ON THE NULLITIES OF KÄHLER C-SPACES IN $P_N(C)$

By

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Let $M$ be a Kähler C-space which is holomorphically and isometrically imbedded in an $N$-dimensional complex projective space $P_N(C)$. Then $M$ is a minimal submanifold of $P_N(C)$. Let $n_a(M)$ be the analytic nullity of $M$ which was defined in [2]. We know that the nullity $n(M)$ of $M$ is equal to $n_a(M)$ if $M$ is a Hermitian symmetric space (Kimura [2]). In this note we prove that $n(M)=n_a(M)$ for any Kähler C-space $M$.

By a theorem of Simons [5], the nullity of a Kähler submanifold coincides with the real dimension of the space of holomorphic sections of a normal bundle of the submanifold. Put $M=G/U$ where $G$ is a complex semi-simple Lie group and $U$ is a parabolic subgroup of $G$. By a result of Nakagawa and Takagi [4], we know that every imbedding of $M$ in $P_n(C)$ is induced by a holomorphic linear representation of $G$. From this result we see that the normal bundle $N(M)$ over $M$ is a homogeneous vector bundle.

We prove Theorem 1 which generalizes the generalized Borel-Weil theorem of Bott [1]. Applying the theorem to calculate the dimension of the space of holomorphic sections of $N(M)$ and prove that $n(M)=n_a(M)$.

The author proved the above result before Professor Takeuchi gave another proof of it. His proof does not use Theorem 1 and is more simple than our proof (c.f. Takeuchi [6]).

§1. The generalization of Bott’s result.

Let $G$ be a simply connected compact semi-simple Lie group with Lie algebra $\mathfrak{g}$. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Denote by $\Delta$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We fix a linear order on the real vector space spanned by the elements $\alpha \in \Delta$. Let $\Delta^+$ (resp. $\Delta^-$) be the set of all positive (resp. negative) roots. Let $\Pi=\{\alpha_1,\cdots,\alpha_l\}$ be the fundamental root system, where $l$ is the rank of $\mathfrak{g}$ and $\Pi_i$ be a subsystem of $\Pi$. We put

$$\Delta_i=\{\alpha \in \Delta; \alpha = \sum_{i=1}^l m_i\alpha_i, m_j=0 \text{ for any } \alpha \notin \Pi_i\}$$
\[ \Delta(\pi^*\beta \in \Delta; \beta = \sum_{i=1}^{l} m_i \alpha_i, m_j > 0 \text{ for some } \alpha \in \Pi_\lambda) \]

\[ \Delta(\pi) = \Delta_1 \cup \Delta(\pi^*) \]

Define Lie subalgebras \( g_1, n^1 \) and \( u \) of \( g \) by

\[ g_1 = \mathfrak{h} + \sum_{\alpha \in \Delta_1} \mathfrak{g}_{\alpha} \]

\[ n^1 = \sum_{\rho \notin \Delta(\pi^*)} \mathfrak{g}_{\rho} \]

\[ u = \mathfrak{h} + \sum_{\alpha \in \Delta(\pi)} \mathfrak{g}_{\alpha} \]

where \( \mathfrak{g}_{\alpha} \) is the root space corresponding to \( \alpha \in \Delta \). Then \( g_1 \) (resp. \( n^1 \)) is a reductive (resp. nilpotent) subalgebra of \( g \) and \( u = g_1 + n^1 \) (semi-direct). Let \( U \) be the connected Lie subgroup of \( G \) with Lie algebra \( u \). Then \( U \) is a parabolic Lie subgroup of \( G \), and \( M=G/U \) is a Kähler C-space.

We denote by \( D \) (resp. \( D_1 \)) the set of dominant integral forms of \( g \) (resp. \( g_1 \)). Let \( \xi \in D_1 \). Then there exists the irreducible representation \((\rho_{-\xi}, W_{-\xi})\) of \( g_1 \) with the lowest weight \( -\xi \). We extend it to a representation of \( u \) so that its restriction to \( n^1 \) is trivial, which will be denoted by \((\rho_{-\xi}, W_{-\xi})\). There exists a representation of \( U \) which induces the representation \((\rho_{-\xi}, W_{-\xi})\) and we denote it by \((\rho_{-\xi}, W_{-\xi})\).

Let \((\nu, V)\) be a holomorphic representation of \( G \). We denote by \((\nu|_U \otimes \rho_{-\xi}, V \otimes W_{-\xi})\) the tensor product of the representations \((\nu|_U, V)\) and \((\rho_{-\xi}, W_{-\xi})\) of \( U \). We also denote by \( E_{\xi} \) the holomorphic vector bundle over \( M \) associated to the principal bundle \( G \rightarrow M \) by a representation of \( U \) on \( S \). For a holomorphic vector bundle \( E \) over \( M \), we denote by \( \Omega E \) the sheaf of germs of local holomorphic sections of \( E \). We shall consider the cohomology groups \( H^j(M, \Omega E \otimes \omega_{-\xi}) \).

Let \( W \) be the Weyl group of \( g \) and \( \Delta_1^+ \) the set of all positive roots of \( \Delta_1 \). We define a subset \( W^1 \) of \( W \) by

\[ W^1 = \{ \sigma \in W; \sigma^{-1}(\Delta_1^+) \subset \Delta^* \} \]

Let \( \delta \) be the half of sum of all positive roots of \( g \).

**Theorem 1.** Let \( \xi \in D_1 \) and \((\nu, V)\) be a holomorphic representation of \( G \). If \( \xi + \delta \) is not regular, then

\[ H^j(M, \Omega E_{\xi} \otimes \omega_{-\xi}) = (0) \quad \text{for all } j = 0, 1, \ldots \]

If \( \xi + \delta \) is regular, \( \xi + \delta \) is expressed uniquely as \( \xi + \delta = \sigma(\lambda + \delta) \), where \( \lambda \in D \) and \( \sigma \in W^1 \), and

\[ H^j(M, \Omega E_{\xi} \otimes \omega_{-\xi}) = (0) \quad \text{for all } j = m(\sigma), \]
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\[
H^{\alpha}(M, \Omega E_{V_{\otimes W_{-\iota}}} = V \otimes V_{-\iota} \text{ (as G-module)},
\]
where \( n(a) \) is the index of \( a \) and \((\nu_{-\lambda}, V_{-\iota})\) is the irreducible \( G \)-module with the lowest weight \( -\lambda \).

If \((\nu, V)\) is the trivial representation of \( G \), the theorem coincides with the generalized Borel-Weil theorem of Bott [1].

We prepare some lemmas to prove this theorem. Let \((f, S)\) be a representation of \( u \) and let \( H^j(n^+, S) \) be the \( j \)-th cohomology group formed with respect to the representation \( f|_{\n^+} \) of \( \n^+ \) on \( S \). We may regard \( H^j(n^+, S) \) as \( \g_{\delta \iota} \)-module in a canonical way. We denote by \( H^j(n^+, S)^\delta \) the subspace of \( H^j(n^+, S) \) annihilated by all \( X \in \g_{\delta \iota} \). We may easily get the following lemma from theorems of Bott [1].

**Lemma 1.** Let \( \lambda \in \mathbf{D} \). Then

\[
\text{the multiplicity of } \nu \text{ in } H^j(M, \Omega E_{V_{\otimes W_{-\iota}}}) = \dim H^j(\n^+, \text{Hom}(V^\iota, V \otimes W_{-\iota}))^\delta \text{ for } j = 0, 1, \ldots,
\]

where \((\nu^\iota, V^\iota)\) is an irreducible representation of \( \g \) with the highest weight \( \lambda \).

Since the representation \( (\rho_{-\iota}|_{\n^+}, W_{-\iota}) \) is trivial, we have

\[
\begin{align*}
H^j(\n^+, \text{Hom}(V^\iota, V \otimes W_{-\iota})) &= H^j(\n^+, V_{-\iota} \otimes V \otimes W_{-\iota}) \\
&= H^j(\n^+, V_{-\iota} \otimes V \otimes W_{-\iota}).
\end{align*}
\]

From Schur's lemma we have

\[
\dim H^j(\n^+, \text{Hom}(V^\iota, V \otimes W_{-\iota}))^\delta = \text{the multiplicity of } \nu^\iota \text{ in } H^j(\n^+, V_{-\iota} \otimes V),
\]

where \( \nu^\iota \) is an irreducible representation of \( \g_{\delta \iota} \) with the highest weight \( \xi \).

**Lemma 2.** Let \( \lambda \in \mathbf{D} \). Then

\[
\text{the multiplicity of } \nu \text{ in } H^j(M, \Omega E_{V_{\otimes W_{-\iota}}}) = \text{the multiplicity of } \nu^\iota \text{ in } H^j(\n^+, V_{-\iota} \otimes V).
\]

Now we recall Kostant's result of Lie algebra cohomology.

**Theorem of Kostant ([3]).** Let \( \lambda \in \mathbf{D} \). Then \( \g_{\delta \iota} \)-module \( H^j(\n^+, V^\iota) \) is decomposed into direct sums:

\[
H^j(\n^+, V^\iota) = \bigoplus_{\xi \in W^{(\lambda,\delta)}} \bigoplus_{\vartheta \in W_{(\lambda,\delta)}} W^{(\lambda,\delta)}(\xi + \vartheta),
\]
where \( W(j) = \{ \alpha \in W_1 ; \hbar(\alpha) = j \} \) and \((\nu'_j, W')\) is the irreducible representation of \( \mathfrak{g}_1 \) with the highest weight \( \mu \).

**Proof of Theorem 1.** Assume that the multiplicity of \( \nu'_j \) in \( H^p(n', V'), \gamma \in D \), is not 0. By the above theorem there exists an element \( \sigma \in W'(j) \) so that \( \xi + \delta = \sigma(\gamma + \delta) \). Since \( \gamma + \delta \) is regular, \( \xi + \delta \) is also regular. Therefore by Lemma 2 we see that if \( \xi + \delta \) is not regular then \( H^p(M, \Omega E_{\nu_0 \mathfrak{w}_q}) = (0) \) for any \( j \).

Assume that \( \xi + \delta \) is regular. Then \( \xi + \delta \) is expressed uniquely as \( \xi + \delta = \sigma(\lambda + \delta) \), where \( \lambda \in D \) and \( \sigma \in W'(\lambda) \) (Kostant [3]). If \( j \neq \hbar(\sigma) \), we see immediately that \( H^p(M, \Omega E_{\nu_0 \mathfrak{w}_q}) = (0) \) by Lemma 2 and Theorem of Kostant.

Let \( G_u \) be a maximal compact subgroup of \( G \). Denote by \( \chi_{\lambda} \) the character of a representation \( \phi \) of \( G \). Then by Theorem of Kostant we get the following:

- the multiplicity of \( \nu'_j \) in \( H^{n^{(\phi)}}(n', \mathfrak{w}_q \otimes \nu) \)
- the multiplicity of \( \nu' \) in \( \mathfrak{w}_q \otimes \nu \)

\[
= \sum_{\sigma \in W'} \chi_{\lambda} \cdot \chi_{\sigma} \cdot \lambda \cdot \sigma \cdot dg
\]
- the multiplicity of \( \nu' \) in \( \mathfrak{w}_q \otimes \nu \),

where \( dg \) is the normalized Haar measure on \( G_u \). Therefore by Lemma 2, we get

\[
H^{n^{(\phi)}}(M, \Omega E_{\nu_0 \mathfrak{w}_q}) = \mathfrak{w}_q \otimes \nu \quad \text{(as } \mathfrak{g}-\text{module).} \quad \text{Q.E.D.}
\]

§ 2. Proof of the main theorem.

We retain the same notations and assumptions introduced in § 1. Let \( \mathfrak{a} \) be an integral form such that \( (\mathfrak{a}, \alpha_i) = 0 \) for \( \alpha_i \in \mathfrak{h}_1 \) and \( (\mathfrak{a}, \alpha_j) > 0 \) for \( \alpha_j \notin \mathfrak{h}_1 \). We denote by \( (\nu', V') \) the irreducible representation of \( G \) with highest weight \( \lambda \). Let \( P(V') \) be the complex projective space consisting of all 1-dimensional subspace of \( V' \). Since the dimension of the weight space \( (\nu) \) in \( V' \) corresponding to the highest weight \( \lambda \) is equal to 1, \( (\nu) \) is an element of \( P(V') \). Moreover \( G \) acts canonically on \( P(V') \) via the representation \( (\nu', V') \), and it is known that \( U \) coincides with the isotropy subgroup of \( G \) at \( (\nu) \). Therefore we get a \( G \)-equivariant imbedding \( f' : M = G/U \rightarrow P(V') \). Since \( \nu' \) is an irreducible representation, \( f' \) is a full imbedding. Conversely every full Kähler imbedding of a Kähler \( \mathfrak{g} \)-space \( M \) in \( P_u(\mathfrak{g}) \) is obtained in this way (Nakagawa and Takagi [4]).

**Theorem 2.** Let \( M = G/U \) be a Kähler \( \mathfrak{g} \)-space fully imbedded in \( P_u(\mathfrak{g}) \). Then the nullity \( n(M) \) of \( M \) in \( P_u(\mathfrak{g}) \) is given by

\[
n(M) = \dim_R(P_u(\mathfrak{g})) - \dim_R(M),
\]
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where $a(P_n(C))$ (resp. $a(M)$) is the vector space of all analytic vector fields on $P_n(C)$ (resp. $M$).

**Proof.** Assume that the imbedding of $M$ in $P_n(C)$ is induced by the irreducible representation $(\nu^A, V^A)$, $A \in D$ and $\dim V^A = n + 1$, of $G$. Denote by $(h, (v))$ the representation of $U$ on $(v)$ induced by $\nu^A$ and denote by $(h^*, (v)^*)$ the contragredient representation of $(h, (v))$. Then we get the following exact sequence of $U$-modules:

$$0 \longrightarrow (v) \otimes (v)^* \longrightarrow V \otimes (v)^* \longrightarrow V \otimes (v)^*/(v) \otimes (v)^* \longrightarrow 0.$$  

It is easy to see that $E_{\otimes (v), \otimes (v)^*} = T(P_n(C))|_M$. Therefore we get the following exact sequence of holomorphic vector bundles over $M$:

$$0 \longrightarrow 1 \longrightarrow E_{\otimes (v), \otimes (v)^*} \longrightarrow T(P_n(C))|_M \longrightarrow 0,$$

where $1$ is the trivial line bundle over $M$. Since $M$ is a Kähler C-space, $H^0(M, \mathcal{O}) = (0)$. Therefore we get the following exact sequence of cohomology groups:

$$0 \longrightarrow H^0(M, \mathcal{O}) \longrightarrow H^0(M, \Omega E_{\otimes (v), \otimes (v)^*}) \longrightarrow H^0(M, \Omega (T(P_n(C))|_M)) \longrightarrow 0.$$

Since the lowest weight of $(h^*, (v)^*)$ is $-A$, it follows, by Theorem 1, that $H^0(M, \Omega E_{\otimes (v), \otimes (v)^*}) = V \otimes V_{-1}$ as $G$-modules. It is obvious that $\dim H^0(M, \mathcal{O}) = 1$. Therefore we get

$$\dim H^0(M, \Omega (T(P_n(C))|_M)) = (n + 1)^2 - 1.$$  

Since $\dim_R a(P_n(C)) = 2((n + 1)^2 - 1)$, we get

(1) \quad $\dim_R H^0(M, \Omega (T(P_n(C))|_M)) = \dim_R a(P_n(C))$.

The exact sequence of holomorphic vector bundles over $M$:

$$0 \longrightarrow T(M) \longrightarrow T(P_n(C))|_M \longrightarrow N(M) \longrightarrow 0$$

and $H^1(M, \Omega T(M)) = (0)$ (Bott [1]) induce the following exact sequence of cohomology groups:

(2) \quad $0 \longrightarrow H^0(M, \Omega T(M)) \longrightarrow H^0(M, \Omega (T(P_n(C))|_M)) \longrightarrow H^0(M, \Omega N(M)) \longrightarrow 0$.

Recall that the nullity $n(M)$ of $M$ is given by

(3) \quad $n(M) = \dim_R H^0(M, \Omega N(M))$

(Kimura [2]). From (1), (2), (3) and $\dim_R H^0(M, \Omega T(M)) = \dim_R a(M)$, we get

$n(M) = \dim_R a(P_n(C)) - \dim_R a(M)$.

Q.E.D.

From the above theorem and Lemma 3.4 in Kimura [2] we have the following result.
Corollary. Let $M$ be a Kähler $C$-space holomorphically and isometrically imbedded in $P_N(C)$. Then

$$n(M) = n_d(M).$$

References


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