REGULAR GAMMA RINGS

By
Shoji KYUNO, Nobuo NOBUSAWA and Mi-Soo B. SMITH

0. Introduction

Let $M$ and $\Gamma$ be additive abelian groups. If for all $a,b,c \in M$ and $\alpha, \beta, \gamma \in \Gamma$, the conditions

1. $aab \in M$, $\alpha \beta \in \Gamma$;
2. $(a+b)ac = aac + bac$, $a(\alpha + \beta)b = aab + a\beta b$, $a\alpha(b + c) = aab + aac$,
   $(a + \beta)\alpha = a\alpha + \beta \alpha$, $\alpha(a + b)\beta = a\alpha \beta + ab \beta$, $\alpha(\beta + \gamma) = a\alpha \beta + a\alpha \gamma$,
3. $(aab)\beta c = a(a \beta b)c = a\alpha \beta \gamma$, $(aa\beta)\beta \gamma = a(\alpha \beta b)\gamma = a\alpha (\beta \gamma)$,

are satisfied, then $M$ is called a weak gamma ring in the sense of Nobusawa and denoted by $(\Gamma, M)_{wn}$. In this note $(\Gamma, M)$ denotes $(\Gamma, M)_{wn}$, unless otherwise specified.

A gamma ring $(\Gamma, M)$ is regular if for each $a \in M$ there exists $\delta \in \Gamma$ such that $a \delta a = a$. For a left $R$-module $M$, letting $\Gamma = \text{Hom}_R(M, R)$, we have a gamma ring $(\Gamma, M)$. A left $R$-module $M$ is called regular, if for any element $m \in M$ there exists $f \in \text{Hom}_R(M, R)$ with $(mf)m = m$, [8]. Thus, the concept of regular gamma rings is a natural generalization of regular modules.

In this note, we study various properties of regular gamma rings. In 1, we obtain a couple of necessary and sufficient conditions that $(\Gamma, M)$ is regular, and then characterize a commutative regular Nobusawa gamma ring as a subdirect sum of gamma fields (Th. 1.7).

In 2, we define a regular ideal and prove a basic theorem: If $J \subseteq K$ are two ideals in $M$, then, $K$ is regular if and only if $J$ and $K/J$ are both regular (Th. 2.2). If $\mathcal{R}$ is the class of all regular gamma rings, then this theorem shows that $\mathcal{R}$ is a radical class. Next, we introduce the concept of a weakly nilpotent element, and we obtain that a non-zero subdirectly irreducible regular gamma ring with no non-zero weakly nilpotent elements is a division gamma ring (Th. 2.11).

In 3, we obtain relations among the regularities of the operator rings $L, R$ and a gamma ring $(\Gamma, M)$ as follows: If $(\Gamma, M)$ has the left and right unities, then the following conditions are equivalent: (1) $L$ is regular; (2) $R$ is regular; (3) $M$ is regular (Th. 3.2). By this theorem, we have that, when Mod-$R \cong$

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Mod-\(L\), \(R\) is regular if and only if \(L\) is regular (Corollary 3.5). Furthermore, we show that if \((T, M)\) is a semi-prime gamma ring with min-\(r\) and min-\(l\) conditions, every left (right) \(L\)-module and every left (right) \(R\)-module are regular. In particular, \(L\), \(M\) and \(R\) are regular (Th. 3.8).

In 4, we consider the regularity of a Morita context \((Q, R, S, T, \mu, \nu)\), where \(\mu, \nu\) are surjective. Here, it is not assumed that \(Q, R\) have unities nor that \(S, T\) are unital. We obtain an extension (Th. 4.1) of Theorem 3.2.

For the definitions of the following basic notions in gamma rings we refer, respectively, to [3] for the right operator ring \(R\), the left operator ring \(L\), a right (left, two-sided) ideal of \(M\), \([a]\), \([N, \Phi]\), where \(N \subseteq M\) and \(\Phi \subseteq \Gamma\) and to [4] for semiprime ideals, nilpotent elements, the right unity and the left unity.

1. Regular Gamma Rings.

1.1 Definition. A gamma ring \((\Gamma, M)\) is regular if for each \(x \in M\) there exists \(\delta \in \Gamma\) such that \(x \delta x = x\). We abbreviate this as \(M\) is regular, when \(\Gamma\) is understood.

1.2 Theorem. For a gamma ring \((\Gamma, M)\) with the left and right unities, the following conditions are equivalent:

(1) \((\Gamma, M)\) is regular.

(2) Every principal right ideal of \(M\) is generated by an idempotent of the left operator ring \(L\).

(2') Every principal left ideal of \(M\) is generated by an idempotent of the right operator ring \(R\).

(3) Every finitely generated right ideal of \(M\) is generated by an idempotent of the left operator ring \(L\).

(3') Every finitely generated left ideal of \(M\) is generated by an idempotent of the right operator ring \(R\).

Proof. We note that for any \(a \in M\), \([a] = a\Gamma M\), since \([a] = Za + a\Gamma M \subseteq a\Gamma M\). (\(Z\) is the set of all integers.)

(1) \(\Rightarrow\) (2): Suppose that for each \(a \in M\) there exists \(\delta \in \Gamma\) such that \(a \delta a = a\). Then \([a, \delta] = [a, \delta] = [a, \delta]\) and so \([a, \delta]\) is an idempotent in \(L\). Since \(a\Gamma M = a\delta a\Gamma M \subseteq a\delta M, a\Gamma M = a\delta M\). Thus, \([a] = a\delta M\).

(2) \(\Rightarrow\) (3): It suffices to show that for any \(a, b \in M\), \([a] + \|b\| = tM\), where \(t\) is an idempotent in \(L\). By (2), \([a] = hM, h^2 = h \in L\), and \([b] = fM\) where \(f^2 = f \in L\). Then, since \(b - hb \in fM + hM, |b - hb| \subseteq fM + hM\), and so \(hM + \|b - hb\| \subseteq hM + fM\). On the other hand, \(b = hb + (b - hb) \in hM + \|b - hb\|\), whence \(fM = \|b\| \subseteq hM + \|b - hb\|\). Thus, \(hM + fM \subseteq hM + \|b - hb\|\). Therefore, \(hM + fM = hM + \|b - hb\|\).
Again by (2) \(|b-hb|=sM\), where \(s^2=s\in L\). Then, \(hsM=h|b-hb|=0\), and it follows that \(hs=hs^2=0\). So if \(g=s-sh\), then \(g\) is an idempotent and orthogonal to \(h\). Since \(sg=g\) and \(gs=s\), we see that \(gM=sM=|b-hb|\). Therefore, \(|a|+|b|=hM+gM\). Since \(h\) and \(g\) are orthogonal, we have \(|a|+|b|=(h+g)M\).

\((3)\Rightarrow(1):\) Suppose that for any \(x\in M\), \(|x|=hM\), where \(h^2=h\in L\). Then, \(x=h^2y=h(y)=hx\), where \(y\in M\). On the other hand, \(hL=[hM, \mathcal{J}]=[|x|, \mathcal{J}]\). Again \(\mathcal{J}=[Zx+x\mathcal{J}M, \mathcal{J}]\in[x, \mathcal{J}]\), which implies \(h=h^2=[x, \delta]\), where \(\delta\in\mathcal{J}\). Hence \(x=hx=[x, \delta]x=x\delta x\). 

1.3 Definition. A gamma ring \((\mathcal{J}, M)\) is right semi-hereditary if every finitely generated right ideal of \(M\) is a projective \(R\)-module. A right ideal \(I\) in \(M\) is called essential if for every non-zero right ideal \(A\) in \(M\), \(I\cap A \neq \emptyset\). Let \(\varphi(M)\) be the set of all essential right ideals in \(M\), and \(Z_r(M)=\{x\in M|x\mathcal{J}I=0\text{ for some }I\in\varphi(M)\}\). \((\mathcal{J}, M)\) is called a right nonsingular gamma ring if \(Z_r(M)=0\). Similarly, a left semi-hereditary gamma ring and a left nonsingular gamma ring are defined.

1.4 Corollary. Let \((\mathcal{J}, M)\) be a regular gamma ring. Then

1. All one-sided ideals in \(M\) are idempotent.
2. All two-sided ideals in \(M\) are semi-prime.
3. The Jacobson radical of \(M\) is zero.
4. \((\mathcal{J}, M)\) with the left and right unities is right and left semi-hereditary.
5. \((\mathcal{J}, M)\) is right and left nonsingular.

Proof. Let \(J\) be a right ideal of \(M\). Since \(M\) is regular, for each \(x\in J\) \(x\gamma x=x\) for some \(\gamma\in \mathcal{J}\). Consequently, \(x=x\gamma x\in J\gamma J\) and so \(J=J\gamma J\). Thus, we have (1).

Let \(I\) be a two-sided ideal of \(M\). If \(A\) is a two-sided ideal in \(M\) such that \(A\gamma A\subseteq I\), then \(A\subseteq I\), because by (1) \(A=A\gamma A\). Hence we have (2).

To show (3), suppose that \(e\) is right quasi-regular and \(e=\delta e\). Then, there exists \(r\in R\) such that \([\delta, e]r=r+[\delta, e]-[\delta, e]r=0\). It follows \([\delta, e]=[\delta, e]=0\)

\([\delta, e]r=(\delta, e)([\delta, e]r)+([\delta, e]r)=0\). Thus, \(e=\delta e\). Recall that \(J(M)=\{e\in M|\langle e\rangle\text{ is right quasi-regular}\}\. Since \(\langle e\rangle=0, e=0\) and so \(J(M)=0\).

Now we prove (4). By Theorem 1.2.(3), every finitely generated right ideal in \(M\) may be written as \(hM\), where \(h^2=h\in L\). Let \(A=\{x\in M|xh=0\}\). Clearly \(A\) is a right ideal in \(M\). For any \(x\in M\), \(x=hx+(x-hx)\), and \(M=hM\oplus A\), because if \(a\in hM\cap A\) then \(a=ha=0\). Thus, \(hM\) is a direct summand of \(M\) and
so every finitely generated right ideal in \( M \) is a projective \( R \)-module. Similarly it can be proved that \( (\gamma, M) \) is left semi-hereditary.

For (5), let \( J \) be an essential right ideal in \( M \). Suppose that \( a\gamma J = 0 \) for some \( a \in M \), and that there exists \( \delta \in \gamma \) such that \( a\delta a = a \). Then, \( a\delta M \cap J = 0 \), for if \( x \in a\delta M \cap J \) then \( x = a\delta x = 0 \). Since \( J \) is essential, \( a\delta M = 0 \) and so \( a = 0 \). Similarly we obtain the same result for left ideals. \( \Box \)

Given an ideal \( I \) in \( M \), we form a residue class gamma ring \( (\gamma/I^*, M/I) \), where \( I^* = \{ \gamma \in \gamma | M/I \leq I \} \).

1.5 Theorem. A gamma ring \( (\gamma, M) \) is regular if and only if the following (1), (2) and (3) hold.

1. \( M \) is semi-prime,

2. The union of any chain of semi-prime ideals of \( M \) is semi-prime,

3. \( M/P \) are regular for all prime ideals \( P \) of \( M \).

Proof. Let \( M \) be regular. Corollary 1.4 (2) shows that all ideals in \( M \) are semi-prime, whence (1) and (2) hold. (3) obviously holds, for, \((x + P)(\gamma + P^*) = x\gamma + P = x + P\).

Conversely, assume that (1), (2) and (3) hold. If \( M \) is not regular, then there is \( a \in M \) such that \( a \in a\gamma a + I \). By (2), there is a semi-prime ideal \( I \) in \( M \) which is maximal among semi-prime ideals such that \( a \in a\gamma a + I \). Note that \([0] \) is a semi-prime ideal of \( M \) such that \( a \in a\gamma a + [0] \). \( M/I \) is not regular, because otherwise, for any \( x \in M \), \((x + I)(\gamma + I^*)(x + I) = x + I \) would imply \( x \in x\gamma x + I \), a contradiction. Hence, by (3) \( I \) is not prime. Thus, there are ideals \( A \) and \( B \) which properly contain \( I \) and \( A\gamma B \leq I \). Indeed, since \( A \not\subseteq I \) and \( B \not\subseteq I \), \( A \not\subseteq A + I \) and \( I \not\subseteq B + I \). If we set \( A + I = A' \) and \( B + I = B' \), then \( A'\gamma B' = A\gamma B + I \subseteq I + I = I \) and \( I \subseteq A' \) and \( I \subseteq B' \). Thus, we can take \( A', B' \) instead of \( A', B' \) from the beginning. Now set \( P = \{ x \in M | x\gamma B \subseteq I \} \) and \( Q = \{ x \in M | P\gamma x \subseteq I \} \). Since \( I \) is semi-prime, \( P \) and \( Q \) are semi-prime. For, \( K\gamma K \subseteq P \Rightarrow K\gamma K\gamma B \subseteq I \Rightarrow K\gamma B \gamma K \gamma B \subseteq K \gamma B \subseteq I \Rightarrow K \gamma B \subseteq I \Rightarrow K \subseteq P \), and \( U \gamma U \subseteq Q \Rightarrow P \gamma U \gamma U \subseteq I \Rightarrow P \gamma U \gamma U \subseteq P \gamma U \gamma U \subseteq I \Rightarrow P \gamma U \subseteq I \Rightarrow U \subseteq Q \).

Since \((P \cap Q)\gamma (P \cap Q) \subseteq P\gamma Q \subseteq I \), we have \( P \cap Q \subseteq I \). Clearly, \( A \subseteq P \) and \( B \subseteq Q \), and hence \( P \) and \( Q \) properly contain \( I \). By the maximality of \( I \), there exist elements \( \gamma, \omega \in \gamma \) such that \( a - a\gamma a \in P \) and \( a - a\omega a \in Q \). Then, \( a - a(\gamma + \omega - \gamma a) = a - a\gamma a - (a - a\gamma a) = a - a\omega a - a\omega = a - a\omega a \in P \). Also \( a - a(\gamma + \omega - \gamma a) = a - a\omega a - a\omega = a - a\omega a \in Q \).

It follows that \( a \in a\gamma + P \cap Q \subseteq a\gamma a + I \), which is a contradiction. Hence, \( M \) is regular. \( \Box \)

1.6 Corollary. A gamma ring \( (\gamma, M) \) is regular if and only if all ideals
of $M$ are idempotent and $M/P$ are regular for all prime ideals $P$ of $M$.

PROOF. If all ideals of $M$ are idempotent, all ideals of $M$ are semi-prime. □

1.7 THEOREM. A commutative regular Nobusawa gamma ring with more than one element is a subdirect sum of gamma fields.

PROOF. A regular gamma ring has no non-zero nilpotent elements. For, suppose $(a\gamma)^n a = 0$ for any $\gamma \in \Gamma$. Then we have $a = (a\delta)^n a = 0$ since there exists $\delta \in \Gamma$ such that $a = a\delta a$. A homomorphic image of a regular gamma ring is regular, and so it has no non-zero nilpotent elements. Then, the theorem follows immediately from Theorems 3 and 4 in [5]. □

2. Regular Ideals

2.1 DEFINITION. A two-sided ideal $J$ in $M$ is regular if for each $x \in J$ there exists $\gamma \in J^*$ such that $x\gamma x = x$, where $J^* = \{\gamma \in \Gamma | M\gamma M \subseteq J\}$.

2.2 THEOREM. Let $J \subseteq K$ be two-sided ideals in $M$. Then $K$ is regular if and only if $J$ and $K/J$ are both regular.

PROOF. Let $J^* = \{\gamma \in \Gamma | M\gamma M \subseteq J\}$ and $K^* = \{\gamma \in \Gamma | M\gamma M \subseteq K\}$. Then $(J^*, J)$, $(K^*, K)$ and $(K^*/J^*, K/J)$ are gamma rings. Suppose that $K$ is regular. For each $k \in K$ there exists $\gamma \in K^*$ such that $k\gamma k = k$. Thus, $(k+j)(\gamma+j^*) = k\gamma k + J = k + J$ and so $K/J$ is regular.

Given $x \in J$, we have $x\delta x = x$ for some $\delta \in K^*$, since $J \subseteq K$. Then, $\omega = \delta x\delta x \in J^*$, for $M\omega M = M\delta x\delta x M \subseteq J\delta M \subseteq J$. Hence, $x\omega x = x\delta x\delta x = x\delta x = x$, and so $J$ is regular.

Conversely, assume that $J$ and $K/J$ are both regular. For a given $a \in K$, $a + J = (a + J)(\gamma + J^*) = a\gamma a + J$, where $\gamma \in K^*$ from the regularity of $K/J$. Hence, $a - a\gamma a \in J$, for some $\gamma \in K^*$. Consequently, $a - a\gamma a = (a - a\gamma a)\omega(a - a\gamma a)$, where $\omega \in J^*$. Then,

$a = a - a\gamma a + a\gamma a$

$= (a - a\gamma a)\omega(a - a\gamma a) + a\gamma a$

$= a(\omega - \gamma \omega)(a - a\gamma a) + a\gamma a$

$= a(\omega - \gamma \omega - \omega \gamma a + \gamma a\omega a) + a\gamma a$

$= a(\omega - \gamma \omega - \omega \gamma a + \gamma a\omega a + \gamma) a$

$= a\lambda a$, where $\lambda = \omega - \gamma \omega - \omega \gamma a + \gamma a\omega a + \gamma \in K^*$,

because $J^* \subseteq K^*$ and $K^*$ is an ideal in $\Gamma$.

Therefore, $K$ is regular. □

2.3 REMARK. Let $\mathcal{R}$ be the class of all regular gamma rings. Theorem 2.2 shows that $\mathcal{R}$ is a radical class, since other two conditions: $\mathcal{R}$ is homomorphically
closed and \( R \) has the inductive property are trivially satisfied.

(See, for instance, [7]) In fact, a radical \( N \) for any gamma ring \( (\Gamma, M) \) may be defined by the conditions in Proposition 2.6.

**2.4 Proposition.** Any finite subdirect sum of regular Nobusawa gamma rings is regular.

**Proof.** It suffices to show that a subdirect sum of two regular Nobusawa gamma rings is regular. Suppose that \( M \) has two ideals \( J \) and \( K \) such that \( J \cap K = 0 \). Then \( J^* \cap K^* = 0 \), where \( J^* = \{ \gamma \in \Gamma | M \gamma M \subseteq J \} \) and \( K^* = \{ \gamma \in \Gamma | M \gamma M \subseteq K \} \). For, if \( \gamma \in J^* \cap K^* \), then \( M \gamma M \subseteq J \cap K = 0 \) and \( \gamma = 0 \). Let the gamma rings \( (\Gamma/J^*, M/J) \) and \( (\Gamma/K^*, M/K) \) be both regular. Consider the homomorphism

\[
(\varphi, \theta) : (J^*, J) \rightarrow (J^* + K^*/K^*, J + K/K)
\]

where

- \( \theta \) is the natural epimorphism: \( J \rightarrow J + K/K, \ x \theta = x + K \) and \( \text{Ker } \theta = J \cap K = 0 \),
- \( \varphi \) is the natural epimorphism: \( J^* \rightarrow J^* + K^*/K^*, \ a \varphi = a + K^* \) and \( \text{Ker } \varphi = J^* \cap K^* = 0 \).

Then

\[
(axy) \theta = axy + K = (x + K)(a + K^*)(y + K) = x \varphi \varphi y \theta, \text{ and } (ax \beta) \varphi = a x \beta + K^* = (a + K^*)(x + K)(\beta + K^*) = a \varphi x \beta \varphi \varphi. \]

Hence, \( (\varphi, \theta) \) is an isomorphism from \( (J^*, J) \) onto \( (J^* + K^*/K^*, J + K/K) \). Since \( J + K/K \) is an ideal in \( M/K \), \( J + K/K \) is regular. Theorem 2.2 shows \( J \) is regular. Hence, \( J \) and \( M/J \) are regular, and again by Theorem 2.2 \( M \) is regular. \( \Box \)

**2.5 Remark.** A subdirect sum of infinitely many regular Nobusawa gamma rings need not be regular. For example, \( (Z, Z) \) is the subdirect sum of infinitely many regular Nobusawa gamma rings \( (Z/(p), Z/(p)) \), where \( p \) runs through all prime numbers.

**2.6 Proposition.** For a gamma ring \( (\Gamma, M) \), set \( N = \{ x \in M | \langle x \rangle \text{ is regular} \} \).

Then,

1. \( N \) is a regular ideal in \( M \),
2. \( N \) contains all regular ideals of \( M \),
3. \( M/N \) has no non-zero regular ideals.

**Proof.** Let \( x, y \in N \). Then \( \langle y \rangle \) is regular and \( \langle x \rangle + \langle y \rangle / \langle y \rangle \) is regular. Hence by Theorem 2.2 \( \langle x \rangle + \langle y \rangle \) is regular. For any \( a \in \langle x \rangle + \langle y \rangle \), \( \langle a \rangle \subseteq \langle x \rangle + \langle y \rangle \). Theorem 2.2 shows \( \langle a \rangle \) is regular, and so \( a \in N \). Thus, \( \langle x \rangle + \langle y \rangle \subseteq N \), whence \( N \) is an ideal in \( M \). For any \( x \in N \), since \( \langle x \rangle \) is regular, there exists
\[ \delta \in \langle x \rangle^*, \text{ where } \langle x \rangle^* = \{ \gamma \in \Gamma | M\gamma M \subseteq \langle x \rangle \}, \text{ such that } x\delta x = x. \]  
Since \( N \) is an ideal, \( \langle x \rangle \subseteq N \) and then \( \langle x \rangle^* \subseteq N^* \). Thus, \( \delta \in N^* \) and \( N \) is regular. This completes the proof of \((1)\).

To prove \((2)\), let \( A \) be any regular ideal in \( M \). For any \( a \in A \), \( \langle a \rangle \subseteq A \). Thus, by Theorem 2.2, \( \langle a \rangle \) is regular and so \( a \in N \). Hence \( A \subseteq N \).

If \( A/N \) is a non-zero regular ideal in \( M/N \), \( A \) is regular by Theorem 2.2, and \( A \) contains \( N \) properly, which contradicts to \((2)\). \( \square \)

2.7 DEFINITION. An element \( a \in M \) is said to be a weakly nilpotent element if there exist a non-zero element \( r \in \mathfrak{r} \) and an integer \( n > 1 \) such that \( (ar)^n = a = 0 \).

2.8 PROPOSITION. In a gamma ring \((\Gamma, M)\) with no non-zero weakly nilpotent elements, every idempotent commutes with every element in \( M \).

PROOF. Let \( e \delta e = e \), \( \delta \in \Gamma \), and \( x \in M \). If \( e = 0 \), \( e \delta x = 0 = x \delta e \). Suppose \( e \neq 0 \). Then \( \delta \neq 0 \). Since
\[
(e \delta x - e \delta x \delta e)(e \delta x - e \delta x \delta e) = (e \delta x - e \delta x \delta e)([\delta, x] - [\delta, x \delta e]) = 0 \text{ and } (\Gamma, M) \text{ has no non-zero weakly nilpotent elements, } e \delta x - e \delta x \delta e = 0 \text{ or } e \delta x = x \delta e. \]
Similarly, \( x \delta x = x \delta e \), and so \( e \delta x = x \delta e \). \( \square \)

2.9 PROPOSITION. Let \((\Gamma, M)\) be a regular gamma ring with no non-zero weakly nilpotent elements. Then

(1) Every principal one-sided ideal is generated by an idempotent which commutes with any element in \( M \).

(2) Every one-sided ideal is a two-sided ideal.

PROOF. Let \( a = \delta a \) for some \( \delta \in \Gamma \). Then, \( |a| = za + a \gamma M = a[\delta, z] + a \gamma M = a \gamma M = \delta a \delta a \gamma M \subseteq a \delta M \), and hence \( |a| = \delta a \). Proposition 2.8 shows that \( a \) commutes with any element in \( M \). Thus we have \((1)\).

To prove \((2)\), let \( A \) be a right ideal in \( M \). For any \( a \in A \), \( a \delta M \subseteq A \), where \( a \delta a = a \) for some \( \delta \in \Gamma \). By Proposition 2.8 \( a \delta M = M \delta a \). Since \( M \delta a = M \Gamma a \), \( M \Gamma a \subseteq A \), and so \( A \) is a left ideal. \( \square \)

2.10 DEFINITION. A gamma ring \((\Gamma, M)\) is said to be a division gamma ring if \((\Gamma, M)\) has the strong left unity \([e, \delta]\) and the strong right unity \([\delta, e]\), and if for each non-zero element \( a \in M \) there exists \( b \in M \) such that \( a \delta b = b \delta a = e \). A gamma ring \((\Gamma, M)\) is said to be subdirectly irreducible if the intersection of all non-zero ideals of \( M \) is not zero.

2.11 THEOREM. A non-zero subdirectly irreducible regular gamma ring with no non-zero weakly nilpotent elements is a division gamma ring.
Shoji Kyuno, Nobuo Nobusawa and Mi-Soo B. Smith

PROOF. Let \((\Gamma, M)\) be a non-zero subdirectly irreducible regular gamma ring with no non-zero weakly nilpotent elements. For each non-zero element \(e \in M\) there exists \(\delta \in \Gamma\) such that \(e\delta e = e\). Proposition 2.8 shows that for any \(x \in M\) \(e\delta x = x\delta e\). Let us consider two ideals \(e\delta M\) and \(A = \{x - e\delta x | x \in M\}\), whose intersection is zero. \(M\) is subdirectly irreducible, so \(e\delta M = 0\) or \(A = 0\). But \(e\delta M + 0\), hence \(A = 0\), and thus \(e\delta x = x\delta e = x\). This means that \([e, \delta]\) and \([\delta, e]\) are the strong left and right unities, respectively. Let \(a\) be a non-zero element of \(M\). Then, there exists \(\omega \in \Gamma\) such that \(a\omega a = a\). By the observation made above, \(a\omega x = x = a\omega a\) for any \(x \in M\) and so \(a\omega e = e = a\omega a\). Therefore, \((\Gamma, M)\) is a division gamma ring. \(\square\)

3. Relations among the regularities of the operator rings and a gamma ring.

Assuming the existence of the left and right unities in a gamma ring \((\Gamma, M)\), we prove that the left (right) operator ring \(L(R)\) is regular if and only if \(M\) is regular. From this, we can conclude that the regularity may be considered one of Morita invariants.

For a ring \(A\) we prepare the following:

3.1 PROPOSITION. For a ring \(A\) with the unity, the following conditions are equivalent:

1. \(A\) is regular.
2. Every principal right (left) ideal of \(A\) is generated by an idempotent.
3. Every finitely generated right (left) ideal of \(A\) is generated by an idempotent.

The proof is analogous to the proof of Theorem 1.2. \(\square\)

3.2 THEOREM. Suppose \((\Gamma, M)\) has the left and right unities. Then, following conditions are equivalent:

1. \(L\) is regular.
2. \(R\) is regular.
3. \(M\) is regular.

PROOF. (2) \(\Rightarrow\) (3): Suppose that \(R\) is regular and let \(MI'm\), where \(m \in M\), be a principal left ideal of \(M\). We shall show that there exists \(e \in R\) such that \(e^2 = e\) and \(MI'm = Me\). Let \(1_L = \sum [e_i, \delta_i]\), where \(e_i \in M\), \(\delta_i \in \Gamma\). Then, \(\Gamma = \sum [e_i, \delta_i] = \sum \Gamma e_i \delta_i \subseteq \sum R\delta_i\). Clearly, \(\sum R\delta_i \subseteq \Gamma\). Hence \(\Gamma = \sum R\delta_i\). So, \([\Gamma, m] = \sum R\delta_i\), where \(r_i = [\delta_i, m] \in R\). Since \(R\) is regular by Proposition 3.1 \(\sum Rr_i = Re\), with \(e \in R\), \(e^2 = e\). Now, \(MI'm = MRe = Me\), as required. By Theorem 1.2, \(M\) is regular.
Suppose that $M$ is regular, and let $Rr$ be a principal left ideal of $R$. Let $1_r = \sum \epsilon_i f_i$, where $\epsilon_i \in \Gamma$ and $f_i \in R$. Then, $M = M(1_r) = \sum (Ms_i)f_i \subseteq \sum Lf_i$. Since $\sum Lf_i \subseteq M$, we have $M = \sum Lf_i$. Then, $M_r = \sum Lm_j$, where $m_j = f_jr \in R$. Therefore, $Rr = \Gamma M_r = \Gamma Me = Re$. By Proposition 3.1, $R$ is regular.

$(1) \Longleftrightarrow (3)$ is proved analogously. \(\Box\)

### 3.3 Corollary

Suppose $(\Gamma, M)$ has the left and right unities, and $R$ and $L$ are the right and left operator rings, respectively. Then, for any positive integers $m, n$, $R_n$ and $L_m$ denote the total matrix rings of $n \times n$ matrices over $R$ and of $m \times m$ matrices over $L$, respectively.

**Proof.** Consider the matrix gamma ring $(\Gamma_{n,m}, M_{m,n})$ over $(\Gamma, M)$. Then $R_n = [\Gamma_{n,m}, M_{m,n}]$ and $L_m = [M_{m,n}, \Gamma_{n,m}]$ are the right and left operator rings of $(\Gamma_{n,m}, M_{m,n})$, respectively. \(\Box\)

### 3.4 Remark

In Corollary 3.3, put $m = 1$, then $R_n$ is regular if and only if $L$ is regular. Also we know $L$ is regular if and only if $R$ is regular. Hence, we have $R_n$ is regular if and only if $R$ is regular. Likewise, $R_n$ is regular if and only if $M_m, n$ is regular, and $R$ is regular if and only if $M$ is regular. Hence, $M$ is regular if and only if $M_m, n$ is regular.

Now, let $R$ and $R'$ be ordinary rings with the unities. Suppose the categories $\text{Mod-}R$ and $\text{Mod-}R'$ are equivalent, written $\text{Mod-}R \cong \text{Mod-}R'$. Then, there exist bimodules $\nu R, \mu P'$ and a Morita context $(R, R', P, P', \tau, \mu)$ for which $\tau$ and $\mu$ are surjective, so Morita I holds (see [2, p. 178]). Thus, $(P', P)$ forms a gamma ring having the right operator ring $R$ and the left operator ring $R'$. Thus, Theorem 3.2 shows the following:

### 3.5 Corollary

If $R$ and $R'$ are rings with the unities and $\text{Mod-}R \cong \text{Mod-}R'$, then $R$ is regular if and only if $R'$ is regular.

By this corollary, the regularity may be considered as one of Morita invariants.

### 3.6 Definition

A left $R$-module $M$ is called regular if, given any element $m \in M$, there exists $f \in \text{Hom}_R(M, R)$ with $(mf)m = m$.

Chung and Luh [1] proved the following:

### 3.7 Theorem

Let $R$ be a ring with unity. For unital left $R$-modules, the following conditions are equivalent:
(1) $R$ is a semi-simple artinian ring.
(2) Every $R$-module is regular.
(3) Every simple $R$-module is regular.

Using Theorem 3.7 we have

3.8 THEOREM. Let $(\Gamma, M)$ be a semi-prime gamma ring with min-$r$ and min-$l$ conditions. Let $L$ and $R$ be the left and right operator rings respectively. Then, every left (right) $L$-module and every left (right) $R$-module are regular. In particular, $L$, $M$ and $R$ are regular.

PROOF. First we note that by Corollaries 3.6 and 3.7 in [4] $M$ has the left unity $1_L$ and the right unity $1_R$. Hence, $1_L = \sum [e_i, \delta_i]$ where $[e_i, \delta_i]$, $\ldots$, $[e_n, \delta_n]$ are mutually orthogonal primitive idempotents. Similarly for $1_R$. Thus,

$L = \oplus [e_i, \delta_i] L = \oplus [e_i, \delta_i]$ where $[e_i, \delta_i] L$ and $L[e_i, \delta_i]$ are right and left minimal ideals respectively. Hence, $L$ is left and right artinian. So, we have

$L = \oplus_i [e_i, \delta_i] L[e_i, \delta_i]$, where $[e_i, \delta_i] L[e_i, \delta_i]$ are division rings. Thus, $L$ is a semi-simple artinian ring. By Theorem 3.7, every left (right) $L$-module is regular. In particular, $L$ is regular as a left (right) $L$-module. Since $L$ has the unity $1_L$, $L = \text{End}(L_L)$ (End$(L_L)$), and so $L$ is regular as a ring, because for any $h \in L$ there exists $h' \in \text{End}(L_L) = L$ such that $hh' = h$. Now by Theorem 3.2 $M$ is regular. Similarly, every left (right) $R$-module is regular, and in particular $R$ is regular. $\square$

4. Regularity of Morita pairs.

Let $(Q, R, S, T, \mu, \nu)$ be a Morita context, where $Q$ and $R$ are rings, $S$ and $T$ are bimodules such that $S = qS_T$ and $T = R_T q$, and $\mu$ and $\nu$ are mappings such that $\mu : S \otimes qT \rightarrow Q$ and $\nu : T \otimes R_T \rightarrow R$. For $s, s' \in S$, and $t, t' \in T$, denote

$$st = \mu(s \otimes t) \in Q, \quad ts = \nu(t \otimes s) \in R,$$

$$sts' = (st)s' \in S, \quad tst' = (ts)t' \in T.$$  

Due to the associative laws in a Morita context, the conditions (1), (2) and (3) of $0$ are satisfied, and we obtain a gamma ring $(T, S)$.

Conversely, if $(\Gamma, M)$ is a gamma ring with the left and the right operator rings $L$ and $R$, we obtain a Morita context $(L, R, M, \Gamma, \mu, \nu)$. However, note that $Q$ and $R$ of a Morita context are not the operator rings of a gamma ring $(T, S)$, because $S$ (or $T$) is not necessarily a faithful module.

For a Morita context, we let $ST = \{ \sum s \otimes t \}$, $TS = \{ \sum t \otimes s \}$. For the case $Q = ST$ and $R = TS$ we say that $Q$ and $R$ are related through a Morita context, or simply $(Q, R)$ is a Morita pair, [6]. Let $(L, R)$ be a Morita pair, where $L = ST$ and
Regular gamma rings

$R=TS$. Define $L_0=\{h \in L|Th=0\}$, $R_0=\{r \in R|rT=0\}$, and $S_0=\{s \in S|TsT=0\}$. $L_0$ and $R_0$ are ideals of $L$ and of $R$, respectively, and $S_0$ is an $L-R$-submodule of $S$. It is easy to see that $S_0T \subseteq L_0$ and $TS_0 \subseteq R_0$. When $S$ is a finitely generated left $L$-module, we simply say that $LS$ is finitely generated. The same convention is used for $S_R$, $R_T$ and $T_L$. With the notations above, we have the following theorem:

4.1 THEOREM. Suppose that $LS$, $S_R$, $R_T$ and $T_L$ are all finitely generated. Then, the following conditions are equivalent.

(1) $L/L_0$ is a regular ring.
(2) $R/R_0$ is a regular ring.
(3) For any element $s \in S$, there exists an element $t \in T$ such that $sts \equiv s$ mod $S_0$.

PROOF. The proof consists of the following four steps.

Step 1. Suppose that $T_L$ is finitely generated. Then (1) implies (3).

Proof of Step 1. Suppose that (1) holds. Since $T_L$ is finitely generated, we have $T=\sum t_iL_i$ ($t_i \in T$). For any element $s \in S$, $sT=\sum s_iL_i$. Here $s_iL_i$ are principal right ideals of $L$, and since $L/L_0$ is regular, there exists $e \in L$ such that $e^2 \equiv e$ mod $L_0$ and $\sum s_iL_i \equiv eL$ mod $L_0$. Thus, $sT \equiv eL$ mod $L_0$. Then, there exists an element $t_0 \in T$ such that $st_0 \equiv e$ mod $L_0$. On the other hand, for any $t \in T$, $st \equiv eh$ mod $L_0$ with some $h \in L$. Therefore, $est \equiv e^2h \equiv eh \equiv st$ mod $L_0$, (as $s=s$) $t \equiv 0$ mod $L_0$, and hence $(st_0s-s)t \equiv 0$. This implies that $T(st_0s-s)t=0$ for any $t$. We have shown that $st_0s-s \in S_0$. So, (3) holds.

Step 2. Suppose that $LS$ is finitely generated. Then, (3) implies (2).

Proof of Step 2. Suppose that (3) holds. Since $LS$ is finitely generated, $S=\sum L_{si}(u_i \in S)$. For any element $r \in R$, $Sr=\sum Lu_ir=\sum L_{si}$, where $u_i=ur \in S$. By (3), there exist $t_i$ such that $st_0s_t \equiv e$ mod $S_0$. Let $e_i=t_is_i \in R$. Then, $e_i^2=t_is_i t_is_i \equiv e_i$ mod $R_0$, as $TS_0 \subseteq R_0$. Hence, $e_i^2 \equiv e_i$ mod $R_0$. Clearly, $Rt_0t_i \equiv TsiLs_iT \subseteq TLs_i$. On the other hand, $TLs_i \equiv TLs_t t_is_i$ mod $R_0$, and $TLs_i t_is_i \equiv TLs_t e_i \subseteq RE_i$. So, $TLs_i \equiv Re_i \equiv R$ mod $R_0$. Hence, $R \equiv \sum Re_i$ mod $R_0$. By a well known argument in ring theory, we have that $\sum Re_i \equiv Re \equiv R$ mod $R_0$ with $e^2 \equiv e$ mod $R_0$. Thus, every principal left ideal of $R/R_0$ is generated by an idempotent and hence $R/R_0$ is regular. Thus, (3) holds.

Step 3. Suppose that $R_T$ is finitely generated. Then, (2) implies (3).

Proof of Step 3. The proof is similar to the proof of the step 1, using $R$ in place of $L$, and changing the order of multiplication. Namely, let $T=\sum R_t$ and $T=\sum R_t$. We can show that there exists $e \in R$ such that $e^2 \equiv e$ mod $R_0$ and...
$T^s = Re \mod R_0$. Then, $e \equiv t_s \mod R_0$ with some $t_0$. We can also show that $t(st_0 - s) \equiv 0 \mod R_0$, and hence $st_0 \equiv s \mod S_0$.

Step 4. Suppose that $S_R$ is finitely generated. Then (3) implies (1).

Proof of Step 4. The proof is similar to the proof of Step 2. □

4.2 Corollary. Suppose that $tS$ and $T_L$ are finitely generated. Assume, further, that $rR = 0$ implies $r = 0$. Then, $R$ is regular if $L$ is regular.

References


Shoji KYUNO
Dept. of Mathematics
Tohoku Gakuin University
Tagajo, Miyagi 985
Japan

Nobuo NOBUSAWA
Dept. of Mathematics
University of Hawaii
Honolulu, Hawaii 96822
U.S.A.

Mi-Soo B. SMITH
Chaminade University
of Honolulu
Honolulu, Hawaii 96816
U.S.A.