ON RINGS WITH FINITE SELF-INJECTIVE DIMENSION II

(Dedicated to Professor Goro Azumaya on his 60th birthday)

By

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For a module $M$ over a ring $R$ (with an identity), $\text{pd}(M)$ and $\text{id}(M)$ denote the projective and injective dimension of $M$, respectively. In the previous paper [5] and [6], we called a (left and right) noether ring $R$ $n$-Gorenstein if $\text{id}(R_R) \leq n$ and $\text{id}(R_R) \leq n$ for an $n \geq 0$, and Gorenstein if $R$ is $n$-Gorenstein for some $n$. This note is concerned with two subjects on Gorenstein rings. In §1, we consider the modules of finite projective or injective dimension over a Gorenstein ring and, first, show that the finiteness of projective dimension coincides with one of injective dimension. Then it follows that the highest finite projective (or injective) dimension is $n$ for modules over an $n$-Gorenstein ring and, next, such modules over an artinian Gorenstein ring are investigated. Finally, we present some example to compare with Auslander’s definition of an $n$-Gorenstein ring.

In §2, for a Gorenstein ring $R$, we consider a quasi-Frobenius extension of $R$ and show it also is a Gorenstein ring. Further we generalize [3, Corollary 8 and 8'] to the case of a quasi-Frobenius extension. Also an example concerning with a maximal quotient ring of a Gorenstein ring is presented.

1. Modules of finite projective or injective dimension

We start with the next proposition which states [4, Korollar 1.12] and [7, Corollary 5] more precisely:

**Proposition 1.** For a noether ring $R$,

$$\text{id}(R_R) = \sup \{\text{flat dim } (E); \text{ } R E \text{ is an injective left } R\text{-module} \}.$$

**Proof.** By [2, Chap. VI, Proposition 5.3],

\begin{equation}
\text{Tor}_i^R (A_R, R E) \cong \text{Hom}_R (\text{Ext}_R^i (A_R, R R_R), R E)
\end{equation}

for any finitely generated right $R$-module $A_R$, injective left $R$-module $R E$ and $i > 0$.

First assume $\text{id}(R_R) = n < \infty$, then $\text{Ext}_R^{n+i} (A, R) = 0$ for any finitely generated

Received May 17, 1979. Revised December 11, 1979.
\[ A_R \text{ and so } \text{Tor}^R_{n+1}(A, E)=0 \text{ for any injective } R E. \] Further, for any \( X \), we can represent \( X=\lim \rightarrow A_x \) such that each \( A_x \) is finitely generated and hence
\[ \text{Tor}^R_{n+1}(X, R E)=\lim \rightarrow \text{Tor}^R_{n+1}(A_n, E)=0. \]
Therefore flat \( \text{dim} \) \( (E)\leq n \).
Conversely, if flat \( \text{dim} \) \( (E)\leq n<\infty \) for any injective \( R E \), (*) induces
\[ \text{Hom}_R(\text{Ext}^R_{n+1}(A, R), E)\cong \text{Tor}^R_{n+1}(A, E)=0 \]
for any finitely generated \( A_R \). Now then, by taking \( R E \) as an injective cogenerator, it holds that \( \text{Ext}^R_{n+1}(A, R)=0 \) for any finitely generated \( A_R \) and hence \( \text{id}(R_E)\leq n \).

The following was shown in [5] and [6] under certain assumption on the dominant dimension, but now we can release this assumption and include completely the commutative case.

**Theorem 2.** For an \( n \)-Gorenstein ring \( R \) and an \( R \)-module \( M \), the following are equivalent:

1. \( \text{pd}(M)<\infty \),  
2. \( \text{pd}(M)\leq n \),  
3. \( \text{id}(M)<\infty \),  
4. \( \text{id}(M)\leq n \).

**Proof.** Since the implications (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (4) are proved in [1] and [5], respectively, we prove only (3) \( \Rightarrow \) (2).

Let
\[
0 \rightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \rightarrow \cdots \rightarrow E_m \rightarrow 0
\]
be an injective resolution of \( M \) and \( K_{i-1}=\ker(f_i) \) \( (i=1, \ldots, m) \), then in the exact sequence
\[
0 \rightarrow K_{m-1} \rightarrow E_{m-1} \rightarrow E_m \rightarrow 0,
\]
if \( \text{pd}(E_{m-1}), \text{pd}(E_m)\leq n \), then \( \text{pd}(K_{m-1})\leq n \) by [5, Lemma 4]. For an arbitrary \( i \), in the exact sequence
\[
0 \rightarrow K_{i-1} \rightarrow E_{i-1} \rightarrow K_i \rightarrow 0,
\]
if \( \text{pd}(K_i), \text{pd}(E_{i-1})\leq n \), then \( \text{pd}(K_{i-1})\leq n \) and therefore \( \text{pd}(M)=\text{pd}(K_0)\leq n \) by the induction. Thus, it is enough to show \( \text{pd}(E)\leq n \) for any injective left module \( R E \).

Now, since flat \( \text{dim} \) \( (E)\leq n \) by Proposition 1, let
\[
0 \rightarrow U_n \xrightarrow{f_n} U_{n-1} \xrightarrow{f_{n-1}} \cdots \rightarrow U_0 \rightarrow E \rightarrow 0
\]
be a resolution of \( R E \) by flat modules \( U_i \) \( (i=0, 1, \ldots, n) \) and \( C_{i-1}=\text{cok}(f_i) \) \( (i=1, \ldots, n) \), then \( \text{pd}(U_i)<\infty \) for \( i=0, 1, \ldots, n \) by [7, Proposition 6]. First,
from the exact sequence

$$0 \rightarrow U_n \rightarrow U_{n-1} \rightarrow C_{n-1} \rightarrow 0$$

with $\text{pd}(U_n), \text{pd}(U_{n-1}) < \infty$, it follows that $\text{pd}(C_{n-1}) < \infty$. For an arbitrary $i$, in the exact sequence

$$0 \rightarrow C_{i+1} \rightarrow U_i \rightarrow C_i \rightarrow 0,$$

if $\text{pd}(C_{i+1}), \text{pd}(U_i) < \infty$, then it follows that $\text{pd}(C_i) < \infty$ and hence $\text{pd}(E) = \text{pd}(C_i) < \infty$ by the induction, which is equivalent to $\text{pd}(E) \leq n$ by the implication $(1) \Rightarrow (2)$. From Theorem 2, we are interested in modules $M$ satisfying $\text{pd}(M) = n$ or $\text{id}(M) = n$ over an $n$-Gorenstein ring. Thus we next consider such modules.

For a module $M$, we define $E^i(M)$ for $i \geq 0$ as the $(i+1)$-th term in a minimal injective resolution of $M$ and $E(M) = E^0(M)$, i.e.

$$0 \rightarrow M \rightarrow E^0(M) \rightarrow \cdots \rightarrow E^i(M) \rightarrow \cdots$$

is a minimal injective resolution of $M$. Dually, if $M$ has a minimal projective resolution, we define $P^i(M)$ for $i \geq 0$, similarly.

**Theorem 3.** Let $R$ be an artinian $n$-Gorenstein ring, $0 \rightarrow R \rightarrow E_0 \rightarrow \cdots \rightarrow E_n \rightarrow 0$ a minimal injective resolution for $R$ and $R$ a left $R$-module.

(1) If $\text{id}(M) = n$, then $\text{id}(M) = \text{pd}(E^i(M)) = n$ and, for any direct summand $RE$ of $E^i(M)$, $\text{pd}(E) = n$.

If $\text{pd}(M) = n$, then $\text{id}(P^n(M)) = \text{pd}(M) = n$ and, for any direct summand $RP$ of $P^n(M)$, $\text{id}(P) = n$.

In particular, $\text{id}(P^n(E_n)) = \text{pd}(E_n) = n$ provided $\text{id}(R) = n$.

(2) If $\text{pd}(M) = n$, then $E^nP^n(M)$ is isomorphic to a direct sum of a direct sum of copies of $E_n$.

Especially, $E^nP^n(E_n)$ is isomorphic to a direct summand of $E_n$.

**Proof.** (1) Suppose $\text{id}(M) = n$ and $RE$ an indecomposable summand of $E^i(M)$, then since $E$ is of the form $E(S)$ for some simple module $_RS$, the exact sequence

$$0 \rightarrow _RS \rightarrow _RE(S) \rightarrow _RE(S)/S \rightarrow 0$$

induces

$$\text{Ext}^k_R(E(S), M) \rightarrow \text{Ext}^k_R(S, M) \rightarrow \text{Ext}^{k+1}_R(E(S)/S, M) \text{ (exact)}.$$

Here, $\text{Ext}^{k+1}_R(E(S)/S, M) = 0$ but $\text{Ext}^k_R(S, M) \neq 0$ by [6, Lemma 1] since $_RS$ is monomorphic to $E^n(M)$, and hence $\text{Ext}^k_R(E(S), M) \neq 0$. So $\text{pd}(E(S)) \geq n$ implies $\text{pd}(E(S)) = n$ by Theorem 2.

Next, assume $\text{pd}(M) = n$ and $RP$ an indecomposable summand of $P^n(M)$, then
for any simple homomorphic image \( \kappa S \) of \( P \), the exact sequence

\[
0 \rightarrow \kappa K \rightarrow \kappa P \rightarrow \kappa S \rightarrow 0
\]

induces

\[
\text{Ext}_R^i(M, P) \rightarrow \text{Ext}_R^{i+1}(M, S) \rightarrow \text{Ext}_R^{i+1}(M, K) \quad \text{(exact)}.
\]

Now, since \( \text{Ext}_R^{i+1}(M, K) = 0 \) but \( \text{Ext}_R^i(M, S) \neq 0 \) by the dual of [6, Lemma 1], \( \text{Ext}_R^i(M, P) \neq 0 \) and hence \( \text{id}(P) = n \) again by Theorem 2.

(2) Decompose \( \kappa R \) into projective indecomposables, then for any projective indecomposable \( \kappa P \) with \( \text{id}(P) = n \), \( E^n(P) \) is isomorphic to a direct summand of \( E^n \).

On the other hand, if \( \text{pd}(M) = n \), \( \text{id}(\text{Ext}_R^i(M, S)) = 0 \) by (1) and hence \( E^n P^n(M) \) is isomorphic to a summand of a direct sum of copies of \( E_n \).

**Corollary 4.** Let \( R \) be an \( n \)-Gorenstein ring with \( \text{dom-dim}_R R \leq n \) and assume \( \kappa M \) a left \( R \)-module with \( \text{id}(M) = n \), then \( E^n(M) \) is isomorphic to a direct summand of a direct sum of copies of \( E_n \).

Now we present an example which seems itself interesting.

**Example.** Let \( R \) be an artinian Gorenstein ring with \( \text{id}(\kappa R) = n \) and \( 0 \rightarrow \kappa R \rightarrow E_0 \rightarrow \cdots \rightarrow E_n \rightarrow 0 \) a minimal injective resolution of \( \kappa R \), then we see from Theorem 3 that \( E_n \) has the largest projective dimension \( n \). Here, we give an example of an \( n \)-Gorenstein ring \( R \) with \( \text{pd}(E_0) = \cdots = \text{pd}(E_n) = n \), which shows that our definition of an \( n \)-Gorenstein ring is different from Auslander's one.

Let \( k \) be a field and \( R \) a subalgebra of \( (k)_8 \), all \( 8 \times 8 \) matrices over \( k \), having \( \{c_{11} + c_{88}, c_{22} + c_{55}, c_{33} + c_{44}, c_{66}, c_{77}, c_{31}, c_{43}, c_{82}, c_{87}, c_{86}, c_{68}, c_{85}, c_{67}\} \) as a \( k \)-basis where \( c_{ij} \) is a matrix unit in \( (k)_8 \) Then \( \text{id}(\kappa R) = \text{id}(R_R) = 2 \), i.e. \( R \) is \( 2 \)-Gorenstein, \( \text{gl-dim} R = \infty \) and \( \text{pd}(E_0) = \text{pd}(E_1) = \text{pd}(E_2) = 2 \). Further any left \( R \)-module of projective dimension \( =2 \) is a summand in a direct sum of copies of \( E_0 \oplus E_1 \oplus E_2 \).

2. A quasi-Frobenius extension of a Gorenstein ring

For rings \( R \subseteq T \), \( T/R \) is called a left quasi-Frobenius (=QF) extension if \( \kappa T \) is finitely generated projective and \( \tau T_R \) is isomorphic to a direct summand in a direct sum of copies of \( \tau \text{Hom}_R(\kappa T, R_R)_R \). A quasi-Frobenius extension is a left and right quasi-Frobenius extension. See [9] for details.

In this section we show a QF extension of a Gorenstein ring is also a Gorenstein ring. First we observe the following.

Let \( R, T \) be rings and \( F : \kappa M \rightarrow \kappa M \) a functor of the category of left \( R \)-modules to one of left \( T \)-modules, which satisfies the condition:

1) \( F \) is exact,
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2) if \( _RE \) is injective, so is \( \tau F(E) \),
then \( \text{id}(\tau F(M)) \leq \text{id}(\_RM) \) for any left \( R \)-module \( _RM \). Further if
3) \( F \) preserves an essential monomorphism
is satisfied, \( \text{id}(\tau F(M)) = \text{id}(\_RM) \) for any \( _RM \).

The next is a generalization of [3, Corollary 8] to a quasi-Frobenius extension and concerns with the case of a Gorenstein order [10, Lemma 5].

**Proposition 5.** Let \( T \) be a left quasi-Frobenius extension of a ring \( R \) and \( _RM \) a left \( R \)-module, then

\[
\text{id}(\tau T \otimes_R M) \leq \text{id}(\_RM).
\]

**Proof.** By [2, VI Proposition 5.2],
\[
\tau T \otimes_R M \cong \tau \otimes_R \text{Hom}_R(\_R, \_RM)
\cong \text{Hom}_R(\_R \text{Hom}_R(\tau T, _R), _RM).
\]

Here, \( T_R \) is projective by [9, Satz 2] and since \( \text{Hom}_R(\tau T, _R) \) is projective ([9, Satz 2]), \( \tau \otimes_R E \) is injective for any injective left \( R \)-module \( _RE \). Therefore the functor \( \tau \otimes_R \rightarrow : _RM \rightarrow _TM \) satisfies the conditions 1)—2) and so

\[
\text{id}(\tau T \otimes_R M) \leq \text{id}(\_RM).
\]

The following should be compared with [9, satz 3].

**Corollary 6.** A quasi-Frobenius extension of an \( n \)-Gorenstein ring also is an \( n \)-Gorenstein ring.

In connection with [1, Example (2)] and [3, Corollary 8'], we state the following.

**Proposition 7.** (1) Let \( T \) be a left quasi-Frobenius extension of a ring \( R \) and suppose \( T_R \) a generator, then

\[
\text{id}(\tau T \otimes_R M) = \text{id}(\_RM)
\]

for any left \( R \)-module \( M \) and especially \( \text{id}(\tau T) = \text{id}(\_R) \).

Moreover, for a finite group \( G \) and a ring \( R \),
\[
\text{id}(\_R G) \rightarrow \_R[G] = \text{id}(\_R).
\]

(2) Let \( T \) be a quasi-Frobenius extension of a ring \( R \) and suppose \( A T \) (or \( T_R \)) a generator, then

\[
\text{id}(\tau T) = \text{id}(\_R) \text{ and } \text{id}(T_R) = \text{id}(R_R).
\]

**Proof.** (1) Let \( F = \tau \otimes_R M \rightarrow _TM \), then \( F \) satisfies the conditions 1)—3) for \( T_R \) is a prognerator by [9, Satz 2].
(2) Let $F=\text{Hom}_R(\tau T, -): \tau M \to \tau M$, then $\tau T$ is a progenerator and so $\text{id}(\tau \text{Hom}_R(\tau T, R)) = \text{id}(\tau F(R)) = \text{id}(R)$. Now, since $T/R$ is a left (resp. right) quasi-Frobenius extension, $\tau \text{Hom}_R(\tau T, R)$ is a generator (resp. finitely generated projective) and therefore $\text{id}(\tau \text{Hom}_R(\tau T, R)) = \text{id}(\tau T)$. Also $\text{id}(\tau T) = \text{id}(R)$ follows from (1).

REMARK. In Proposition 7, if we replace a ring $T$ by an $R$-module and its endomorphism ring, then we obtain the following.

Let $R$ be a ring, $RP$ a projective left $R$-module, $T=\text{End}_R(P)$ and assume $P$ flat, then the functor $F=\text{Hom}_R(RP, -): RP \to \tau M$ satisfies 1–2) by [2, VI Proposition 5.1] and hence $\text{id}(\tau F(P)) \leq \text{id}(RP)$. Observing this fact,

(i) Let $R$ be a left noether ring, $RP$ a projective generator and $T=\text{End}_R(P)$, then $\text{id}(\tau T) \leq \text{id}(R)$. Therefore it follows immediately that an endomorphism ring of a faithful finitely generated projective module over a quasi-Frobenius ring also is a quasi-Frobenius ring. (Curtis and Morita)

(ii) If rings $R$ and $T$ are Morita equivalent, then $\text{id}(R) = \text{id}(\tau T)$ and $\text{id}(R) = \text{id}(\tau T)$.

Now, if rings $R$ and $T$ are Morita equivalent, there exists a finitely generated projective generator (i.e. progenerator) $RP$ and $T=\text{End}_R(P)$. However, if we delete that $RP$ is a generator, it happens that $R$ is Gorenstein but $T$ is not and we see also that faithfulness in Curtis-Morita theorem above is necessary. For example, let $R$ be a self-basic serial ring and $R=Re_1 \oplus Re_2 \oplus Re_3$ a decomposition into primitive left ideals such that $|Re_1|=|Re_2|=|Re_3|=5$ and $Re_1$, (resp. $Re_2$) is epimorphic to $Ne_1$ (resp. $Ne_2$) where $N$ is the radical of $R$. Then $R$ is a quasi-Frobenius ring, but $\text{id}(Re_1Re_2)$ is infinite for $e=e_1+e_2$.

Finally we state an example concerning with a maximal quotient ring of a Gorenstein ring.

EXAMPLE. It is easily seen that a classical quotient ring or more generally a flat epimorphic extension of a Gorenstein ring also is a Gorenstein ring, but it is not known yet that a maximal quotient ring of a Gorenstein ring is also so. (See [11] in the special case.) Here we present an example of a Gorenstein ring $R$ whose left maximal quotient ring $Q$ has id $(\tau Q) > \text{id}(R)$.

Let $k$ be a field, $R$ a subalgebra of $(k)^5$ whose $k$-basis consists of $c_{11}+c_{23}$, $c_{25}+c_{44}$, $c_{33}$, $c_{31}$, $c_{22}$, $c_{44}$ and $Q$ (resp. $Q_1$) a left (resp. right) maximal quotient ring of $R$. Then $R$ is 1-Gorenstein, id $(\tau Q_1)$ = 2 and $Q_1$ is a quasi-Frobenius ring.

ACKNOWLEDGEMENT The author wishes to thank the referee for valuable advices and simplification of the proof of Proposition 5.
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References


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