ROOTS OF KAC-MOODY LIE ALGEBRAS

By

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0. Introduction.

The notion of Kac-Moody Lie algebras has recently been introduced and studied as a natural generalization of a finite dimensional split semisimple Lie algebra with successful applications to Macdonald type identities (cf. Lepowsky [3]). Such a Lie algebra has a root system, which is a natural analogue of the usual root systems in the sense of Bourbaki [1]. In this paper, we will give a characterization of positive root systems of Kac-Moody Lie algebras as a subset of a lattice. (See Proposition 1 and Theorem 1 below.)

Let $A$ be a generalized Cartan matrix and $L$ the Kac-Moody Lie algebra associated with $A$, and let $\Delta$ (resp. $\Delta_+$) be the root system (resp. the positive root system) of $L$ (for the definition, see §1). In §2, we will consider the special positive root system $P(A)$ associated with $L$. This system $P(A)$ satisfies the conditions $(X1)$, $(X2)$, $(Y1)$, $(Y2)$ and $(Y3)$, which are specified in §2. Conversely we will show that any set satisfying these conditions coincides with the system $P(A)$ arising from some Kac-Moody Lie algebra. In particular, $\Delta_+$ is uniquely determined by $(Y1)$, $(Y2)$ and $(Y3)$ when $A$ is given.

On the other hand, there are two kinds of roots in $\Delta$, called real roots and imaginary roots respectively. In §3, we will present a characterization of imaginary roots. In §4, we will give a way to produce the roots of $L$ inductively from simple roots.

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In this section, we will review the notion of a Kac-Moody Lie algebra and its root system (cf. Kac [2], Lepowsky [3], Moody [4]). Let $I$ be a positive integer, and set $I=\{1, \cdots, l\}$. Let $A=(a_{ij})$ be an $l\times l$ generalized Cartan matrix—that is $a_{ij} \in \mathbb{Z}$ for all $i, j \in I$, $a_{ij}=2$ for all $i \in I$, $a_{ij} \leq 0$ for distinct $i, j \in I$. Received July 30, 1980.
\[ a_{ij} = 0 \text{ whenever } a_{ji} = 0 \text{ for each } i, j \in I. \]

For any generalized Cartan matrix \( A = (a_{ij}) \) and for any field \( K \) of characteristic zero, \( L \) denotes the Lie algebra over \( K \) generated by 3l generators \( e_1, \ldots, e_l, h_1, \ldots, h_l, f_1, \ldots, f_l \) with the defining relations \([h_i, h_j] = 0, [e_i, f_j] = \delta_{ij} h_i, [h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} f_j \) for all \( i, j \in I \), and \((\text{ad} e_i)^{a_{ij}+1} e_j = 0, (\text{ad} f_i)^{a_{ij}+1} f_j = 0 \) for distinct \( i, j \in I \). We call the algebra \( L \) the (standard) Kac-Moody Lie algebra over \( K \) associated with \( A \).

Let \( \Gamma = \sum_{\alpha \in I} \mathbb{Z} \alpha_i \) be the free \( \mathbb{Z} \)-module with free generators \( \alpha_1, \ldots, \alpha_l \). We give the structure of a \( \Gamma \)-graded Lie algebra to \( L \) by defining \( \deg(e_i) = \alpha_i \), \( \deg(h_i) = 0 \) and \( \deg(f_i) = -\alpha_i \) for all \( i \in I \). For any \( \alpha \in \Gamma \), let \( L_{\alpha} \) be the subspace of \( L \) consisting of all elements with degree \( \alpha \). Set \( H = L_0 \), which equals \( K h_1 \oplus \cdots \oplus K h_l \). We call a nonzero element \( \alpha \in \Gamma \) a root of \( L \) if \( L_{\alpha} \neq 0 \). Let \( \Delta \) be the set of all roots of \( L \), called the root system of \( L \), so \( L = H \oplus \sum_{\alpha \in \Gamma} L_{\alpha} \). Then \( \Delta \) is contained in \( \Gamma_+ \cup (-\Gamma_+) \), where \( \Gamma_+ = \{ \alpha = \sum_{i \in I} k_i \alpha_i | k_i \geq 0, \alpha \neq 0 \} \). Set \( \Delta_+ = \Delta \cap \Gamma_+ \) and \( \Delta_- = \Delta \cap (-\Gamma_+) \). We call \( \Delta_+ \) the positive root system of \( L \). We note \( \Delta_- = -\Delta_+ \). For each \( \alpha = \sum_{i \in I} k_i \alpha_i \in \Delta_+ \) (resp. \( \Delta_- \)), \( L_{\alpha} \) is the subspace of \( L \) spanned by the elements

\[
[\alpha_1, [\alpha_2, \ldots, [\alpha_{l-1}, \alpha_l] \ldots]] \\
(\text{resp. } [\alpha_1, [\alpha_2, \ldots, [\alpha_{l-1}, \alpha_l] \ldots]]),
\]

where \( \alpha_j \) (resp. \( f_j \)) occurs \( |k_j| \) times. In particular, \( L_{\alpha_i} = Ke_i \) and \( L_{-\alpha_i} = Kf_i \).

Set \( H' = \{ \alpha_1, \ldots, \alpha_l \} \), called simple roots. For each \( i \in I \), let \( U_i \) be the subalgebra of \( L \) generated by \( e_i, h_i, f_i \), which is isomorphic to \( \text{sl}(2, K) \).

**Lemma 1.** (cf. Lepowsky [3, Proposition 1.4]). The subspace \( \sum_{\alpha \in \Lambda_{\alpha, \alpha}} L_{\alpha} \) is a direct sum of finite dimensional irreducible \( U_i \)-modules for each \( i \in I \).

**Lemma 2.** Let \( \alpha \in \Delta_+ \setminus \Pi \). Then there is \( \alpha_i \in \Pi \) such that \( \alpha - \alpha_i \in \Delta_+ \).

**Proof.** Since \( L_{\alpha} \neq 0 \), there is a nonzero generator \( [\alpha_1, [\alpha_2, \ldots, [\alpha_{l-1}, \alpha_l] \ldots]] \) in \( L_{\alpha} \). Then \( [\alpha_1, [\alpha_2, \ldots, [\alpha_{l-1}, \alpha_l] \ldots] \) is a nonzero element of \( L_{\alpha - \alpha_i} \). Therefore such \( \alpha - \alpha_i \) is in \( \Delta_+ \).

q.e.d.

2. Abstract positive root systems.

Let \( \Gamma = \sum_{i \in I} \mathbb{Z} \alpha_i \) be the free \( \mathbb{Z} \)-module with free generators \( \Pi = \{ \alpha_1, \ldots, \alpha_l \} \), and let \( \Gamma_+ = \{ \alpha = \sum_{i \in I} k_i \alpha_i \in \Gamma | k_i \geq 0, \alpha \neq 0 \} \) as in §1. Set \( \Gamma^* = \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{Z}) \), the dual of \( \Gamma \). A pair \( \Phi = (X, Y) \) consisting of a subset \( X \) of \( \Gamma^* \) and a subset \( Y \) of \( \Gamma \) is called an abstract positive root system (in the lattice \( \Gamma \)) if the following axioms (X1), (X2), (Y1), (Y2) and (Y3) are satisfied.

(X1) \( X \) consists of \( l \) distinct elements \( \phi_1, \ldots, \phi_l \), labeled by \( I \).
(X2) $\phi_i(\alpha_i)=2$ for all $i \in I$.

(Y1) $\Pi \subseteq Y \subseteq \Delta$.

(Y2) For each $i \in I$, if $\alpha \in Y - \{\alpha_i\}$, then there are nonnegative integers $p=p(i, \alpha)$ and $q=q(i, \alpha)$ satisfying

\[
\begin{align*}
\begin{array}{c}
\phi_i(\alpha_i) = 2 \\
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{c}
\phi_i(\alpha_i) = 2 \\
\end{array}
\end{align*}
\]

(Y3) If $\alpha \in Y - \Pi$, then there exists $\alpha_i \in \Pi$ for which $\alpha - \alpha_i \in Y$.

Let $L$ be the (standard) Kac-Moody Lie algebra associated with a generalized Cartan matrix $A$, and let $\Delta_+$ be the positive root system of $L$ (see §1). Let $\phi_1, \ldots, \phi_I$ be elements of $\Gamma^*$ defined by $\phi_i(\alpha_j) = a_{ij}$ for all $j \in I$. Set $\Psi = \{\phi_1, \ldots, \phi_I\}$ and $P(A) = (\Psi, \Delta)$. We call $P(A)$ the special positive root system of $A$ or of $L$. For each $\alpha = \sum_{i \in I} k_i \alpha_i \in \Gamma$, let $ht(\alpha) = k_1 + \cdots + k_I$, the height of $\alpha$.

**Proposition 1.** Let $A$ be a generalized Cartan matrix. Then the special positive root system $P(A)$ is an abstract positive root system.

**Proof.** By Lemma 1 and Lemma 2, we see that (Y2) and (Y3) hold. The other conditions are easily verified. q.e.d.

For each abstract positive root system $\Phi$, set $C_\Phi = (e_{ij})$, where $e_{ij} = \phi_i(\alpha_j)$ for all $i, j \in I$.

**Theorem 1.** Let $\Phi$ be an abstract positive root system. Then:

1. $C_\Phi$ is a generalized Cartan matrix,
2. $\Phi = P(C_\Phi)$.

**Proof.** (1) By the definition, $\phi_i(\alpha_i) \in \mathbb{Z}$. The axiom (X2) says $\phi_i(\alpha_i) = 2$. For distinct $i, j \in I$, we see $\alpha_j - \alpha_i \in Y$ by (Y1), so $\phi(i, \alpha_i) = 0$ and $\phi(i, \alpha_j) = -\phi(i, \alpha_i) \leq 0$ by (Y2). Furthermore, for distinct $i, j \in I$, the condition $\phi_i(\alpha_j) = 0$ means $\alpha_i + \alpha_j \notin Y$. Therefore $\phi_i(\alpha_i) = 0$ if and only if $\phi_j(\alpha_i) = 0$.

(2) This follows from Proposition 2 below.

**Proposition 2.** Let $\Phi = (X, Y)$ and $\Phi' = (X', Y')$ be abstract positive root systems in the lattices $\Gamma$ and $\Gamma'$ respectively. Suppose $C_\Phi = C_{\Phi'}$. Then there is an isomorphism $\lambda : \Gamma \to \Gamma'$ such that $\lambda(\Phi) = \Phi'$.

**Proof.** Let $l = \text{rank } \Gamma = \text{rank } \Gamma'$. Since $C_\Phi = C_{\Phi'}$, we have $\phi_i(\alpha_j) = \phi'_i(\alpha'_j)$ for all $i, j \in I$, where $\phi_i \in X$, $\phi'_i \in X'$, $\alpha_j \in \Pi$, and $\alpha'_j \in \Pi'$. Let $\lambda$ be the isomorphism of $\Gamma$ to $\Gamma'$ defined by $\lambda(\alpha_i) = \alpha'_i$. Then $\phi'_i = \phi_i \cdot \lambda^{-1}$. Put $Y_n = \{\alpha \in Y | \text{ht}(\alpha) \leq n\}$. 


and \( Y'_n = \{ \alpha \in Y' \mid \text{ht}(\alpha) \leq n \} \), where \( n \in \mathbb{Z}_{\geq 0} \). If \( n=1 \), then \( \lambda(Y'_1) = Y'_1 = Y' \) by (Y1).
Assume \( n > 1 \) and \( \lambda(Y'_{n-1}) = Y'_{n-1} \). Let \( \alpha \in Y_n \). By (Y3), there exists \( \alpha_i \in \Pi \) for which \( \alpha - \alpha_i \in Y'_{n-1} \). Since \( \lambda(Y'_{n-1}) = Y'_{n-1} \), we have \( \lambda(\alpha) \in Y'_{n} \) by (Y2). Hence \( \lambda(Y_n) \subseteq Y'_n \). Similarly \( \lambda^{-1}(Y'_n) \subseteq Y_n \). Therefore \( \lambda(Y_n) = Y'_n \) for all \( n \in \mathbb{Z}_{>0} \), which implies \( \lambda(Y) = Y' \).

q.e.d.

Moreover we will prove the following two results without recourse to the theory of Lie algebras.

**Proposition 3.** Let \( \Phi=(X, Y) \) be an abstract positive root system. Then \( Z\alpha_i \cap Y = \{ \alpha_i \} \) for any \( \alpha_i \in \Pi \).

**Proof.** Let \( \alpha_i \in \Pi \). Clearly \( m\alpha_i \in Y \) if \( m \leq 0 \). Suppose \( m\alpha_i \in Y \) for some \( m \in \mathbb{Z}_{>1} \). By (Y2), we see \( \alpha_i, 2\alpha_i, \ldots, m\alpha_i \in Y \). Thus \( p(i, m\alpha_i) = m-1 \) by (Y1) and (Y2). Then \( m-1 \geq p(i, m\alpha_i) - q(i, m\alpha_i) = \phi(p(m\alpha_i)) = 2m \) by (Y2), so \( m \leq -1 \), which is a contradiction.

q.e.d.

For each \( i \in I \), let \( w_i \) be an involutive endomorphism of \( \Gamma' \) defined by \( w_i(\alpha) = \alpha - \phi_+(\alpha)\alpha_i \) for all \( \alpha \in \Gamma' \).

**Proposition 4.** Let \( \Phi=(X, Y) \) be an abstract positive root system. Then \( Y \setminus \{ \alpha_i \} \) is \( w_i \)-stable for any \( i \in I \).

**Proof.** Let \( \beta \in Y \setminus \{ \alpha_i \} \). Then \( w_i(\beta) = \beta - \phi_+(\beta)\alpha_i = \beta + (q - p)\alpha_i \). By (Y2), we see \( w_i(\beta) \in Y \) since \( -p \leq q - p \leq q \). Suppose \( w_i(\beta) = \alpha_i \), then \( \beta = \phi_+(\beta) + 1)\alpha_i \in Z\alpha_i \cap Y = \{ \alpha_i \} \), which is a contradiction. Therefore \( w_i(\beta) \in Y \setminus \{ \alpha_i \} \).

q.e.d.

**3. Imaginary roots.**

Let \( W \) be the subgroup of \( GL(\Gamma') \) generated by \( w_i \) for all \( i \in I \). The group \( W \) is called the Weyl group. Set \( \Delta^r = W(\Pi) \), real roots, and set \( \Delta^i = \Delta - \Delta^r \), imaginary roots. Put \( \Delta^p = \Delta^r \cap \Delta^i \). For each \( \alpha = \sum_{i \in I} k_i \alpha_i \in \Gamma' \), let \( S_\alpha \) be the diagram, called the support of \( \alpha \), with vertices \( v_i \) for all \( i \) satisfying \( k_i \neq 0 \) such that \( v_i \) and \( v_j \) are joined whenever \( \phi_+(\alpha) \neq 0 \) for distinct \( i, j \). Let \( J \) be the set consisting of all elements \( \alpha \) of \( \Gamma' \) such that the support \( S_\alpha \) is connected, and let \( J^p = \bigcap_{w \in W} (wJ) \). We call a generalized Cartan matrix indecomposable if the corresponding Dynkin diagram is connected (cf. Kac [2], Lepowsky [3], Moody [4]).
THEOREM 2. Let $A$ be an indecomposable generalized Cartan matrix. Then $D^\mu = J^\mu$.

PROOF. We note that the support $S_\alpha$ of a root $\alpha \in A$ is connected. Therefore $D^\mu \subseteq J$, and $D^\mu \subseteq J^\mu$ since $D^\mu$ is $W$-invariant. Conversely let $\alpha \in J^\mu$. Then choose an element $\omega \alpha$ in the orbit $W(\alpha)$ of minimal height. Then $\omega \alpha \in J$ and in fact in the fundamental set $M$, where $M = \{ \alpha \in J | \phi_i(\alpha) \leq 0 \text{ for all } i \in I \}$. We know $M \subseteq D^\mu$ (see Kac [2], or Remark (3) below). Hence $\omega \alpha \in D^\mu$ and $\alpha \in D^\mu$.

q.e.d.


Let $A$ be a generalized Cartan matrix. For each $\alpha \in \Gamma_+$, it is difficult to determine whether $\alpha$ is in $\Delta_+$ or not, since in general the Weyl group is infinite. Here we will give an actual method of constructing the positive roots inductively from $\Pi$. Let $P(A)=(X, Y)$ be the special positive root system of $A$. Let $Y_n = \{ \alpha \in Y | \text{ht}(\alpha) \leq n \}$ for each $n \in \mathbb{Z}_{\geq 0}$.

PROPOSITION 5. Suppose $n \geq 1$. Let $\phi_i \in X$ and $\alpha_i \in \Pi$ for each $i \in I$, and let $\alpha \in Y_n$.

1. If $\phi_i(\alpha) < 0$, then $\alpha + \alpha_i \in Y_{n+1}$.
2. If $\phi_i(\alpha) \geq 0$, then $\alpha + \alpha_i \in Y_{n+1}$ if and only if $\alpha - (\phi_i(\alpha) + 1) \alpha_i \in Y_n$.

Therefore we can construct the set of roots recursively. (That an inductive construction is possible is already known—cf. Moody [4, Proposition 1].)

REMARKS.

1. We can prove Propositions 2 and 3 without the condition (Y3).
2. Let

$$A = \begin{pmatrix}
2 & -2 & 0 & 0 & 0 \\
-2 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -2 \\
0 & 0 & 0 & -2 & 2
\end{pmatrix}$$

and $S = \cap_{w \in \mathbb{W}(w \Gamma_+)}$. Set $Y = \Delta_+ \cup S$. Then we see that $Y$ satisfies the conditions (Y1) and (Y2). Take a minimal height element $\alpha$ in $Y - \Delta_+$. This is possible since $\delta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in Y - \Delta_+$. For such an element $\alpha$, there is no element $\alpha_i \in \Pi$ such that $\alpha - \alpha_i \in Y$. (This example appears in a letter from Mr. M. Kaneda to Prof. N. Iwahori.)
(3) (Kac [2]). Let $\Delta^m$, $W$ and $M$ be the positive imaginary roots, the Weyl group and the fundamental set respectively. Then $\Delta^m = W(M)$.

(4) In Moody and Yokonuma [5], a geometric axiomatic foundation for real root systems of Kac-Moody Lie algebras has been developed.

References


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