A CHARACTERIZATION OF COMPLEX PROJECTIVE SPACES BY LINEAR SUBSPACE SECTIONS

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1. Introduction

It is conjectured in [2] that a complex projective space will be characterized from the standpoint of the positivity of sectional curvature. This conjecture is partially supported. Namely, a compact Kähler manifold \((M, g)\) with positive curvature is biholomorphically homeomorphic to a complex projective space, if, for examples, one of the following conditions is satisfied:

\begin{enumerate}
  \item \(\text{dim}_CM = 2\) ([2]),
  \item the Kähler metric \(g\) is Einstein ([1]),
  \item the group of holomorphic transformations acts on \(M\) transitively ([6]) and \(\text{dim}_CM = 3\) or 4 and \(H^*(M; \mathbb{Z}) \cong H^*(\mathbb{P}_n(\mathbb{C}); \mathbb{Z})\), \(n = \text{dim}_CM\) ([4]).
\end{enumerate}

These conditions play essential role in each result.

In this connection, we are in a position to consider the following assertion.

**Assertion** If a compact complex manifold \(M\) admits a closed complex submanifold, in particular, a closed complex hypersurface which is biholomorphically homeomorphic to a complex projective space, then \(M\) itself is biholomorphically homeomorphic to a complex projective space.

If this assertion is verified, the conjecture due to Frankel can be reduced to the following conjecture.

**Conjecture** A compact Kähler manifold with positive curvature will admit a closed complex submanifold endowed with a Kähler metric of positive curvature.

Of course, the submanifold of positive curvature may not be a Kähler submanifold of the ambient manifold.

In general, the assertion is false. For example, a product manifold \(\mathbb{P}_n(\mathbb{C}) \times M\), where \(M\) is a compact complex manifold, has \(\mathbb{P}_n(\mathbb{C})\) as a closed complex submanifold, but the total manifold can never be biholomorphically homeomorphic to a complex projective space. Hence, the submanifold in the assertion must satisfy further as-

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A compact Kähler manifold with positive curvature has the positive definite Ricci tensor, hence its first Chern class $c_1$ is positive. It is, then, an algebraic variety of a complex projective space by the aid of Kodaira's imbedding theorem. Therefore, the compact complex manifold stated in the assertion is furthermore assumed to be a closed submanifold of a complex projective space.

The assertion is held under the conditions that the submanifold is given as a section by a linear subspace in an ambient projective space and that it is biholomorphically homeomorphic to a complex projective space. This fact is precisely stated in Theorem 1.

The main purpose of this paper is to give a proof of Theorem 1. It is shown by the aid of a generalized Lefschetz's theorem ([5]) together with a characterization theorem of a complex projective space in terms of Chern classes ([7]).

2. Theorem and Corollaries

The following theorem characterizes a complex projective space by a linear subspace section in an ambient complex projective space.

**Theorem 1.** Let $M$ be an $n$-dimensional closed complex submanifold in an $N$-dimensional complex projective space $\mathbf{P}_N(\mathbb{C})$.

Assume that there is a linear subspace $V$ in $\mathbf{P}_N(\mathbb{C})$ of codimension $r$ ($\leq n-2$) such that a section $M \cap V$ of $M$ by $V$ is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbb{C})$. Then, $M$ itself is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbb{C})$.

Note that $r \leq n-2$ is necessary in proving Theorem 1, since the surjectivity of $\tilde{\iota}_*: H_2(M \cap V; \mathbb{Z}) \to H_2(M; \mathbb{Z})$ is guaranteed under the requirement of $r$.

The following is an immediate conclusion from Theorem 1.

**Corollary 2.** Let $M$ be as in Theorem 1. If there is a sequence of linear subspaces $\{V^1, \ldots, V^k\}$ of $\mathbf{P}_N(\mathbb{C})$, $r = \sum_{i=1}^{k} r_i \leq n-2$, $r_i = \text{codim}_V V^i$ such that

i) $M^{(i)}$ is a closed complex submanifold of $M^{(i-1)}$, $i=1, \ldots, k$,

ii) $M^{(k)}$ is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbb{C})$,

where $M^{(i)} = M \cap V^1 \cap \cdots \cap V^i$, $i=1, \ldots, k$ and $M^{(k)} = M$, then $M$ is biholomorphically homeomorphic to $\mathbf{P}_n(\mathbb{C})$.

Since $M^{(k)}$ is biholomorphically homeomorphic to $\mathbf{P}_{n-r}(\mathbb{C})$, $M^{(k-1)}$ is also biholomorphically homeomorphic to a complex projective space by the result of Theorem 1. Hence an inductive argument verifies Corollary 2.
Corollary 3. Let $M$ be as in Theorem 1. If there is a closed complex hypersurface $S$ in $\mathbb{P}_n(C)$ such that a section $M \cap S$ is biholomorphically homeomorphic to $\mathbb{P}_{n-1}(C)$, then $M$ is also biholomorphically homeomorphic to $\mathbb{P}_n(C)$.

If, moreover, there is a sequence of closed complex hypersurfaces $\{S^1, \cdots, S^k\}$, $k \leq n-2$ such that

i) $M^{(i)}$ is a hypersurface of $M^{(i-1)}$, $i=1, \cdots, k$ and

ii) $M^{(k)}$ is biholomorphically homeomorphic to $\mathbb{P}_{n-k}(C)$,

where $M^{(i)} = M \cap S^i \cap \cdots \cap S^1$, $i=1, \cdots, k$ and $M^{(0)} = M$, then $M$ itself is biholomorphically homeomorphic to $\mathbb{P}_n(C)$.

Corollary 3 is shown by the aid of Veronese mapping. Veronese mapping $v_m : \mathbb{P}_n(C) \to \mathbb{P}_{N}(C)$, $N = \left( \begin{array}{c} N+m \\ m \end{array} \right) - 1$, is defined as follows ([8]). Let $u_{i_0}, \cdots, u_{i_N}$ be homogeneous coordinates in $\mathbb{P}_n(C)$ where $i_0, i_1, \cdots, i_N$ are nonnegative integers such that $i_0 + i_1 + \cdots + i_N = m$. $v_m$ is defined by $u_{i_0}u_{i_1} \cdots u_{i_N} = z_0u_{i_0}x_{i_1} \cdots x_{i_N}$, where $z_0, z_1, \cdots, z_N$ are the homogeneous coordinates in $\mathbb{P}_N(C)$. It follows from the definition that the Veronese mapping is an imbedding.

Since the hypersurface $S$ of $\mathbb{P}_N(C)$ in Corollary 3 is given as zero points of a homogeneous polynomial of degree $m$, $\sum a_{i_0}u_{i_1} \cdots u_{i_N}z_0^{x_{i_1}} \cdots x_{i_N}$, $S$ is imbedded onto $v_m(S) = H \cap v_m(\mathbb{P}_N(C))$, where $H$ is a hyperplane in $\mathbb{P}_N(C)$ defined by $\sum a_{i_0}u_{i_1} \cdots u_{i_N}z_0^{x_{i_1}} \cdots x_{i_N} = 0$. Thus, $M \cap S$ is imbedded onto $v_m(M) \cap v_m(S) = v_m(M) \cap H$ which is biholomorphically homeomorphic to $\mathbb{P}_{n-1}(C)$ by the assumption. From Theorem 1, $v_m(M)$, hence, $M$ is biholomorphically homeomorphic to $\mathbb{P}_n(C)$. Hence we have the first part of Corollary 3. The second part of the corollary is easily obtained.

3. Proof of Theorem 1

Let $i : M \to M$ and $j : M \to \mathbb{P}_n(C)$ be the imbeddings, where $M = M \cap V$. Let $\tau_M$, $\tau_V$, and $\nu$ be the tangent bundle of $M$, the tangent bundle of $M'$ and the normal bundle of $M'$ in $M$, respectively.

If we denote by $[V]$ the vector bundle over $\mathbb{P}_n(C)$ defined by $V$, then the normal bundle of $V$ in $\mathbb{P}_n(C)$ is the pullback of $[V]$. Moreover, it is well-known that $\nu$ is isomorphic to the pullback of the normal bundle of $V$ in $\mathbb{P}_n(C)$. Therefore we have

$$i^*\tau_M = \tau_M \oplus j^*([V]).$$

Since $V$ is a linear subspace of codimension $r$, there is a hyperplane $H$ in $\mathbb{P}_n(C)$ such that $[V] = r[H]$, where $[H]$ is the line bundle over $\mathbb{P}_n(C)$ defined by $H$. Hence we have

$$i^*c_i(M) = c_i(M') + r^*j^*c_i([H]),$$

(1)
where \( c_i \)'s denote the first Chern classes.

Since \([V]\) is positive in the sense of Griffiths ([5]) and \( M' = M \cap V \) is a non-singular zero locus of a non-trivial global section of \( \mathcal{O}(j^*[V]) \), by the aid of a generalized Lefschetz's theorem (see Theorem \( H \) in [5]), we obtain the following two exact sequences under the condition \( r \leq n-2 \):

\[
H_2(M'; \mathbb{Z}) \xrightarrow{f_*} H_2(M; \mathbb{Z}) \to 0
\]

and

\[
0 \to H_3(M'; \mathbb{Z}) \xrightarrow{f_*} H_3(M; \mathbb{Z}) \to 0.
\]

Since \( M' \) is homeomorphic to a complex projective space, we have \( H_2(M'; \mathbb{Z}) \cong \mathbb{Z} \) and \( H_3(M'; \mathbb{Z}) = 0 \). Hence we obtain the following exact sequence:

\[
0 \to H_2(M'; \mathbb{Z}) \xrightarrow{f_*} H_2(M; \mathbb{Z}) \to 0
\]

which, together with \( H_3(M; \mathbb{Z}) = H_3(M; \mathbb{Z}) = 0 \), implies that \( \varepsilon^*: H^3(M; \mathbb{Z}) \to H^3(M'; \mathbb{Z}) \) is an isomorphism.

If \( \alpha \) is a positive generator of \( H^3(M'; \mathbb{Z}) \cong \mathbb{Z} \), then \( \varepsilon^* \alpha \) is also a positive generator of \( H^3(M'; \mathbb{Z}) \). Thus we have \( \varepsilon^* \varepsilon^*_M(\alpha) \cong \varepsilon^* \alpha \), and hence, \( \varepsilon^* c_1(M') \cong c_1(M') + r \varepsilon^* \alpha \). Since \( c_1(M') = (n-r+1) \varepsilon^* \alpha \), which is derived from the fact that \( M' \) is biholomorphically homeomorphic to an \((n-r)\)-dimensional complex projective space, we have \( c_1(M') \cong (n-r+1) \alpha + r \alpha = (n+1) \alpha \) by the injectivity of \( \varepsilon^*: H^3(M; \mathbb{Z}) \to H^3(M'; \mathbb{Z}) \).

Therefore, Theorem 1 follows from a result of [7].

4. Further Remarks

1) A linear subspace of a complex projective space is also a complex projective space. And its section by another linear subspace gives a linear subspace again. This is a trivial example which supports Theorem 1. We have a non-trivial example for Theorem 1 as follows. Recall the Veronese mapping \( v_m: \mathbb{P}_n(\mathbb{C}) \to \mathbb{P}_m(\mathbb{C}) \), \( N = \binom{n+m}{m} -1 \). The section of \( v_m(\mathbb{P}_n(\mathbb{C})) \) by the hyperplane of a form; \( u_{m0\ldots0} = 0 \), in \( \mathbb{P}_m(\mathbb{C}) \) gives a hyperplane \( z_0 = 0 \) in \( \mathbb{P}_m(\mathbb{C}) \). On the contrary, the section by the hyperplane of a form; \( u_{m0\ldots0} + u_{0m0\ldots0} + \ldots + u_{0\ldots0m} = 0 \), gives the hypersurface of degree \( m \);

\[
\sum_{j=0}^n x_j^m = 0 \text{ in } \mathbb{P}_n(\mathbb{C})
\]

2) In [3], a pair \((V, L)\) of a compact variety \( V \) and a line bundle \( L \) is called a polarized variety, if \( L \) is ample. If a compact complex manifold \( M \) is imbedded in a complex projective space, then a hyperplane section \( M \cap H \) of \( M \) induces a polarized variety \((M, [M \cap H])\), since \([M \cap H]\) is very ample. Theorem 1 is, if es-
3) With respect to Conjecture stated in Introduction, we have the following consideration.

Let $(M, g)$ be a compact Kähler manifold with positive curvature. Then, $M$ is a closed complex submanifold of a complex projective space. A hyperplane section $M'$ of $M$ gives a hypersurface which is defined by a certain holomorphic function, which we denote by $f$, locally. A relation between the holomorphic bisectional curvature $H\sigma, \tau$ of $M'$ with respect to the induced metric and $H_\sigma, \tau$ of $(M, g)$ is given as follows;

\[
H\sigma, \tau = H_\sigma, \tau - \frac{|H_\sigma(Z, W)|^2}{||df||^2 ||Z||^2 ||W||^2}.
\]

Here $\sigma$ and $\tau$ are holomorphic planes tangent to $M'$, $\sigma = X \wedge IX$, $\tau = Y \wedge IY$ and $Z = X - \sqrt{1} IX$, $W = Y - \sqrt{1} IY$. $H_f$ denotes the complex Hessian of $f$, i.e., $H_f = (\Gamma_0 f, f)$ and $||df||^2 = \sum g^{i\bar{j}} \frac{\partial f}{\partial z^i} \frac{\partial f}{\partial \bar{z}^j}$.

(2) is obtained by the similar argument as that in [9].

From (2), we have the following statement which locally supports Conjecture with respect to the holomorphic bisectional curvature.

For any point $p$ of $M$ and an arbitrary positive number $\varepsilon$, there are a neighborhood $U$ of $p$ and a holomorphic function $f$ defined on $U$ which satisfy the following;

i) \{(q \in U; f(q) = 0)\} is a hypersurface of $M$ which contains $p$

and

ii) on the hypersurface endowed with the induced metric,

\[|H_\sigma, \tau - H_\sigma, \tau| < \varepsilon\] for any pair of holomorphic planes $\sigma$ and $\tau$ tangent to the hypersurface.

This statement is observed as follows. Among all charts around $p$ we can choose a certain normal chart $(U', x')$, $x'(p) = 0$, with respect to which the components $\varrho_{ij}$ of the metric $g$ satisfy

\[
\varrho_{ij}(x') = \delta_{ij} + \sum \varrho_{ij}(p) x'^i x'^j + o(r),
\]

where $r = (\sum |x'|^2)^{1/2}$, and $\varrho_{ij}$'s are the components of the curvature tensor $R$.

Assume that a holomorphic function $f$ on $U'$ is of the form, $f(x') = \sum a^i x'^i + o(r)$, $(a^i) \equiv 0$, of course, such an $f$ exists indeed. Then, $f(x') = \varrho_{ij}(p) x'^i x'^j + o(r)$, where $\varrho_{ij}$'s are the Christoffel's symbols, that is, $\varrho_{ij} = \sum g^{ik} \partial \varrho_{ij}/\partial x'^k$. Hence, the complex Hessian $H_f = (\Gamma_0 f, f)$ vanishes at $p$. Therefore we can choose a sufficiently small neighborhood $U$ around $p$ such that
\[ |H^e\sigma, \tau - H^e\tau, \sigma| = \frac{|H^e(Z, W)|^2}{|df|^2||Z||^2||W||^2} < \varepsilon \]

for any pair of holomorphic planes \(\sigma\) and \(\tau\) tangent to \(q \in U; f(q) = 0\).

Since \(p\) is arbitrary, \(M\) is covered with such a system \(\{(U_p, f_p)\}_{p \in M}\) which gives local hypersurfaces. In order for the system to define a global hypersurface in \(M\), it must satisfy the property that there is a subsystem \(\{(U_p, f_p)\}_{p \in A}\), which covers \(M\) and \(f_p f_q\) gives a non-vanishing holomorphic function on \(U_p \cap U_q(\neq \phi)\), that is, the subsystem induces a non-singular holomorphic divisor.

It should be noticed that if this is verified, Conjecture can be supported with respect to the holomorphic bisectional curvature, since we only need to set \(\varepsilon = 1/2 \cdot \min H^e\sigma, \tau\) over all pairs of holomorphic planes of \(M\).

References