ON THE CAUCHY PROBLEM FOR ANALYTIC SEMIGROUPS WITH WEAK SINGULARITY

By

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I. Introduction and Results

Let $X$ be a Banach space with norm $\| \cdot \|$ and $\mathbb{A}$ a linear operator defined in $X$. We consider the following initial-value problem: Given an element $u_0 \in X$ and an $X$-valued function $f$ defined on an interval $I=[0, T]$, find an $X$-valued function $u$ defined on $I$ such that

\[
\begin{cases}
\frac{du}{dt}(t)=\mathbb{A}u(t)+f(t), & 0 < t \leq T, \\
u(0)=u_0.
\end{cases}
\]

In this paper, under the condition that the operator $\mathbb{A}$ generates an analytic semigroup with weak singularity, we give sufficient conditions on the function $f$ for the existence and uniqueness of solutions of the problem $(\ast)$.

We say that a function $u(t)$ is a strict solution or simply a solution of the problem $(\ast)$ if it satisfies the following three conditions:

\begin{align*}
(1.1) & \quad u \in C([0, T]; X) \cap C^{1}((0, T]; X), \\
(1.2) & \quad u(t) \text{ is in the domain } \mathcal{D}(\mathbb{A}) \text{ of the operator } \mathbb{A} \text{ for } 0 < t \leq T, \\
(1.3) & \quad u(0)=u_0 \quad \text{and} \quad \frac{du}{dt}(t)=\mathbb{A}u(t)+f(t), \quad 0 < t \leq T.
\end{align*}

Here $C([0, T]; X)$ denotes the space of continuous functions on $[0, T]$ taking values in $X$, and $C^{1}((0, T]; X)$ denotes the space of continuously differentiable functions on $(0, T]$ taking values in $X$, respectively.

We recall the following fundamental result in the theory of analytic semigroups (cf. Pazy [2]; Tanabe [4]):

**Theorem 1.0.** Assume that the following three assumptions are satisfied:

(A.1) The operator $\mathbb{A}$ is a densely defined, closed linear operator in $X$.

(A.2) There exist constants $0 < \omega < \pi/2$ and $\lambda_{0} > 0$ such that the resolvent set of $\mathbb{A}$ contains the region $\Sigma(\omega)=\{\lambda \in \mathbb{C}; |\text{arg}(\lambda-\lambda_{0})| < \pi/2 + \omega\}$.

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(A.3) If $0<\varepsilon<\omega$, then there exists a constant $C(\varepsilon)>0$ such that the resolvent $(\mathcal{A}-\lambda)^{-1}$ satisfies the estimate:

\[
\| (\mathcal{A}-\lambda)^{-1} \| \leq \frac{C(\varepsilon)}{1+|\lambda|}, \quad \lambda \in \Sigma(\varepsilon).
\]

Then the operator $\mathcal{A}$ generates a semigroup $e^{\mathcal{A}t}$ in $X$ which is analytic in the sector $\Delta(\omega) = \{ z = t + is \in \mathbb{C} : z \neq 0, |\arg z| < \omega \}$.

If $0<\gamma<1$, we let

\[ C^\gamma([0,T];X) = \text{the space of } X\text{-valued, continuous functions } f(t) \text{ on } [0,T] \text{ such that we have } \| f(t)-f(s) \| \leq M|t-s|^{\gamma}, \ t, s \in [0,T] \text{ for some constant } M>0. \]

Now it is known (cf. Pazy [2], Theorem 3.2) that the following theorem holds.

**Theorem 1.1.** Assume that the operator $\mathcal{A}$ satisfies Assumptions (A.1), (A.2) and (A.3). If $f \in C^\gamma([0,T];X)$ with $0<\gamma\leq 1$, then, for any $u_0 \in X$, the problem (*) has a unique solution which takes the following form:

\[
u(t) = e^{\mathcal{A}t}u_0 + \int_0^t e^{(t-s)\mathcal{A}}f(s)ds.
\]

The next Besov space version of Theorem 1.1 is due to Muramatu [1] (see [1], Theorem B).

**Theorem 1.2.** Assume that the operator $\mathcal{A}$ satisfies Assumptions (A.1), (A.2) and (A.3). If $f$ belongs to the Besov space $B^\gamma_{\infty,1}((0,T);X)$, then, for any $u_0 \in X$, the problem (*) has a unique solution which takes the form of (1.4).

**Remark 1.1.** Theorem 1.2 is a generalization of Theorem 1.1. In fact, the following inclusion holds:

\[
\bigcup_{0<\gamma\leq 1} C^\gamma([0,T];X) \subseteq B^\gamma_{\infty,1}((0,T);X).
\]

**Example 1.1.** The following function $f$ belongs to the space $B^\gamma_{\infty,1}((0,T);\mathbb{R})$, but does not belong to the spaces $C^\gamma([0,T];\mathbb{R})$ for any $0<\gamma\leq 1$.

\[
f(t) = \begin{cases} 
1 / \log t & \text{if } 0<t\leq T, \\
0 & \text{if } t=0.
\end{cases}
\]

For the precise definition of the Besov space $B^\gamma_{\infty,1}((0,T);X)$, we refer to Section 2.
We say that the operator $\mathfrak{A}$ satisfies Assumption $(AS)_{\theta}$ with $0<\theta<1$ if it satisfies Assumptions $(A.1)$ and $(A.2)$ and the following weaker assumption than $(A.3)$:

$(A.3)_{\theta}$ If $0<\varepsilon<\omega$, then there exists a constant $C(\varepsilon)>0$ such that the resolvent $(\mathfrak{A} - \lambda)^{-1}$ satisfies the estimate:

$$
\| (\mathfrak{A} - \lambda)^{-1} \| \leq \frac{C(\varepsilon)}{(1 + |\lambda|)^{\theta}}, \quad \lambda \in \Sigma(\varepsilon).
$$

By Theorem 5.3 of Taira [3], we know that the operator $\mathfrak{A}$ which satisfies Assumption $(AS)_{\theta}$ with $0<\theta<1$ generates an analytic semigroup $e^{t\mathfrak{A}}$ such that

$$
\| e^{t\mathfrak{A}} \| \leq \frac{M_{\theta}}{|z|^{1-\theta}}, \quad z \in \Delta(\omega).
$$

Thus, such an analytic semigroup as $e^{t\mathfrak{A}}$ may be called an analytic semigroup with weak singularity. We remark that Assumption $(A.3)_{1}$ is nothing but Assumption $(A.3)$.

A concrete example of $\mathfrak{A}$ which satisfies Assumption $(AS)_{\theta}$ is given by Taira [3]. Furthermore, Taira [3] has demonstrated that the operator $\mathfrak{A}$ generates an analytic semigroup $e^{t\mathfrak{A}}$ which does not necessarily have the following property:

$$
\lim_{t \to 0^+} e^{t\mathfrak{A}} u_0 = u_0 \quad \text{for all } u_0 \in X.
$$

Here $\Delta(\omega) = \{ \lambda \in C : |\arg \lambda| < \omega \}$. More precisely, using fractional powers of the operator $\mathfrak{A}$, Taira [3] has proved that if Assumption $(AS)_{\theta}$ is satisfied, then the operator $\mathfrak{A}$ generates an analytic semigroup $e^{t\mathfrak{A}}$ which has the property

$$
\lim_{t \to 0^+} e^{t\mathfrak{A}} u_0 = u_0
$$

for all $u_0 \in \mathcal{D}(\mathfrak{A})$ with $1-\theta<\alpha<1$. Here if the operator $\mathfrak{A}$ satisfies Assumptions $(A.1)$, $(A.2)$ and $(A.3)_{\theta}$, we can define the fractional powers $(-\mathfrak{A})^{-\alpha}$ of $\mathfrak{A}$ for $1-\theta<\alpha<1$ by

$$
(-\mathfrak{A})^{-\alpha} = \frac{\sin \alpha \pi}{\pi} \int_{0}^{\infty} t^{-\alpha} (t-\mathfrak{A})^{-1} dt,
$$

and also define the fractional powers $(-\mathfrak{A})^{\alpha}$ by

$$
(-\mathfrak{A})^{\alpha} = \text{the inverse of } (-\mathfrak{A})^{-\alpha}.
$$

By the definition of $(-\mathfrak{A})^{\alpha}$, we have the following:

$$
\mathcal{D}(\mathfrak{A}) \subset \mathcal{D}(\mathfrak{A})^{\alpha} \subset X, \quad 1-\theta<\alpha<\theta,
$$

$$
\mathcal{D}(\mathfrak{A})^{\alpha} = X.
$$
The following theorem is due to Taira [3] (cf. [3], Theorem 8.2). In the case $\theta=1$, the theorem coincides with Theorem 1.1.

**Theorem 1.3.** Assume that the operator $\mathfrak{A}$ satisfies Assumption $(AS)_{\theta}$ with $1/2<\theta<1$. If $f \in C^\gamma([0, T]; X)$ with $1-\theta<\gamma\leq 1$, then, for any $u_0 \in \mathcal{D}((-\mathfrak{A})^{\alpha})$ with $1-\theta<\alpha<\theta$, the problem $(\ast)$ has a unique solution which takes the form of (1.4).

In this paper, using Besov space theory, we prove the following result:

**Theorem 1.4.** Assume that the operator $\mathfrak{A}$ satisfies Assumption $(AS)_{\theta}$ with $1/2<\theta<1$. If $f$ belongs to the Besov space $B^\alpha_{\infty, r}((0, T); X)$, then, for any $u_0 \in \mathcal{D}((-\mathfrak{A})^{\alpha})$ with $1-\theta<\alpha<\theta$, the problem $(\ast)$ has a unique solution which takes the form of (1.4).

**Remark 1.2.** Theorem 1.4 is a generalization of Theorem 1.3 and Theorem 1.2. In fact, the following inclusion holds (cf. Corollary 2.1 and Remark 2.2):

\[ \bigcup_{1-\theta<\gamma<1} C^\gamma([0, T]; X) \subseteq B^\alpha_{\infty, r}((0, T); X). \]

**Example 1.2.** The following function $f$ belongs to the space $B^\alpha_{\infty, r}((0, T); \mathbb{R})$, but does not belong to the spaces $C^\gamma([0, T]; \mathbb{R})$ for any $1-\theta<\gamma\leq 1$.

\[ f(t) = \begin{cases} \frac{t^{1-\theta}}{\log t} & \text{if } 0<t\leq T, \\ 0 & \text{if } t=0. \end{cases} \]

The rest of this paper is organized as follows:

In Section 2 we state the basic definition and properties of Besov spaces that will be used in the sequel.

In Section 3 we present a brief description of the analytic semigroups with weak singularity generated by the operator $\mathfrak{A}$ which satisfies Assumption $(AS)_{\theta}$ with $0<\theta<1$.

Section 4 is devoted to the proof of our main Theorem 1.4 by following the argument in the proof of Theorem B of Muramatu [1].

2. Besov spaces

This section is devoted to a description of the definition and properties of Besov spaces (for the details, see Muramatu [1]). We define Besov spaces on an open set $\Omega$ in $\mathbb{R}^N$, but, in this paper, only use the case when $\Omega$ is an open interval $I(N=1)$. 
Let $\mathcal{Q}$ be an open set in $\mathbb{R}^N$, $X$ a Banach space with norm $\| \cdot \|$, $1 \leq p \leq \infty$ and $m$ a non-negative integer. For an $X$-valued function $f$ on $\mathcal{Q}$, we define

$$
\| f \|_{L^p(\mathcal{Q}; X)} = \begin{cases} 
\left( \int_{\mathcal{Q}} |f(x)|^p \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty, \\
\text{ess sup} \|f(x)\| & \text{if } p = \infty,
\end{cases}
$$

$$
\| f \|_{L^\infty(\mathcal{Q}; X)} = \begin{cases} 
\left( \int_{\mathcal{Q}} \frac{|f(x)|^p}{|x|^N} \, dx \right)^{1/p} & \text{if } 1 \leq p < \infty, \\
\text{ess sup} \|f(x)\| & \text{if } p = \infty,
\end{cases}
$$

$$
\| f \|_{H^m(\mathcal{Q}; \mathcal{Q}; X)} = \sum_{|\alpha| \leq m} \| \partial^\alpha f \|_{L^p(\mathcal{Q}; X)}.
$$

Here all the derivatives $\partial^\alpha f$ are taken in the sense of distributions. If $X = \mathbb{R}$, we simply write $\| f \|_{L^p(\mathcal{Q}; \mathcal{Q}; X)}$, $\| f \|_{L^\infty(\mathcal{Q}; \mathcal{Q}; X)}$ and $\| f \|_{H^m(\mathcal{Q}; \mathcal{Q}; X)}$ as $\| f \|_{L^p}$, $\| f \|_{L^\infty}$ and $\| f \|_{H^m}$ respectively.

We introduce function spaces as follows:

$L^p(\mathcal{Q}; X)$ = the space of $X$-valued functions such that $\| f \|_{L^p(\mathcal{Q}; X)}$ is finite.

$L^\infty(\mathcal{Q}; X)$ = the space of $X$-valued functions such that $\| f \|_{L^\infty(\mathcal{Q}; X)}$ is finite.

$H^m(\mathcal{Q}; X)$ = the space of functions $f \in L^p(\mathcal{Q}; X)$ whose derivatives $\partial^\alpha f$, $|\alpha| \leq m$, in the sense of distributions, belong to $L^p(\mathcal{Q}; X)$.

The spaces $L^p(\mathcal{Q}; X)$ and $H^m(\mathcal{Q}; X)$ are Banach spaces with the norms $\| \cdot \|_{L^p(\mathcal{Q}; X)}$ and $\| \cdot \|_{H^m(\mathcal{Q}; X)}$, respectively.

**Definition of Besov spaces.** Let $X$ be a Banach space with norm $\| \cdot \|$, $\mathcal{Q}$ an open set in $\mathbb{R}^N$, $1 \leq p, q \leq \infty$ and $\sigma$ a real number such that $\sigma = m + \theta$ with an integer $m$ and $0 < \theta \leq 1$.

(a) The case $m \geq 0$ and $0 < \theta < 1$: The Besov space $B^\sigma_{p,q}(\mathcal{Q}; X)$ is the set of all functions $f \in H^m(\mathcal{Q}; X)$ such that the seminorm

$$
|f|_{B^\sigma_{p,q}(\mathcal{Q}; X)} = \sum_{|\alpha| \leq m} \left( \int_{\mathcal{Q}} \| \partial^\alpha f(x+y) - \partial^\alpha f(x) \|_{L^p(\mathcal{Q}; X)} \, dy \right)^{\frac{1}{q}}
$$

is finite. Here $\mathcal{Q}_{k,v} = \left\{ \frac{k}{v} \mathcal{Q} - jy : j \in \mathbb{Z} \right\}$. $\mathcal{Q} - jy = \{z - jy : z \in \mathcal{Q}\}$.

(b) The case $m \geq 0$ and $\theta = 1$: The Besov space $B^\sigma_{p,q}(\mathcal{Q}; X)$ consists of all functions $f \in H^m(\mathcal{Q}; X)$ such that the seminorm

$$
|f|_{B^\sigma_{p,q}(\mathcal{Q}; X)} = \sum_{|\alpha| \leq m} \left( \int_{\mathcal{Q}} \| \partial^\alpha f(x+y) - 2\partial^\alpha f(x+y) + \partial^\alpha f(x) \|_{L^p(\mathcal{Q}; X)} \, dy \right)^{\frac{1}{q}}
$$
The space $B_{p,q}^s(\Omega; X)$ is a Banach space with the norm

$$\|f\|_{B_{p,q}^s(\Omega; X)} = \|f\|_{H^m_p(\Omega; X)} + \|f\|_{B_{p,q}^s(\Omega; X)}.$$  

(c) The case $m<0$: The Besov space $B_{p,q}^s(\Omega; X)$ is the set of all distributions $f$ of the form

$$f = \sum_{\alpha \subset \mathbb{Z}^n} \partial^\alpha f_\alpha, \quad f_\alpha \in B_{p,q}^s(\Omega; X).$$

The space $B_{p,q}^s(\Omega; X)$ is a Banach space with the norm

$$\|f\|_{B_{p,q}^s(\Omega; X)} = \inf \sum_{\alpha \subset \mathbb{Z}^n} \|f_\alpha\|_{B_{p,q}^s(\Omega; X)},$$

where the infimum is taken over all expressions of the form (2.1).

In the rest of this section we describe a characterization theorem of Besov spaces. In the following we denote the interval $(0, T)$ by $I$.

We introduce two function spaces.

(i) $\mathcal{K}_d(I)$ is the set of all functions $\phi \in C^\infty(\mathbb{R}^n)$ which satisfy the following conditions:

(2.2) For any $t \in \mathbb{R}$, there exists a compact set $K_t$ in $\mathbb{R}$ such that $K_t$ contains the support of $\phi(t, \cdot)$.

(2.3) For any compact set $K$ in $I$, there is a compact set $K_t \subset I$ such that $\text{supp} \phi(t, (t-\cdot)/\tau) \subset K_t$ for $t \in K$ and $0 < \tau \leq 1$.

(ii) $\mathcal{K}_m(I)$ is the set of $m$-th derivatives $\partial^m \phi(t, s)$ of the functions in $\mathcal{K}_d(I)$.

Let $\phi_0$ be a function in $C^\infty(\mathbb{R})$ which satisfies the conditions:

$$\text{supp} \phi_0 \subset I, \int_\mathbb{R} \phi_0(t) dt = 1.$$  

If $0 < c \leq 1$, we define $\phi, e_m, e_m^*$ as follows:

$$\phi(t, s) = \frac{m}{m!} s^m \phi_0(t-s),$$

$$e_m(t, s) = \sum_{k=0}^{m!} \frac{1}{k!} s^k \phi_0(t-s), \quad m=1, 2, \ldots,$$

$$e_m^*(t, s) = 2e_m(t, s) - \int e_m(t, r)e_m(t, s-r) dr, \quad m=1, 2, \ldots.$$  

Then we have the following results:

**Lemma 2.1.** The functions $\phi, e_m$ and $e_m^*$ introduced above belong to the space $\mathcal{K}_d(I)$. Further $\phi, e_m$ and $e_m^*$ belong to the space $\mathcal{K}_d(J)$ for any open interval $J \supset I$.  

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Lemma 2.2 (Integral representation of distributions). Let $0 < c \leq 1$ and $m = l + h$ where $l$ and $h$ are non-negative integers. Let $\phi$, $e^s_m$ be the functions as above. If $f$ is an $X$-valued distribution on $I$, then it can be represented as follows:

$$f(t) = \int_0^t \left( \frac{1}{r^l} \phi_{h,k}(t, \frac{t-s}{r}) u_{l}(\tau, s) \right) \frac{d\tau}{r}$$

$$+ \sum_{j=0}^{h} \int_0^t \left( \frac{1}{r^l} \phi_{h,m+e}(t, \frac{t-s}{r}) u_{j}(\tau, s) \right) \frac{d\tau}{r}$$

$$+ \frac{1}{c} \left( \langle e^s_m(t, \frac{t-s}{c}), f(s) \rangle \right)$$

where $\langle , \rangle_s$ denotes the pairing of $\mathcal{D}(R) \times \mathcal{D}'(R; X)$ and

$$\phi_{h,k}(t, s) = \partial^{h} \partial^{l} \phi(t, s),$$

$$u_{l}(\tau, t) = \left( \frac{1}{r^l} \sum_{k=0}^{l} \right) \frac{1}{r^l} \phi_{h,l-m+e}(t, \frac{t-s}{r}) f(s) \frac{d\tau}{r},$$

$$u_{j}(\tau, t) = \left( \frac{1}{r^l} \sum_{k=0}^{l} \right) \frac{1}{r^l} \phi_{h,l-m+e}(t, \frac{t-s}{r}) f(s) \frac{d\tau}{r}.$$

Theorem 2.1 (Characterization of Besov spaces). Let $1 \leq p, q \leq \infty$, $\sigma \in R$ and $m$ a non-negative integer such that $m > \sigma$, and $0 < c \leq 1$. An $X$-valued distribution $f$ on $I$ belongs to the space $B^{\sigma}_{p,q}(I; X)$ if and only if the following conditions are satisfied:

$$\langle \phi(t, \frac{t-s}{c}), f(s) \rangle_s \in L^p(I; X) \quad \text{for any } \phi \in \mathcal{J}(I),$$

$$\tau^{-\sigma} \langle \phi(t, \frac{t-s}{c}), f(s) \rangle_s \in L^q((0, c); L^p(I; X)) \quad \text{for any } \phi \in \mathcal{J}(I).$$

Remark 2.1. (A) Let $m$, $h$ and $l$ be integers such that $-h < \sigma < l$, $m = l + h$. Set

$$\phi_{k}(t, s) = \partial^{k} \phi^{*}_{m}(t, s), \quad k = 0, \ldots, l.$$  

Then $f \in B^{\sigma}_{p,q}(I; X)$ if the following conditions are satisfied:

$$\tau^{-\sigma} \left( \frac{1}{r^l} \phi_{h,m+e}(s, \frac{s-r}{r}), f(r) \right) \in L^q((0, c); L^p(I; X))$$

for $k = 0, \ldots, l,$

$$\tau^{-\sigma} \left( \frac{1}{r^l} \phi_{h-l-e}(s, \frac{s-r}{r}), f(r) \right) \in L^q((0, c); L^p(I; X))$$

for $j = 0, \ldots, h,$

$$\langle \phi_{k}(t, \frac{t-s}{c}), f(s) \rangle_s \in L^p(I; X) \quad \text{for } k = 0, \ldots, l.$$
Furthermore, the norm of $f$ in $B_{p,q}(I, X)$ is equivalent with the sum of the corresponding norms of the above functions.

**Corollary 2.1.** We have the following inclusions:

(2.6) \[ B_{p,q_1}^1(I; X) \subset B_{p,q_2}^2(I; X) \] for $1 \leq q_1, q_2 \leq \infty$, $\sigma_1 < \sigma_2$.

(2.7) \[ B_{p,q_1}(I; X) \subset B_{p,q_2}(I; X) \] for $1 \leq q_1, q_2 \leq \infty$, $\sigma \in \mathbb{R}$.

(2.8) \[ B_{p,1}(I; X) \subset L^\omega(I; X). \]

(2.9) \[ B_{p,1}(I; X) \subset C^m([0, T]; X) \] if $m$ is a non-negative integer.

(2.10) \[ B_{p,\infty}(I; X) = C^\theta([0, T]; X) \] for $0 < \theta < 1$.

Further the inclusions (2.6), (2.7) and (2.8) are continuous.

**Remark 2.2.** From the inclusions (2.6) and (2.10), it follows that

\[ C([0, T]; X) \subset B_{\infty, \infty}^1(I; X) \] for $1 - \theta < \gamma \leq 1$.

**Theorem 2.2.** Let $1 \leq p, q \leq \infty$ and $\sigma \in \mathbb{R}$. If $g \in B_{p,q}^\sigma(I; X)$, then there exists a sequence $\{g_n\}_{n=1}^\infty$ such that

\[ g_n \in B_{p,q}^\sigma(I; X) \cap C^\gamma([0, T]; X), \]

\[ g_n \longrightarrow g \text{ in } B_{p,q}^\sigma(I; X) \cap L^\gamma(I; X) \text{ as } n \longrightarrow \infty. \]

3. Analytic semigroups with weak singularity

In this section we briefly state properties of analytic semigroups with weak singularity which will be used in the following section.

**Theorem 3.1.** Assume that a linear operator $A$ satisfies conditions (A.1), (A.2) and (A.3) for $0 < \theta < 1$. Then we have the following:

(3.1) The operator $A$ generates a semigroup $e^{tA}$ on $X$ which is analytic in the sector $\mathcal{D}(\omega)$.

(3.2) The operators $A^m e^{tA}$ and $(d^m/dz^m)e^{tA}$ are bounded operators on $X$ for any non-negative integer $m$ and $z \in \mathcal{D}(\omega)$, and satisfy the following relation and estimate.

\[ \frac{d^m}{dz^m} e^{tA} = A^m e^{tA}, \quad z \in \mathcal{D}(\omega). \]

\[ \|A^m e^{tA}\| \leq M_m |z|^\theta - m, \quad z \in \mathcal{D}(\omega). \]

Here the letter $M_m$ is a constant depending on $m$ and $\omega$. 
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Proof. We can define the semigroup $e^{zt}$ for any $0 < t < \omega$ as follows:

$$e^{zt} = -\frac{1}{2\pi i} \int_{\Gamma} e^{zt}(\mathbb{U} - \lambda)^{-1} d\lambda.$$ 

Here $\Gamma$ is a path in the set $\Sigma(\varepsilon)$ such that $\Gamma = -\Gamma_1 + \Gamma_2$ where

$$\Gamma_1 = \{re^{i(s/2 + \varepsilon)}; \ 0 \leq r < \infty\}.$$ 

$$\Gamma_2 = \{re^{i(s/2 + \varepsilon)}; 0 \leq r < \infty\}.$$ 

Then, according to Theorem 5.3 of Taira [3], we have the conditions (3.1) and (3.2) for $m=0,1$. In the following we show the condition (3.2) for general $m \geq 2$.

First we show the following formula:

$$d^m e^{zt} = -\frac{1}{2\pi i} \int_{\Gamma} \lambda^m e^{zt}(\mathbb{U} - \lambda)^{-1} d\lambda, \quad m \geq 1, \ z \in \mathcal{A}(\varepsilon).$$

For $z \in \mathcal{A}(\varepsilon)$ and $\lambda \in \Gamma_1$, we set

$$z = |z| e^{i\alpha}, \quad 0 \leq \alpha < \varepsilon,$$

$$\lambda = re^{i(s/2 + \varepsilon)}, \quad 0 \leq r < \infty.$$ 

Then we have

$$|e^{zt}| = |e^{iz} r (\cos(\alpha - \pi/2 - \varepsilon) + i \sin(\alpha - \pi/2 - \varepsilon))|$$

$$= e^{-iz} r \cdot \sin(\varepsilon - \alpha).$$ 

Hence it follows that for $z \in \mathcal{A}(\varepsilon)$ and $\lambda \in \Gamma_1$

$$\|\lambda^m e^{zt}(\mathbb{U} - \lambda)^{-1}\| \leq r^m e^{-iz} r \cdot \sin(\varepsilon - \alpha) \frac{C(\varepsilon)}{(1+r)^\theta}.$$ 

Similarly, for $z \in \mathcal{A}(\varepsilon)$ and $\lambda \in \Gamma_2$, we let

$$z = |z| e^{i\alpha}, \quad 0 \leq \alpha < \varepsilon,$$

$$\lambda = r e^{i(\pi/2 + \varepsilon)}, \quad 0 \leq r < \infty.$$ 

Then we have

$$|e^{zt}| = |e^{iz} r (\cos(\alpha + \pi/2 + \varepsilon) + i \sin(\alpha + \pi/2 + \varepsilon))|$$

$$= e^{-iz} r \cdot \sin(\varepsilon + \alpha).$$ 

Hence it follows that for $z \in \mathcal{A}(\varepsilon)$ and $\lambda \in \Gamma_2$

$$\|\lambda^m e^{zt}(\mathbb{U} - \lambda)^{-1}\| \leq r^m e^{-iz} r \cdot \sin(\varepsilon + \alpha) \frac{C(\varepsilon)}{(1+r)^\theta}.$$ 

If $z \in \mathcal{A}(\varepsilon)$, we have by the estimates (3.4) and (3.5)
Let $$p = \frac{\kappa}{\sigma}$$.

By interchanging the integral order, we have

$$
\int_0^\infty \frac{r^m}{(1+r)^\theta} \left( e^{-\left| z \right| r \cdot \sin(\sigma - \Omega)} + e^{-\left| z \right| r \cdot \sin(\sigma + \Omega)} \right) dr
\leq |z|^\theta \int_0^\infty \rho^{m-\theta} \left( e^{-\rho \cdot \sin(\sigma - \Omega)} + e^{-\rho \cdot \sin(\sigma + \Omega)} \right) d\rho.
$$

Since $$\sin(\sigma - \Omega) > 0$$ and $$\sin(\sigma + \Omega) > 0$$, we obtain that

$$
\int_0^\infty \rho^{m-\theta} \left( e^{-\rho \cdot \sin(\sigma - \Omega)} + e^{-\rho \cdot \sin(\sigma + \Omega)} \right) d\rho < \infty.
$$

This implies that the operator $$\int_r \lambda^m e^{i\lambda} (\mathcal{A} - \lambda)^{-1} d\lambda$$ is bounded on $$X$$ for $$z \in \mathcal{A}$$.

Further we have

$$
(3.6) \quad \frac{d^m}{dz^m} (e^{i\lambda}) = -\frac{1}{2\pi i} \int_r \lambda^m e^{i\lambda} (\mathcal{A} - \lambda)^{-1} d\lambda, \quad z \in \mathcal{A}.
$$

and

$$
(3.7) \quad \left\| \frac{d^m}{dz^m} (e^{i\lambda}) \right\| \leq C |z|^\theta, \quad z \in \mathcal{A}.
$$

Here the letter $$C$$ is a constant depending on $$m$$ and $$\omega$$.

Next, using induction on $$m$$, we show that

$$
(3.8) \quad \frac{d^m}{dz^m} (e^{i\lambda}) = \mathcal{A}^m e^{i\lambda}, \quad z \in \mathcal{A}.
$$

By Theorem 5.3 of [3], we have the equality (3.8) for $$m=1$$. We assume that the equality (3.8) holds for $$m \geq 1$$. Then it follows from (3.6) that

$$
\frac{d^{m+1}}{dz^{m+1}} (e^{i\lambda}) = -\frac{1}{2\pi i} \int_r \lambda^{m+1} e^{i\lambda} (\mathcal{A} - \lambda)^{-1} d\lambda
= -\frac{1}{2\pi i} \int_r \lambda^m e^{i\lambda} (\mathcal{A} - \lambda)^{-1} d\lambda.
$$

By Remarks that $$\mathcal{A} (\mathcal{A} - \lambda)^{-1} = 1 + \lambda (\mathcal{A} - \lambda)^{-1}$$, it follows that
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\[
\frac{d^{m+1}}{dz^{m+1}}(e^{z\lambda}) = -\frac{1}{2\pi i} \int_{\Gamma} \lambda^m \lambda^{m+i} \psi(\lambda) \lambda^{-i} d\lambda - \frac{1}{2\pi i} \int_{\Gamma} \lambda^m e^{z\lambda} d\lambda.
\]

The closedness of \( \mathfrak{M} \) tells us that

\[
-\frac{1}{2\pi i} \int_{\Gamma} \lambda^m e^{z\lambda} (\mathfrak{M} - \lambda)^{-1} d\lambda = \mathfrak{M} \left( -\frac{1}{2\pi i} \int_{\Gamma} \lambda^m e^{z\lambda} (\mathfrak{M} - \lambda)^{-1} d\lambda \right) = \mathfrak{M} \frac{d^m}{dz^m}(e^{z\lambda}) = e^{z\lambda}.
\]

Note that

\[
\int_{\Gamma} \lambda^m e^{z\lambda} d\lambda = 0 \quad \text{for} \quad m \geq 1.
\]

Hence it follows that

\[
\frac{d^{m+1}}{dz^{m+1}}(e^{z\lambda}) = \mathfrak{M}^{m+1} e^{z\lambda}, \quad z \in \Lambda(z).
\]

The statements (3.7) and (3.8) imply that

\[
\|\mathfrak{M}^{m} e^{z\lambda}\| \leq M_m |z|^\delta - m, \quad z \in \Lambda(z), \quad m \geq 1
\]

with a constant \( M_m > 0 \) depending on \( m \) and \( \omega \).

The proof of Theorem 3.1 is complete.

4. Proof of Theorem 1.4

In this section we prove Theorem 1.4 by following the proof of Theorem B of Muramatu [1]. If there exists a solution \( u \) of the problem (*) for \( u_0 \in \mathcal{D}(-\mathfrak{M}) \) with \( 1 - \theta < \alpha < \theta \), we can uniquely write the solution in the following form:

\[
u(t) = e^{t\mathfrak{M} u_0} + \int_0^t e^{(t-s)\mathfrak{M}} f(s) ds, \quad 0 \leq t \leq T.
\]

First we verify that \( u \) satisfies the condition (1.1). Theorem 1.3 tells us that

\[
e^{t\mathfrak{M} u_0} \in C([0, T]; X) \cap C^\prime([0, T]; X).
\]

So, it suffices to show that

\[
F(t) = \int_0^t e^{(t-s)\mathfrak{M}} f(s) ds \in C([0, T]; X) \cap C^\prime([0, T]; X).
\]

Since it is clear that \( f \in B_{\alpha, \lambda}^2(I; X) \) implies \( F \in C([0, T]; X) \), we have only to verify that \( F \in C^\prime([0, T]; X) \). By Corollary 2.1, we have

\[
B_{\alpha, \lambda}^2([\varepsilon, T]; X) \subset C^\prime([\varepsilon, T]; X) \quad \text{for any} \quad 0 < \varepsilon < T.
\]
Therefore, if \( F \in B_{\phi, \kappa}(\varepsilon, T); X \) for any \( 0 < \varepsilon < T \), it follows that \( F \in C^{1}(\{(0, T); X\}) \).

Let \( I_\varepsilon \) be the open interval \((\varepsilon, T)\). In the following we simply write \( \int_r \) as \( \int \). In order to verify that \( F \in B_{\phi, \kappa}(I_\varepsilon; X) \), we apply Theorem 2.1 with \( I = I_\varepsilon \) and \( m = 4 \). That is, we show that the function \( F \) satisfies the following conditions for \( 0 < c \leq 1 \):

\[
\int \phi(\cdot, \frac{t-s}{c})F(s)ds \in L^\infty(I_\varepsilon; X) \quad \text{for} \quad \phi \in \mathcal{K}_0(I_\varepsilon),
\]

for \( \phi \in \mathcal{K}_4(I_\varepsilon) \cap \Phi(I) \) (cf. Lemma 2.1 and Remark 2.1(A)).

First, we show that \( F \) satisfies the condition (4.1). Since \( \phi \) satisfies the condition (2.3), we have

\[
\int \phi(t, \frac{t-s}{c})F(s)ds = \int_0^\tau \phi(t, \frac{t-s}{c})(\int_0^s e^{(t-r)\varepsilon}f(r)dr)ds.
\]

By interchanging the integral order of \( s \) and \( r \) and by integration by substitution with \( s-r=s' \), the right hand of (4.3) becomes

\[
\int \phi(t, \frac{t-s}{c})(\int_0^s e^{(t-r)\varepsilon}f(r)dr)ds \quad \text{for} \quad \phi \in \mathcal{K}_0(I_\varepsilon),
\]

Again, by interchanging the integral order of \( s \) and \( r \), it follows that

\[
\int \phi(t, \frac{t-s'-r}{c})e^{hr}ds'f(r)dr = \int_0^\tau \phi(t, \frac{t-s'-r}{c})e^{hr}ds'f(r)dr,
\]

Hence we have

\[
\int \phi(t, \frac{t-s}{c})F(s)ds = \int_0^\tau \phi(t, \frac{t-s-r}{c})f(r)dr.
\]

Now we cite a lemma which we use in order to estimate the right term (cf. Muramatu [1], Lemma 3).

**Lemma 4.1.** Suppose that \( 1 \leq p \leq \infty \), \( 0 < \tau \leq 1 \), \( f \in L^1(I; X) \) and \( \phi \in \mathcal{K}_0(I_\varepsilon) \).
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Then there exists a constant $M_i > 0$ such that

$$\left\| \frac{1}{\tau} \int_0^\tau \phi(t, \cdot, \frac{t-s-r}{\tau}) f(r) dr \right\|_{L^p(I;X)} \leq M_i \tau^{-1+1/p} \| f \|_{L^1(I;X)}$$

for $0 \leq s \leq T$.

By making use of Lemma 4.1 and the estimate:

$$\| e^{sR} \| \leq M s^{\theta-1}, \quad s > 0,$$

it follows that

$(4.4)$

$$\left\| \int_0^\tau e^{sR} ds \right\|_{L^p(I;X)} \leq C \| f \|_{L^1(I;X)}.$$

Here and in the following the letter $C$ is a general constant independent of $f$.

Next we show that $F$ satisfies the condition (4.2). Let $0 < \tau \leq c$, $\phi \in \mathcal{K}_d(I) \cap \mathcal{K}_e(I)$ and

$$U(\tau, t) = \int_0^\tau \phi(t, \frac{t-s}{\tau}) F(s) ds.$$

We divide the integral with respect to $s$ into two parts as follows:

$$U(\tau, t) = \int_0^\tau \phi(t, \frac{t-s}{\tau}) F(s) ds$$

$$= \int_0^\tau e^{sR} ds \int_0^\tau \phi(t, \frac{t-s-r}{\tau}) f(r) dr$$

$$= \left( \int_0^\tau e^{sR} ds \right) \int_0^\tau \phi(t, \frac{t-s-r}{\tau}) f(r) dr$$

$$= U_1(\tau, t) + U_2(\tau, t).$$

We cite a lemma which is used in order to estimate $U_1$ and $U_2$ (cf. Muramatu [1], Lemma 4).

**Lemma 4.2.** Assume that $1 \leq p \leq \infty$, $0 < \tau \leq c$, $f \in L^p(I;X)$ and $\phi \in \mathcal{K}_d(I)$. Then there exists a constant $M_2 > 0$ such that

$$\left\| \frac{1}{\tau} \int_0^\tau \phi(t, \cdot, \frac{t-s-r}{\tau}) f(r) dr \right\|_{L^p(I;X)}$$

$$\leq \sum_{j=0}^\infty \int_0^\tau \| u_j(\tau, \cdot) \|_{L^p(I;X)} + M_2 s^2 \tau^{-1+1/p} \| f \|_{L^1(I;X)}.$$
for $0 \leq s \leq \varepsilon$. Here

$$u_j(t, s) = \int_0^s \frac{1}{r} \phi_j, 0(t, \frac{t-r}{s}) f(r) dr,$$

$$\phi_j, k(t, s) = \partial_i \partial_j \phi_j(t, s).$$

Now we may assume that $0 < c \leq \varepsilon$. Lemma 4.2 gives that

\begin{equation}
\|U_\varepsilon(t, \cdot)\|_{L^\infty(I_x; x)} \leq \int_0^T \|e^{s\varepsilon}\|_{H^1} ds \|\int_0^{T-s} \frac{1}{r} \phi(t, \frac{t-s-r}{r}) f(r) dr\|_{L^\infty(I_x; x)}
\end{equation}

\begin{equation}
\leq \int_0^T M \varepsilon^{-1} \left( \sum_{j=0}^{5} \varepsilon^j \|u_j(t, \cdot)\|_{L^\infty(I_x; x)} + M \varepsilon^{-1} \|f\|_{L^1(I; x)} \right) ds
\end{equation}

\begin{equation}
\leq C \varepsilon^\theta \left( \sum_{j=0}^{5} \varepsilon^j \|u_j(t, \cdot)\|_{L^\infty(I_x; x)} + \varepsilon^\tau \|f\|_{L^1(I; x)} \right).
\end{equation}

Since $\phi \in \mathcal{K}(I_x) \cap \mathcal{K}(I)$, we can represent $\phi$ as $\phi(t, s) = \partial_1 \phi_j(t, s)$ where $\phi \in \mathcal{K}(I_x) \cap \mathcal{K}(I)$. By interchanging the integral order, we have

\begin{equation}
U_\varepsilon(t, s) = \int_0^s e^{s\varepsilon} ds \int_0^{T-s} \frac{1}{r} \phi(t, \frac{t-s-r}{r}) f(r) dr
\end{equation}

\begin{equation}
= \int_0^{T-s} \left( \int_0^s \frac{1}{r} \phi(t, \frac{t-s-r}{r}) e^{s\varepsilon} ds \right) f(r) dr
\end{equation}

\begin{equation}
= \int_0^{T-s} \left( \int_0^s \frac{1}{r} \phi(t, \frac{t-s-r}{r}) e^{s\varepsilon} ds \right) f(r) dr
\end{equation}

where $\phi_{j, k}(t, s) = \partial_1 \partial_j \phi_j(t, s)$. By integration by parts, it follows that

\begin{equation}
\int_0^{T-s} \frac{1}{r} \phi(t, \frac{t-s-r}{r}) e^{s\varepsilon} ds = \sum_{k=0}^{5} \phi_{0, k}(t, \frac{t-s-r}{r}) (\varepsilon \delta^k e^{s\varepsilon})
\end{equation}

\begin{equation}
+ \int_0^{T-s} \varepsilon^k \phi(t, \frac{t-s-r}{r}) \delta^k \varepsilon^{s\varepsilon} ds.
\end{equation}

Hence we obtain that

\begin{equation}
U_\varepsilon(t, s) = \int_0^{T-s} \left( \sum_{k=0}^{5} \phi_{0, k}(t, \frac{t-s-r}{r}) (\varepsilon \delta^k e^{s\varepsilon})
\end{equation}

\begin{equation}
+ \int_0^{T-s} \varepsilon^k \phi(t, \frac{t-s-r}{r}) \delta^k \varepsilon^{s\varepsilon} ds \right) f(r) dr.
\end{equation}

We write the first and second terms of (4.6) as

\begin{equation}
V_k(t, s) = \varepsilon^k \delta^k e^{s\varepsilon} \int_0^{T-s} \frac{1}{r} \phi_{0, k}(t, \frac{t-s-r}{r}) f(r) dr, \quad k = 0, 1, 2, 3,
\end{equation}

\begin{equation}
V_j(t, s) = \int_0^{T-s} \varepsilon^k \phi(t, \frac{t-s-r}{r}) \delta^k \varepsilon^{s\varepsilon} ds f(r) dr.
\end{equation}
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respectively. That is, \( U_2(\tau, t) \) is written as

\[
U_2(\tau, t) = \sum_{k=0}^{3} V_k(\tau, t) + V_3(\tau, t).
\]

By noting that

\[
\| A^m e^{itH} \| \leq M_m t^{\theta - 1 - m}, \quad t > 0
\]

with a constant \( M_m > 0 \) for \( m = 0, 1, 2, \ldots \), Lemma 4.2 gives that

\[
(4.7) \quad \| V_k(\tau, \cdot) \|_{L^\infty(I; x)} \leq C \tau^k \left( \sum_{j=0}^{3} \| \tau^j v_{jk}(\tau, \cdot) \|_{L^\infty(I; x)} + \| f \|_{L^1(I; x)} \right)
\]

for \( k = 0, 1, 2, 3 \).

Here

\[
v_{jk}(\tau, t) = \int_0^T \frac{1}{\tau} \phi_j(s(t, \frac{t-s}{\tau})) f(r) dr,
\]

for \( j = 0, 1, 2, k = 0, 1, 2, 3 \).

\( V_3(\tau, t) \) is, by interchanging the integral order of \( s \) and \( r \), written by the following form:

\[
V_3(\tau, t) = \tau^{s} \left[ \sum_{j=0}^{3} \frac{1}{\tau} \int_0^T \phi_j(s(t, \frac{t-s}{\tau})) f(r) dr \right].
\]

Lemma 4.1 and Lemma 4.2 give that

\[
(4.8) \quad \| V_k(\tau, \cdot) \|_{L^\infty(I; x)} \leq C \tau^k \left( \sum_{j=0}^{3} \frac{1}{\tau} \int_0^T \phi_j(s(t, \frac{t-s}{\tau})) f(r) dr \right)
\]

for \( k = 0, 1, 2, 3 \).

Hence we have

\[
(4.9) \quad \| U_3(\tau, \cdot) \|_{L^\infty(I; x)} \leq \sum_{k=0}^{3} \| V_k(\tau, \cdot) \|_{L^\infty(I; x)}
\]

\[
\leq C \tau^\theta \left( \sum_{k=0}^{3} \| V_k(\tau, \cdot) \|_{L^\infty(I; x)} + (\tau^3 + \tau^{3-\theta}) \| f \|_{L^1(I; x)} \right).
\]

By the estimates (4.5) and (4.9), we have
\[ \| \tau^{-1} U(t, \cdot) \|_{L^1((0, \tau); L^\infty(I; X))} \]
\[
= \int_0^\tau \tau^{-1}\| U(t, \cdot) \|_{L^\infty(I; X)} \frac{d\tau}{\tau} 
\leq C \left( \sum_{j=0}^\infty \| \tau^{-(1-\theta)} U_j(t, \cdot) \|_{L^1((0, \tau); L^\infty(I; X))} 
+ \sum_{j=0}^\infty \sum_{k=0}^\infty \| \tau^{-(1-\theta)} T^{j,k}(t, \cdot) \|_{L^1((0, \tau); L^\infty(I; X))} + \| f \|_{L^1(I; X)} \right).
\]

By Remark 2.1 (B), it follows that
\[ \| \tau^{-1} U(t, \cdot) \|_{L^1((0, \tau); L^\infty(I; X))} \leq C(\| f \|_{B_{\Omega}^{-\theta}(t; X)} + \| f \|_{L^1(I; X)}).\]

It has been proved that \( F \) satisfies the condition (4.2).

Now, by making use of Remark 2.1 (B), the estimates (4.6) and (4.11) imply that
\[ \| F \|_{B_{\Omega}^{1-\theta}(t; X)} \leq C(\| f \|_{B_{\Omega}^{-\theta}(t; X)} + \| f \|_{L^1(I; X)}).\]

Now we verify that \( u \), given by the formula
\[ u(t) = e^{t\Omega} u_0 + \int_0^t e^{(t-s)\Omega} f(s) \, ds, \]
satisfies the conditions (1.2) and (1.3). Theorem 1.3 tells us that \( e^{t\Omega} u_0 \) satisfies the conditions (1.2) and (1.3). By virtue of Theorem 2.2, there exists a sequence \( \{ f_n \}_{n=1}^\infty \) such that
\[ f_n \in B_{\Omega}^{-\theta}(t; X) \cap C([0, T]; X), \]
\[ f_n \longrightarrow f \text{ in } B_{\Omega}^{1-\theta}(t; X) \cap L^1(I; X). \]

We let
\[ F_n(t) = \int_0^t e^{(t-s)\Omega} f_n(s) \, ds. \]

Then we have by Theorem 1.3
\[ F_n \in C([0, T]; X), \]
\[ F_n(t) \in \mathcal{D}(\Omega), \quad 0 < t \leq T, \]
\[ \frac{dF_n(t)}{dt} = \Omega F_n(t) + f_n(t), \quad 0 < t \leq T. \]

By applying the inequality (4.12) to \( f - f_n \) and \( F - F_n \), we have
\[ \| F - F_n \|_{B_{\Omega}^{1-\theta}(t; X)} \leq C(\| f - f_n \|_{B_{\Omega}^{-\theta}(t; X)} + \| f - f_n \|_{L^1(I; X)}). \]

Using the statement (4.14), we obtain that
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\[ \| F_n - F \|_{\mathcal{B}_{\infty,1}(t_0; X)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

On the other hand, we have

\[
\begin{align*}
\| \mathcal{U} F_n - \frac{dF}{dt} + f \|_{\mathcal{B}_{\infty,1}(t_0; X)} \\
= \| -f_n + \frac{dF_n}{dt} - \frac{dF}{dt} + f \|_{\mathcal{B}_{\infty,1}(t_0; X)} \\
\leq \| f_n - f \|_{\mathcal{B}_{\infty,1}(t_0; X)} + \left\| \frac{dF_n}{dt} - \frac{dF}{dt} \right\|_{\mathcal{B}_{\infty,1}(t_0; X)}.
\end{align*}
\]

We estimate the two terms of the right. The inclusion (2.6) and the statement (4.14) tell us that

\[ \| f_n - f \|_{\mathcal{B}_{\infty,1}(t_0; X)} \leq C \| f_n - f \|_{\mathcal{B}_{1-\delta,1}(t_0; X)} \leq C \| f_n - f \|_{\mathcal{B}_{1-\delta,1}(t_0; X)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

The definition of Besov spaces and (4.15) give that

\[ \| \frac{dF_n}{dt} - \frac{dF}{dt} \|_{\mathcal{B}_{\infty,1}(t_0; X)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

From (4.16) and (4.17), it follows that

\[ \| \mathcal{U} F_n - \frac{dF}{dt} - f \|_{\mathcal{B}_{\infty,1}(t_0; X)} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

By using the inclusion (2.8), if \( t \in I_\varepsilon \), the statements (4.15) and (4.18) imply that as \( n \rightarrow \infty \)

\[ F_n(t) \rightarrow F(t) \quad \text{in} \quad X, \]

\[ \mathcal{U} F_n(t) \rightarrow \frac{dF}{dt}(t) - f(t) \quad \text{in} \quad X. \]

By virtue of the closedness of \( \mathcal{U} \), it follows that

\[ F(t) \in \mathcal{D}(\mathcal{U}), \quad 0 < t \leq T, \]

\[ \mathcal{U} F(t) = \frac{dF}{dt}(t) - f(t), \quad 0 < t \leq T. \]

The proof of Theorem 1.4 is now complete.

REMARK 4.1. The proof of Theorem 1.4 tells us that for any \( \varepsilon > 0 \)

\[ f \in \mathcal{B}^\sigma_{\infty,q}(\varepsilon, T); X) \implies F \in \mathcal{B}^\sigma_{\infty,q}(\varepsilon, T); X). \]

This implies that the regularity of \( F \) is as maximal as possible. In other words, if \( \sigma > 1 \) and \( 1 \leq q \leq \infty \), it does not necessarily hold that \( F \in \mathcal{B}^\sigma_{\infty,q}(\varepsilon, T); X) \) if
\[ f \in B_{\infty,1}^{{\mathbf{H}}^{-\theta}}((0, \; \tau); \; X). \]

References


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