{0} - 1 \otimes 1, \]

\[ k'_1 = f_i f_i \otimes 1 + 1 \otimes g_{0} - 1 \otimes 1. \]

Then

\[
\begin{align*}
(f_i \otimes g_0) k_0 &= k'_0 (f_i \otimes g_0) \\
(f_i \otimes g_0) k'_0 &= k_0 (f_i \otimes g_0) \\
(f_i \otimes g_0) k_1 &= k'_1 (f_i \otimes g_0) \\
(f_i \otimes g_0) k'_1 &= k_1 (f_i \otimes g_0).
\end{align*}
\]

As in (2) there are isomorphisms in \( \mathcal{O} \)

\[
\begin{align*}
A_0 \otimes B_0 &\xrightarrow{f_0 \otimes g_0} A_1 \otimes B_0 \xrightarrow{k^{-1}(f_i \otimes g_{0})} A_0 \otimes B_1 \\
A_0 \otimes B_0 &\xrightarrow{k^{-1}(f_i \otimes g_{0})} A_1 \otimes B_0 \xrightarrow{f_0 \otimes g_0} A_0 \otimes B_1.
\end{align*}
\]

Thus

\[ γ \cong \begin{pmatrix} 1 & f_i \otimes g_1 \\ f_i \otimes g_0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & g_i \otimes f_i \\ f_i \otimes g_{0} & 1 \end{pmatrix}. \]

This proves (2.6).

3. **Tensor product of graded \( k[\underline{x}] \)-modules.**

Throughout this section we fix \( \omega \in k \) a primitive \( e^{th} \) root of unity with \( e \geq 2 \).

By a graded \( k[\underline{x}] \)-module we mean a \( k[\underline{x}] \)-module \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) such that \( \dim M < \infty \), \( x M_i \subset M_{i+1} \) for all \( i \in \mathbb{Z} \). If \( M, N \) are graded \( k[\underline{x}] \)-modules we make the vector space \( M \otimes N \) a graded \( k[\underline{x}] \)-module in the following way.

\[
(M \otimes N)_i = \bigoplus_{i+j=i} M_i \otimes N_j,
\]

\[
x(a \otimes b) = xa \otimes b + \omega^i a \otimes xb \quad a \in M_i, b \in N.
\]

This operation \( \otimes \) on graded \( k[\underline{x}] \)-modules is associative. For each \( m \geq 0 \) and \( i \in \mathbb{Z} \), let \( V^m_i \) be a graded \( k[\underline{x}] \)-module of dimension \( m+1 \) generated by an element of degree \( i \). The modules \( V^m_i \) for \( m \geq 0, i \in \mathbb{Z} \) furnish a complete list of indecomposable graded \( k[\underline{x}] \)-modules. The main result of this section is the
following.

**Theorem 3.1.** For any \( m, n \geq 0 \) we have an isomorphism of graded \( k[x] \)-modules

\[
V_m^r \otimes V_n^s \cong \bigoplus_{l \in \mathbb{N}} V_{l}^t,
\]

where \( l \mapsto l^t \) is defined in the following way. Write \( m = re+i, n = se+j, l = qe+h \) with \( r, s, q \in \mathbb{N}, 0 \leq i, j, h < e \).

\[
l^t = \begin{cases} 
 m+n-2l & \text{if } \max(i+j-e+2, 0) \leq h \leq \min(i, j) \\
 or & \min(i, j) + 1 \leq h \leq \min(i+j+1, e-1) \\
 (r+s-2q+1)e-1 & \text{if } 0 \leq h \leq i+j-e+1 \\
 (r+s-2q)e-1 & \text{if } \min(i, j) + 1 \leq h \leq \max(i, j) \\
 (r+s-2q-1)e-1 & \text{if } i+j+2 \leq h \leq e-1.
\end{cases}
\]

Here we understand \( V_{-1}^t = 0 \).

Proposition 1.4 (i) follows from this, by letting \( e = 2 \) and reducing the grading modulo 2. See also Lemma 3.5 and the end of this section.

The proof of Theorem 3.1 goes as follows. We first decompose \( V_m \otimes V_n, V_m \otimes V_n, V_m \otimes V_n \) directly. In the Grothendieck ring we can express all \( [V_m^t] \) as polynomials of \([V_0^t], [V_1^t], [V_2^t]\). Then a straightforward computation gives the desired formula.

We begin with preliminary observation. Let \( m, n \geq 0 \) and let \( G = k[s, t] \) be a graded \( k \)-algebra with defining relations \( ts = st, s^m = t^n = 0 \) and \( \deg s = \det t = 1 \). Let \( G_k \) be the degree \( k \) part of \( G \) for each \( k \geq 0 \). Put \( x = s + t \). Since

\[
x \cdot s^{t} = s^{t+1} + \omega^{t} s^{t+1},
\]

when \( G \) is viewed as a graded \( k[x] \)-module by left multiplication, \( G \) is isomorphic to \( V_m^r \otimes V_n^s \). Since \( tx = oxt + (1 - o)t^2 \) and

\[
0 = s^{m+1} = (x-t)^{m+1} = x^{m+1} + c_1 x^{m} t + \cdots + c_{m+1} t^{m+1}
\]

for some \( c_1, \ldots, c_{m+1} \in k \), \( G \) has a basis \( x^t \), \( 0 \leq i \leq m, 0 \leq j \leq n \). Assume \( m \geq n \) and put

\[
z = x^m + c_1 x^{m-1} t + \cdots + c_m t^m.
\]

Then the following hold.

(i) The left multiplication \( x: G_k \rightarrow G_{k+1} \) is injective for \( k < n \), bijective for \( n \leq k < m \), and surjective for \( m \leq k \).
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(ii) $G/xG$ has a basis $t^j \mod xG$, $0 \leq j \leq n$.

(iii) $\text{Ker}(x : G \rightarrow G)$ has a basis $zt^j$, $0 \leq j \leq n$.

(iv) For each $0 \leq j \leq n$, put

$$l_j = \sup\{l : zt^j \in x^l G_{m+j-l}\}.$$

Then

$$G \cong \bigoplus_{j=0}^{n} V_{m+j-l_j}$$

as graded $k[x]$-modules.

(i) is clear and (ii), (iii) follow from (i). To see (iv), decompose $G = \bigoplus_i k[x]u_i$ with $u_i$ homogeneous elements such that $x^{m_i}u_i \neq 0$, $x^{m_i+1}u_i = 0$. Then the elements $x^{m_i}u_i$ form a basis of $\text{Ker}(x : G \rightarrow G)$. Since $zt^j$, $0 \leq j \leq n$, have mutually different degrees $m + j$, the bases $\{zt^j\}$ and $\{x^{m_i}u_i\}$ of $\text{Ker}(x : G \rightarrow G)$ are equal up to a permutation and scalar multiples. Hence $\{l_j\}$ is a permutation of $\{m_i\}$. This proves (iv).

**Lemma 3.2.** For any $m \geq 0$ we have

$$V_n \otimes V_n^* \cong \begin{cases} V_n \oplus V_n^* & \text{if } m+1 \equiv 0 \pmod{e} \\ V_n^* \oplus V_n & \text{if } m+1 \equiv 0 \pmod{e}. \end{cases}$$

**Proof.** We may assume $e > 0$. In the above observation we specialize $(m, n)$ to $(m, 1)$. Then $t^2 = 0$, $tx = oxt$ and

$$0 = (x-t)^{m+1} = x^{m+1} - \frac{\omega^{m+1}-1}{\omega-1} x^m t,$$

so

$$z = x^{m+1} \frac{\omega^{m+1}-1}{\omega-1} x^{m-1} t \quad zt = x^{m-1} t.$$

If $m+1 \equiv 0$, then $(\omega^{m+1}-1)/(\omega-1) \neq 0$, so

$$z \in x^{m-1} G_1, \quad z \notin x^{m} G_0$$

$$zt = \frac{\omega-1}{\omega^{m+1}-1} x^{m+1} \in x^{m+1} G_0.$$

Thus, by (iv) of the observation, $G \cong V_{m+1} \oplus V_{m+1}^* \cong V_{m} \oplus V_{m}^*$ as graded $k[x]$-modules.

If $m+1 \equiv 0$, then $z = x^{m}$, $x^{m+1} = 0$. So $zt \notin x^{m+1} G_0$. Thus $G \cong V_{m} \oplus V_{m}^*.$

**Lemma 3.3.** For any $r > 0$ we have

$$V_{e} \otimes V_{e}^* \cong V_{e} \otimes V_{e}^* \cong V_{2e} \otimes V_{2e}^* \cong V_{e} \otimes V_{e}^* \cong \cdots \cong V_{re} V_{re} \oplus V_{re}^*.$$
PROOF. We specialize $(m, n)$ in the previous observation to $(r, e)$. Then $t^{e+1} = 0$, $x^e = s^e + t^e$ and $s^e, t^e$ are central elements in $G$. We have

$$0 = (x - t)^{e+1} = (x^e - t^e)(x - t) = x^{r+1} - x^et - r x^{(r-1)e+1} t^e,$$

so

$$z = x^e - x^{e-1} t - r x^{(r-1)e+1} t^e$$

and

$$zt^j = x^e t^j - x^{e-1} t^{j+1}, \quad 1 \leq j \leq e - 1$$

$$zt^e = x^e t^e.$$

Let us determine the integers $l_j := \sup \{l : zt^l \in x^l G_{r+e-1} \}$ for $0 \leq j \leq e$. Clearly $l_e = (r-1)e$. By induction on $j$, we see easily that

$$x^{r+e+j} = x^e t^j + r x^{(r-1)e+j} t^e, \quad j \geq 1$$

$$x^e G_j = \langle x^e t^j, x^{(r-1)e+j} t^e \rangle, \quad j \geq 1.$$

It follows that $x^{r+e+j} e = x^e G_j$ for $1 \leq j < e - 1$, hence $l_j = r e - 1$. We have

$$x^{r+e+1} - (r+1) x^{r+e} t = -r z t^{e-1},$$

and $x^{r+e+1}, z t^{e-1}$ are linearly independent. So $l_i = r e + e - 2$. Finally, since $x^{r+e} = (r+1) z t^e$, we have $l_e = r e + e$. Thus

$$G \cong V_{(r+1)e} \oplus V_{(r+1)e-2} \oplus V_{r-1} \oplus \cdots \oplus V_{r-1} \oplus V_{(r-1)e}$$

as graded $k[x]$-modules.

**Lemma 3.4.** $V^i \otimes V_m \cong V_m \otimes V^i$ for all $m \geq 0$.

**Proof.** We can decompose $V^i \otimes V_m$ in the same manner as $V_m \otimes V^i$.

**Lemma 3.5.** $V^i \otimes V_m \cong V^i \otimes V^j \cong V^i \otimes V^j$ for all $n \geq 0$ and $i, j \in \mathbb{Z}$.

**Proof.** Let $u, v, w$ be homogeneous generators of $V^i, V^i, V^i \otimes V^j$, respectively. The correspondences $u^k u \otimes x^k v \otimes x^k w \mapsto x^k v \otimes u, 0 \leq k \leq n$, give the isomorphisms.

Let $Q$ be the Grothendieck ring of the category of graded $k[x]$-modules with respect to $\oplus, \otimes$. The classes $[V^i]$ in $Q$ form a basis of $Q$. We set

$$\varepsilon = [V^i],$$

$$\phi_n = [V^i] \quad n \geq 0,$$

$$\phi_{-1} = 0.$$

Then $\phi_1 = 1$ and by Lemma 3.5 $\varepsilon$ is a central invertible element in $Q$ and
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By Lemma 3.4 \( \phi_1 \) is also central and by Lemma 3.2

\[
(3.6) \quad \phi_i \phi_j = \begin{cases} 
\phi_{m+1} + \varepsilon \phi_{m-1} & \text{if } m+1 \not\equiv 0 \pmod{e} \\
(1+\varepsilon) \phi_m & \text{if } m+1 \equiv 0 \pmod{e}
\end{cases}
\]

for \( m \geq 0 \) and by Lemma 3.3

\[
(3.7) \quad \phi_r \phi_s = \phi_{(r+s)} + \varepsilon \phi_{(r+s)-2} + (\varepsilon^2 + \cdots + \varepsilon^{r-1}) \phi_{(r-1)} + \varepsilon \phi_{(r-1)s}
\]

for \( r > 0 \). It follows that \( Q \) is generated by \( \varepsilon, \, \phi_1, \, \phi_e \) and in particular \( Q \) is commutative.

For each integer \( n \geq -1 \), define a polynomial \( H_n(s, t) \) with integral coefficients by

\[
H_n(x + y, xy) = \frac{x^{n+1} - y^{n+1}}{x - y}
\]

with \( x, \, y \) indeterminates. Then \( H_{-1} = 0, \, H_0 = 1 \) and we have a formula

\[
H_m(s, t)H_n(s, t) = \sum_{l=0}^{\min(m, n)} t^l H_{m+n-2l}(s, t)
\]

for \( m, \, n \geq -1 \). Put

\[
\theta_n = H_n(\phi_{-e} - \varepsilon \phi_{-2}, \, \varepsilon^2) \in Q
\]

\[
\sigma_n = H_n(\phi_1, \, \varepsilon) \in Q
\]

for \( n \geq -1 \). Then

\[
(3.8) \quad \theta_m \theta_n = \sum_{l=0}^{\min(m, n)} \varepsilon^{l+m-n-2l} \theta_{m+n-2l}
\]

\[
(3.9) \quad \sigma_m \sigma_n = \sum_{l=0}^{\min(m, n)} \varepsilon^l \sigma_{m+n-2l}.
\]

By an easy induction it follows from (3.6) and (3.9) that

\[
(3.10) \quad \sigma_i = \phi_i, \quad 0 \leq i \leq e - 1
\]

\[
(3.11) \quad \sigma_{e-1-i} = (1 + \varepsilon^i) \phi_{e-1} - \varepsilon^i \phi_{e-1-i}, \quad 0 \leq i \leq e - 1.
\]

**Lemma 3.12.** We have

\[
\phi_i \phi_j = \sum_{h=\max(i+j-e+2, 0)}^{\min(i+j, e)} \varepsilon^h \phi_{i+j-2h} + \sum_{h=0}^{i+j-e+1} \varepsilon^h \phi_{e-1}
\]

for \(-1 \leq i, \, j \leq e - 1\).

**Proof.** We may assume \( i \geq j \geq 0 \). When \( i + j \leq e - 2 \), the formula results from (3.9), (3.10). Let \( i + j = e - 1 + l \) with \( 0 \leq l \leq e - 1 \). Then by (3.9) and (3.11)
we have
\[ \phi_i \phi_j = \sigma_i \sigma_j \]
\[ = \sum_{k=0}^{j} e^h \sigma_{i+j-k} \]
\[ = \sum_{0 \leq h \leq l} (1 + e^{l-2h}) \phi_{e-1} - e^{l-2h} \phi_{e-1+1+2h} + \sum_{l \leq h \leq j} e^h \phi_{e-1+1+2h} \]
\[ = \sum_{0 \leq h \leq l} (e^h + e^{l-h}) \phi_{e-1} - \sum_{0 \leq h \leq l} e^{l-h} \phi_{e-1+1+2h} \]
\[ + \sum_{l \leq h \leq j} e^h \phi_{e-1+1+2h} + \sum_{l \leq h \leq j} e^h \phi_{e-1+1+2h} \]
\[ = \sum_{k=0}^{j} e^h \phi_{e-1} + \sum_{h=\lceil l+1 \rceil}^{j} e^h \phi_{i+j-k} , \]
which proves the lemma.

**Lemma 3.13.** \( \phi_{r+1} = \theta_r \phi_i + e^{i+1} \theta_{r+1} \phi_{r+1} \) for \( r \geq 0, 0 \leq i \leq e-1 \).

**Proof.** Denoting by \( \phi_{r+1} \) the right hand side, it is enough to show that
\[ \phi_i = 1 \]
\[ \phi_{r+1} \phi_i = \phi_{r+1} \phi_i + e^{i+1} \theta_{r+1} \phi_{r+1} \]
\[ 0 \leq i \leq e-2, r \geq 0 \]
\[ \phi_{r+1} \phi_i = \phi_{r+1} + e^{i+1} \theta_{r+1} \phi_{r+1} \]
\[ r > 0 . \]
The second equality follows from the definition of \( \theta_i \), and the third follows from (3.6) without difficulty. For the last, using (3.8) and Lemma 3.12, we have
\[ \phi_{r+1} \phi = (\theta_r + e \theta_{r+1} \phi_{r+1})(\theta_r + e \phi_{r+1}) \]
\[ = \theta_r \phi + e \theta_{r+1} \phi_{r+1} + e \theta_{r+1} \phi_{r+1} + e^2 \theta_{r+1} \phi_{r+1} \]
\[ = \theta_r \phi + e \theta_{r+1} + e \theta_{r+1} \phi_{r+1} + e^2 \theta_{r+1} \phi_{r+1} \]
\[ + e \theta_{r+1} \phi_{r+1} + e^2 \theta_{r+1} \phi_{r+1} + e^3 \theta_{r+1} \phi_{r+1} + e^4 \theta_{r+1} \phi_{r+1} + e^5 \theta_{r+1} \phi_{r+1} \]
\[ + \cdots + e^{r-1} \theta_{r-1} \phi_{r-1} + e^r \theta_{r+1} \phi_{r+1} \]
\[ + (e^r + \cdots + e^{r-1}) \theta_{r+1} \phi_{r+1} + e^r \theta_{r+1} \phi_{r+1} , \]
as required.

**Proof of Theorem 3.1.** From Lemmas 3.12 and 3.13 we can deduce easily that
\[ \phi_{r+1} \psi_j = \min \{ i, j \} \sum_{h=\max(1, j) - e+2}^{i+j+1} e^h \phi_{r+1} + \sum_{h=\max(1, j) - e+2}^{i+j+1} e^h \phi_{r+1} + \sum_{h=\max(1, j) - e+2}^{i+j+1} e^h \phi_{r+1} \]
The Grothendieck ring of vector species

for \( r \geq 0, 0 \leq i \leq e - 1, -1 \leq j \leq e - 1 \). Replacing \( j \) by \( e - 2 - j \) and multiplying \( \varepsilon^{j+1} \), we have

\[
\phi_{re+i\varepsilon^{j+1}} \phi_{r-2-i} = \sum_{h = \max(i, j)+1}^{\min(r+1, e-1)} \varepsilon^h \phi_{re+i+j+e-2h} + \sum_{h = j+1}^{r-e-1} \varepsilon^h \phi_{(r+1)h-1} + \sum_{h = r+j+2}^{e} \varepsilon^h \phi_{re-1}
\]

for \( r \geq 0, 0 \leq i, j \leq e - 1 \). Using (3.8) and Lemma 3.13, we can also see

\[
\phi_{re+k} = \sum_{q, h} \varepsilon^q \phi_{(r+s-2q)e+i} \]

if \( r \geq 0, r \geq s \geq -1, 0 \leq k \leq e - 1 \) or if \( r, s \geq -1, k = e - 1 \).

Now let \( m = re+i, n = se+j \) with \( r, s \geq 0, 0 \leq i, j \leq e - 1 \). The formula to prove is symmetric in \( m, n \), so we may assume \( r \geq s \). By the above three formulas, we have

\[
\phi_{re+i} \phi_{re+j} = \phi_{re+i} \phi_{re+j} + \phi_{re+i} \phi_{re-j} \phi_{se-1}
\]

\[
= \sum_{q, h} \varepsilon^{q+2h} \phi_{(r+s-2q)e+i+j-2h} + \sum_{q, h} \varepsilon^{q+2h} \phi_{(r+s-1-2q)e+i+j+e-2h}
\]

\[
+ \sum_{q, h} \varepsilon^{q+2h} \phi_{(r+s-2q)e+i+e-1} + \sum_{q, h} \varepsilon^{q+2h} \phi_{(r+s-1-2q)e+i+e-1},
\]

where the \( k \)th summation \( \sum_{(q, h)} \) is over the elements \( (q, h) \) in the set \( I_k \) defined below.

\[
I_1: 0 \leq q \leq \min(r, s), \quad \max(i+j-e+2, 0) \leq h \leq \min(i, j)
\]

\[
I_2: 0 \leq q \leq \min(r, s-1), \quad \max(i, j)+1 \leq h \leq \min(i+j+1, e-1)
\]

\[
I_3: 0 \leq q \leq \min(r, s), \quad 0 \leq h \leq i+j-e+1
\]

\[
I_4: 0 \leq q \leq \min(r-1, s), \quad i+1 \leq h \leq j
\]

\[
I_5: 0 \leq q \leq \min(r, s-1), \quad j+1 \leq h \leq i
\]

\[
I_6: 0 \leq q \leq \min(r-1, s-1), \quad i+j+2 \leq h \leq e-1.
\]

As observed earlier, \( (V_m \otimes V_n) / x(V_m \otimes V_n) \) has a basis consisting of homogeneous elements of degrees 0, 1, \( \cdots \), \( \min(m, n) \). Therefore the map \( \mathcal{I}_1 \cdots \mathcal{I}_k \rightarrow [0, \min(m, n)] \) taking \( (q, h) \) to \( qe+h \) must be a bijection. Since the ranges of \( h \) in \( I_1, \ldots, I_6 \) give a partition of \([0, e-1]\), putting \( l = qe+h \), we have

\[
\phi_m \phi_n = \sum_{l \in \mathcal{I}_k} \varepsilon^l \phi_l
\]

with \( l_\# \) as described in Theorem 3.1. This proves the theorem.

**Proposition 3.14.** The ring \( Q \) is a commutative ring generated by \( \varepsilon, \varepsilon^{-1}, \phi, \phi_\# \) with a defining relation
\[ H_{e_i}(\phi_1, \epsilon)(\phi_1 - 1 - \epsilon) = 0. \]

**Proof.** This follows from (3.6) and the fact that \( \{ \epsilon^k \phi_i : k \in \mathbb{Z}, 0 \leq i \leq e - 1, r \geq 0 \} \) is a basis of \( Q \). Details are omitted.

Finally we pass from the \( \mathbb{Z} \)-graded case to the \( \mathbb{Z}_e \)-graded case. We consider only \( \mathbb{Z}_e(=\mathbb{Z}/e\mathbb{Z}) \)-graded \( k[x] \)-modules \( M = \bigoplus_{i \in \mathbb{Z}_e} M_i \) such that \( x M_i \subseteq M_{i+1} \) for all \( i \in \mathbb{Z}_e \) and \( x \) acts on \( M \) nilpotently. For such modules \( M, N \), we make the space \( M \otimes N \) a \( \mathbb{Z}_e \)-graded \( k[x] \)-module in the same manner as in the beginning of this section. For a graded \( k[x] \)-module \( M \), let \( \pi_\# M \) be the \( \mathbb{Z}_e \)-graded \( k[x] \)-module such that \( \pi_\# M = M \) as \( k[x] \)-modules and \( (\pi_\# M)_j = \bigoplus_{i \in \mathbb{Z}_e} M_i \) for \( j \in \mathbb{Z}_e \), where \( \pi : \mathbb{Z} \rightarrow \mathbb{Z}_e \) is the natural projection. Then the assignment \( M \mapsto \pi_\# M \) commutes with \( \otimes \), and the objects \( \pi_\# V_n, n \geq 0, 0 \leq j \leq e - 1 \), form a complete list of indecomposable \( \mathbb{Z}_e \)-graded \( k[x] \)-modules. Therefore the Grothendieck ring of the category of \( \mathbb{Z}_e \)-graded \( k[x] \)-modules is isomorphic to \( Q/(\epsilon^e - 1) \). When \( e = 2 \), we obtain Proposition 1.4 (ii) from Proposition 3.14.

**References**


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