ON CONNECTION ALGEBRAS OF HOMOGENEOUS
CONVEX CONES

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§ 1. Introduction.

Let $V$ be a homogeneous convex cone in an $n$-dimensional vector space $X$ over the real number field $\mathbb{R}$. If the dual cone of $V$ with respect to a suitable inner product on $X$ coincides with $V$, then $V$ is said to be self-dual. By using the characteristic function of $V$, we can define a canonical $G(V)$-invariant Riemannian metric $g_V$ on $V$, where $G(V)$ is the Lie group of all linear automorphisms of $X$ leaving $V$ invariant. Let us take a point $e \in V$ and a system of linear coordinates $(x^1, x^2, \ldots, x^n)$ on $X$. Then, a commutative multiplication $\Box$ is defined in $X$ by

$$x^i(a \Box b) = - \sum_{j,k} \Gamma^i_{jk}(e)x^j(a)x^k(b) \quad (1 \leq i \leq n)$$

for every $a, b \in X$, where $\Gamma^i_{jk}$ means the Christoffel symbols for the canonical metric $g_V$ with respect to $(x^1, x^2, \ldots, x^n)$. The structure of the algebra $(X, \Box)$ is independent of choosing the point $e$ and the system of linear coordinates $(x^1, x^2, \ldots, x^n)$. This algebra $(X, \Box)$ is called the connection algebra of $V$ (cf. [13], [14]). A commutative (but not necessarily associative) algebra $A$ over $\mathbb{R}$ is said to be power-associative if the subalgebra $\mathbb{R}[a]$ of $A$ generated by any element $a \in A$ is associative.

The aim of the present note is to prove the following assertion: If the connection algebra of a homogeneous convex cone $V$ is power-associative, then $V$ is self-dual (Theorem 1).

It is known that any Jordan algebra over $\mathbb{R}$ is power-associative (cf. e.g. [3] or [7]). So, from this, we have the known result by Dorfmeister [2]: A homogeneous convex cone $V$ is self-dual if the connection algebra of $V$ is Jordan. On the other hand, it is known that a commutative power-associative algebra over $\mathbb{R}$ having no nilpotent element is Jordan (cf. chap. 5 of [7]). From this, we can see that a power-associative connection algebra is necessarily Jordan. Therefore, the above assertion is contained in [2], but our method used here is

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elementary and quite different from that of [2]. In fact, we will start out from the theory of $T$-algebras developed by E.B. Vinberg and use an identity for a power-associativity condition on a connection algebra. And also, we will make use of the results on the invariant Riemannian connection for the canonical metric obtained in the previous papers [9], [10], [11].

Throughout this note, the same terminologies and notation as those in the author's previous papers will be employed.

§ 2. Preliminaries.

In this section, we will recall the fundamental results on homogeneous convex cones and $T$-algebras due to Vinberg. Detailed description for them may be found in [12], [13], [14].

Let $\mathfrak{A} = \sum_{1 \leq i, j \leq r} \mathfrak{A}_{ij}$ be a $T$-algebra of rank $r$ provided with an involutive antiautomorphism *. A general element of $\mathfrak{A}_{ij}$ will be denoted as $a_{ij}$, and also an element of $\mathfrak{A}$ will be denoted like as a matrix $a=(a_{ij})$, where $a_{ij}$ is the $\mathfrak{A}_{ij}$-component of $a \in \mathfrak{A}$. From now on, the following notation will be used:

\[ n_{ij} = \dim \mathfrak{A}_{ij} \quad (1 \leq i, j \leq r), \]
\[ n_i = 1 + \frac{1}{2} \sum_{1 \leq k < l \leq r} n_{kl} + \frac{1}{2} \sum_{1 \leq k \leq r} n_{kk} \quad (1 \leq i \leq r), \]
\[ \text{Sp} \ a = \sum_{1 \leq i, j \leq r} n_{ij}a_{ij} \quad (a=(a_{ij}) \in \mathfrak{A}), \]
\[ (a, b) = \text{Sp} \ ab^* \quad (a, b \in \mathfrak{A}). \]

(2.1)

From the axiom of $T$-algebra (cf. p. 380 in [13]), it follows that the scalar product $(,)$ defined by (2.1) is positive definite and the numbers \(\{n_{ij}\}_{1 \leq i, j \leq r}\) satisfy the following condition:

\[ \max \{n_{ij}, n_{jk}\} \leq n_{ik} \]

for every triple \((i, j, k)\) of indices \(i < j < k\) satisfying \(n_{ij}n_{jk} \neq 0\).

Let us define subsets $T = T(\mathfrak{A})$, $V = V(\mathfrak{A})$ and $X = X(\mathfrak{A})$ of $\mathfrak{A}$ by

\[ T = \{t=(t_{ij}) \in \mathfrak{A} ; t_{ii} > 0 \quad (1 \leq i \leq r), \quad t_{ij} = 0 \quad (1 \leq j < i \leq r)\} \]

and

\[ V = \{tt^* ; t \in T\} \subset X = \{x \in \mathfrak{A} ; x^* = x\}. \]

Then $V = V(\mathfrak{A})$ is a homogeneous convex cone in the real vector space $X$ and $T$ is a connected Lie group which acts linearly and simply transitively on $V$. Conversely, every homogeneous convex cone is realized in this form up to linear equivalence.
Let $e = (e_{ij})$ be the unit element of the Lie group $T$. Then $e_{ij} = \delta_{ij}$ (Kronecker delta) and $e \in V$. The tangent space $T_e(V)$ of $V$ at the point $e$ may be naturally identified with the ambient space $X$ and also with the Lie algebra $\mathfrak{t}$ of $T$. On the other hand, the Lie algebra $\mathfrak{t}$ may be identified with the subspace $\sum_{i \neq j \neq r} \mathfrak{u}_{ij}$ of $\mathfrak{u}$ provided with the bracket product: $[a, b] = ab - ba$. A canonical linear isomorphism between $\mathfrak{t}$ and $X$ is given by

$$\xi: a \in \mathfrak{t} = \sum_{i \neq j \neq r} \mathfrak{u}_{ij} \rightarrow a + a^* \in X = T_e(V).$$

The canonical Riemannian metric $g_V$ at the point $e$ determines an inner product $\langle , \rangle$ on $\mathfrak{t}$ via the isomorphism $\xi$ by

$$\langle a, b \rangle = g_V(e)(\xi(a), \xi(b))$$

for every $a, b \in \mathfrak{t}$. Concerning two inner products $( , )$ and $\langle , \rangle$, we have the following relations (cf. p. 389, p. 391 and p. 392 in [13]):

$$\langle a_{ij}, b_{ij} \rangle = 2(a_{ij}, b_{ij}) = 2(a^*_i, b^*_j) \quad (1 \leq i < j \leq r),$$

$$\langle a_{ii}, b_{ii} \rangle = 4(a_{ii}, b_{ii}) \quad (1 \leq i \leq r).$$

$$\langle a_{ij}, b_{jk}, c_{ik} \rangle = \langle a^*_i c_{ik}, b_{jk} \rangle = \langle a_{ij}, c_{ik} b^*_k \rangle \quad (1 \leq i < j < k \leq r).$$

$$\langle \mathfrak{u}_{ij}, \mathfrak{u}_{kl} \rangle = 0 \quad ((i, j) \neq (k, l)).$$

We now put

$$e_i = \frac{1}{2\sqrt{n_i}} e_{ii} \in \mathfrak{u}_{ii} \quad (1 \leq i \leq r).$$

Then

$$\|e_i\| = 1.$$  

Here, $\|a\|$ denotes the norm of an arbitrary element $a \in \mathfrak{t}$ with respect to the inner product $\langle , \rangle$.

The connection function $\alpha$ and the curvature tensor $R$ for the canonical Riemannian metric $g_V$ are described in terms of the Lie algebra $\mathfrak{t}$ and the inner product $\langle , \rangle$ as follows (cf. Nomizu [4]):

$$\alpha: \mathfrak{t} \times \mathfrak{t} \rightarrow \mathfrak{t},$$

$$2\langle \alpha(a, b), c \rangle = \langle [c, a], b \rangle + \langle a, [c, b] \rangle + \langle [a, b], c \rangle$$

and

$$R: \mathfrak{t} \times \mathfrak{t} \times \mathfrak{t} \rightarrow \mathfrak{t},$$

$$R(a, b, c) = R(a, b)c = \alpha(a, \alpha(b, c)) - \alpha(b, \alpha(a, c)) - \alpha([a, b], c).$$
for every \(a, b, c \in t\). The multiplication \(\square\) in \(X\) defined in §1 determines a multiplication \(\circ\) in \(t\) via the isomorphism \(\xi\) (cf. (2.3)) as follows:

\[ a \circ b = \xi^{-1}(\xi(a) \square \xi(b)) \]

for every \(a, b \in t\). Then it is known that the identity

\[
(2.10) \quad a \circ b = \frac{1}{2} \langle \xi^{-1}(\xi(a) \xi(b)) + \xi(\xi(b) \xi(a)) \rangle
\]

holds for every \(a, b \in t\) (cf. Theorem 3 in p. 389 of [13]). In the present note, the algebra \((t, \circ)\) thus obtained is called the connection algebra of \(V = V(\mathfrak{H})\). It is known in Proposition 1 of Shima [8] that the curvature tensor \(R\) has the following expression:

\[
(2.11) \quad R(a, b, c) = b \circ (a \circ c) - a \circ (b \circ c)
\]

for every \(a, b, c \in t\).

### §3. Power-associativity.

In this section, \((t, \circ)\) always denotes the connection algebra of a homogeneous convex cone \(V = V(\mathfrak{H})\) in \(X(\mathfrak{H})\) given in §2. By making use of the results obtained in [9], [10] and [11], we will calculate a condition for the connection algebra \((t, \circ)\) to be power-associative in terms of the curvature tensor \(R\).

It is known in Albert [1] that a commutative algebra \((A, \circ)\) over \(R\) is power-associative if and only if the identity

\[
(3.1) \quad (a \circ a) \circ (a \circ a) = a \circ (a \circ (a \circ a))
\]

holds for every \(a \in A\). Therefore, by (2.11) and (3.1), the connection algebra \((t, \circ)\) is power-associative if and only if the identity

\[
(3.2) \quad R(a \circ a, a, a) = 0
\]

holds for every \(a \in t\).

From now on, we will prove two lemmas on the necessary conditions for the connection algebra to be power-associative. We first prove the following

**Lemma 1.** If the connection algebra \((t, \circ)\) is power-associative, then the equality \(n_i = n_j\) holds for every pair \((i, j)\) of indices \(i < j\) satisfying \(n_{ij} \neq 0\).

**Proof.** By (2.3) and (2.10), we have

\[
(3.3) \quad a \circ a = \xi^{-1}(\xi(a) \xi(a)) = \xi^{-1}((a + a^*)(a + a^*))
\]

for every \(a \in t\). Putting \(a = a_{ij} (\neq 0)\) in (3.3), we have
On Connection Algebras of Homogeneous Convex Cones

\[ a \odot a = \frac{1}{2} (a_{ij}a_{ij}^* + a_{ij}^*a_{ij}) = \mathbb{U}_{ij} + \mathbb{U}_{jj}. \]

By (2.4) and (2.7), we have
\[ \langle a_{ij}a_{ij}^*, e_i \rangle = 4 \text{Sp}((a_{ij}a_{ij}^*)e_i) = \frac{2}{\sqrt{n_i}} \text{Sp}(a_{ij}a_{ij}^*) = \frac{1}{\sqrt{n_i}} \|a_{ij}\|^2 \]
and
\[ \langle a_{ij}^*a_{ij}, e_i \rangle = \frac{1}{\sqrt{n_j}} \|a_{ij}\|^2. \]

By using the formulas (1) in Lemmas 3.1 or 3.2 of [10] and the formula (2.9), we get
\[ (3.4) \quad R(e_i, a_{ij}, e_i) = \frac{1}{4} \|a_{ij}\|^2 \left( \frac{1}{\sqrt{n_i n_j}} e_i - \frac{1}{n_i} e_i \right) \]
and
\[ R(e_j, a_{ij}, a_{ij}) = \frac{1}{4} \|a_{ij}\|^2 \left( \frac{1}{\sqrt{n_i n_j}} e_i - \frac{1}{n_j} e_j \right). \]

Therefore, by the condition (2.8), we have
\[ R(a \odot a, a, a) = \frac{1}{2} \langle a_{ij}a_{ij}^*, e_i \rangle R(e_i, a, a) + \langle a_{ij}^*a_{ij}, e_i \rangle R(e_j, a, a) \]
\[ = \frac{1}{2} \|a\|^2 \left( \frac{1}{\sqrt{n_i n_j}} R(e_i, a_{ij}, e_i) + \frac{1}{\sqrt{n_j}} R(e_j, a_{ij}, a_{ij}) \right). \]

From this and (3.2), we get
\[ R(a \odot a, a, a) = \frac{1}{8} \|a\|^2 \left( \frac{1}{n_j} - \frac{1}{n_i} \right) \left( \frac{1}{\sqrt{n_i n_j}} e_i - \frac{1}{\sqrt{n_j}} e_j \right) = 0, \]
which means \( n_i = n_j \).

We next show the following

**Lemma 2.** If the connection algebra \((\cdot, \odot)\) is power-associative, then the following two identities hold:

1. \[ \|a_{ij}^*a_{ij}\|^2 = \frac{1}{2n_i} \|a_{ij}\|^2 \|a_{ij}\|^2 \]
2. \[ \|a_{ij}a_{ik}\|^2 = \frac{1}{2n_k} \|a_{jk}\|^2 \|a_{ik}\|^2 \]

for every \( a_{ij} \in \mathbb{U}_{ij} \), \( a_{jk} \in \mathbb{U}_{jk} \) and \( a_{ik} \in \mathbb{U}_{ik} \) (\( i < j < k \)).

**Proof.** We first show the identity (1). Since the equality in (1) holds trivially for the case of \( n_i n_j = 0 \), we may assume that \( n_i n_j \neq 0 \). By Lemma 1,
we can put \( n_i = n_j = n_k = m \). Let us put \( a = a_{ij} + a_{ik} \) in (3.3). Then, by (2.3) and (2.10), we have

\[
a \circ a = x_{ii} + x_{jj} + x_{kk} + x_{jk},
\]

where

\[
x_{ii} = \frac{1}{2} (a_{ij} a^*_{ij} + a_{ik} a^*_{ik}), \quad x_{jj} = \frac{1}{2} a^2_{ij},
\]

\[
x_{kk} = \frac{1}{2} a^2_{ik} \quad \text{and} \quad x_{jk} = a^*_{ij} a_{ik}.
\]

Similarly as in the proof of Lemma 1, we have

\[
x_{ii} = \frac{1}{2} \sqrt{m} \|a\|^2 e_i \quad \text{and} \quad x_{pp} = \frac{1}{2} \sqrt{m} \|a_{ip}\|^2 e_p \quad (p = j, k).
\]

We now consider the \( \Xi_{ii} \)-component of \( R(a \circ a, a, a) \). Using a well-known identity on the curvature tensor (cf. the formula (1.14) of [11]), we get

\[
\langle R(a \circ a, a, a), e_i \rangle = -\langle R(a \circ a, a, e_i), a \rangle.
\]

From the condition (1.12) of [11] and the formula (2.9), it follows that the identity

\[
R(a \circ a, a, e_i) = R(x_{ii}, a_{ip}, e_i) + R(x_{ij}, a_{ik}, e_i) + R(x_{jk}, a_{ij}, e_i) + R(x_{kk}, a_{ik}, e_i) + R(x_{jk}, a_{ik}, e_i) + R(x_{ij}, a_{ik}, e_i) + R(x_{kk}, a_{ij}, e_i)
\]

holds. On the other hand, by using Lemmas 1.1 and 2.2 of [9], the formulas (2.9) and (3.5), we obtain the following formulas:

\[
R(x_{ii}, a_{ip}, e_i) = \frac{1}{8m \sqrt{m}} \|a\|^2 a_{ip}
\]

and

\[
R(x_{pp}, a_{ip}, e_i) = -\frac{1}{8m \sqrt{m}} \|a_{ip}\|^2 a_{ip} \quad (p = j, k).
\]

Furthermore, we have

\[
R(x_{jk}, a_{ij}, e_i) = -\frac{1}{4 \sqrt{m}} a_{ij} x_{jk} = -\frac{1}{4 \sqrt{m}} a_{ij} (a^*_{ij} a_{ik})
\]

and

\[
R(x_{jk}, a_{ik}, e_i) = -\frac{1}{4 \sqrt{m}} a_{ik} x_{jk}^* = -\frac{1}{4 \sqrt{m}} a_{ik} (a^*_{ik} a_{ij})
\]

(cf. the condition (1.14) of [11] and the formula used in the proof of Proposition 5.1 of [11]). Hence, from the conditions (2.5) and (2.6), it follows that

\[
\langle R(a \circ a, a, a), e_i \rangle = \frac{1}{2 \sqrt{m}} \left( \|a_{ij} a_{ik}\|^2 - \frac{1}{2m} \|a_{ij}\|^2 \|a_{ik}\|^2 \right)
\]
On Connection Algebras of Homogeneous Convex Cones

holds. From this, we have the equality (1).

We proceed to showing the equality (2). Similarly as in the above case, we may assume that $n_jk n_k a_i^k = 0$ and also we may put $n_i = n_j = n_k = m$. By putting $a = a_{jk} + a_{ki}$, we have

$$a \circ a = x_{kk} + x_{kj} + x_{kj} + x_{kj},$$

where

$$x_{kk} = \frac{1}{2 \sqrt{m}} \|a_k\|^2 e_k, \quad x_{kj} = \frac{1}{2 \sqrt{m}} \|a_{jk}\|^2 e_j,$$

$$x_{jj} = \frac{1}{2 \sqrt{m}} \|a\|^2 e_j \quad \text{and} \quad x_{ij} = a_{ik} a_{jk}.$$ 

Similarly as in the above case, we have

$$R(a \circ a, a, e_k) = \frac{1}{2 \sqrt{m}} \|a_{ik}\|^2 \|e_i\|^2 R(e_i, a_{ik}, e_k) + \frac{1}{2 \sqrt{m}} \|a_{jk}\|^2 \|e_j\|^2 R(e_j, a_{jk}, e_k)$$

$$+ \frac{1}{2 \sqrt{m}} \|a\|^2 \left( R(e_k, a_{jk}, e_k) + R(e_k, a_{ik}, e_k) \right)$$

$$+ R(x_{ij}, a_{jk}, e_k) + R(x_{ij}, a_{ik}, e_k).$$

By using the following formulas (cf. Lemmas 1.1 and 2.2 of [9] and the condition (2.9)):

$$R(e_i, a_{ik}, e_k) = - R(e_k, a_{ik}, e_k) = \frac{-1}{4} a_{ik},$$

$$R(x_{ij}, a_{jk}, e_k) = \frac{-1}{4 \sqrt{m}} x_{ij} a_{jk} = \frac{-1}{4 \sqrt{m}} (a_{ik} a_{jk}^*) a_{jk},$$

and

$$R(x_{ij}, a_{ik}, e_k) = \frac{-1}{4 \sqrt{m}} x_{ij} a_{ik} = \frac{-1}{4 \sqrt{m}} (a_{jk} a_{ik}^*) a_{ik},$$

we have

$$\langle R(a \circ a, a, a), e_k \rangle = \frac{1}{2 \sqrt{m}} \left( \|a_{ik} a_{jk}^*\|^2 - \frac{1}{2m} \|a_{ik}\|^2 \|a_{jk}\|^2 \right).$$

Therefore, by (3.2), $\|a_{ik} a_{jk}^*\|^2 = (1/2m) \|a_{ik}\|^2 \|a_{jk}\|^2$ holds. q.e.d.

§ 4. Main result.

In this section, we prove the theorem stated in § 1 by making use of the lemmas obtained in § 3.

We now have the following

**Theorem 1.** If the connection algebra of a homogeneous convex cone $V$ is power-associative, then $V$ is self-dual.
Proof. By the result of Vinberg [13] recalled in §2, we can assume that $V$ is realized as the cone $V(\mathfrak{A})$ in terms of a $T$-algebra $\mathfrak{A} = \sum_{i \neq j} \mathfrak{A}_{ij}$. We first show that the equality $n_{ik} = n_{jk}$ holds for every triple $(i, j, k)$ of indices $i \neq j \neq k$ satisfying the condition $n_{ij} \neq 0$. In fact, let us consider the case of $i \neq j < k$. Then, by (1) of Lemma 2, the linear mapping: $x \in \mathfrak{A}_{ik} \rightarrow a_{ij}^* x \in \mathfrak{A}_{jk}$ is injective for an arbitrary non-zero element $a_{ij} \in \mathfrak{A}_{ij}$. Hence, we have $n_{ik} \leq n_{jk}$. Combining this with the condition (2.2), we get the equality $n_{ik} = n_{jk}$. We proceed to the case of $i < k < j$. By (1) and (2) of Lemma 2, we can see that both of the linear mappings:

$$x \in \mathfrak{A}_{ik} \rightarrow x^* a_{ij} \in \mathfrak{A}_{kj} \quad \text{and} \quad y \in \mathfrak{A}_{kj} \rightarrow a_{ij} y^* \in \mathfrak{A}_{ik}$$

are injective for every non-zero element $a_{ij} \in \mathfrak{A}_{ij}$. Therefore, we have the equality $n_{ik} = n_{jk}$. Finally, we consider the case of $k < i < j$. Similarly as in the above cases, by using (2) of Lemma 2, we can easily see that the equality $n_{ik} = n_{jk}$ holds in this case. Therefore, the kernel of the $T$-algebra $\mathfrak{A}$ coincides with $\mathfrak{A}$ (cf. p. 69 of Vinberg [14] or Lemma 2.2 of [11]). On the other hand, it is known in [14] that $V = V(\mathfrak{A})$ is self-dual if and only if the kernel of $\mathfrak{A}$ coincides with $\mathfrak{A}$. Hence, $V$ is self-dual.

q.e.d.

Several characterizations of homogeneous self-dual cones are known. Combining the result obtained above with them, we can state the following

Theorem 2. For a homogeneous convex cone $V$ in $X = \mathbb{R}^n$, the following six conditions are equivalent:

1. The connection algebra of $V$ is power-associative.
2. $V$ is self-dual.
3. The connection algebra of $V$ is Jordan.
4. $V$ is Riemannian symmetric with respect to the canonical metric $g_V$.
5. The tube domain $D(V) = \{ z \in \mathbb{C}^n ; \text{Im} z \in V \}$ is Hermitian symmetric with respect to the Bergman metric of $D(V)$.
6. The level surface of the characteristic function of $V$ is Riemannian symmetric with respect to the metric induced from $(V, g_V)$.

In fact, the implications (2)$\rightarrow$(3)$\rightarrow$(1) have been proved by [3] and (4)$\rightarrow$(2) has been obtained in [8], [9] or [11]. It is known in [5], [6] that the conditions (2) and (5) are equivalent and the condition (2) implies the condition (4). The implications (4)$\rightarrow$(6) are found in [10]. By Theorem 1, we have the implication (1)$\rightarrow$(2) (For (3)$\rightarrow$(2), see also [2]), and so the conditions stated above are mutually equivalent.
On Connection Algebras of Homogeneous Convex Cones

Theorem 3. For a homogeneous convex cone $V$ in $R^n$ ($n \geq 2$), the following three conditions are equivalent.

1. The connection algebra of $V$ is associative.
2. The curvature tensor for the canonical metric $g_V$ is identically zero.
3. $V$ is linearly isomorphic to the product cone of the half-lines of positive real numbers.

Proof. As was stated in §2, we can assume that $V$ is realized as the cone $V(\mathcal{A})$ by means of a $T$-algebra $\mathcal{A} = \sum_{i=1}^{r} \mathcal{A}_i$ of rank $r$. The implications $(1) \rightarrow (2)$ follow from the formula due to Shima [8] recalled by (2.11). The condition (3) implies that $V$ is isometric to the product Riemannian manifold of the half-lines of positive real numbers. Hence, we get $(3) \rightarrow (2)$. By the formula (3.4) in the proof of Lemma 1, we can see that the condition (2) implies $n_{ij} = 0$ for every pair $(i, j)$ of indices $1 \leq i < j \leq r$. Hence, $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_r$. From this and the construction theorem of homogeneous convex cones due to Vinberg [13] recalled in §2, it follows that the implication $(2) \rightarrow (3)$ holds. q.e.d.

References


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